EC202: Microeconomic Principles 2
2012/13

http://darp.lse.ac.uk/EC202

			Office hour
Lecturers	Frank Cowell x7277 f.cowell@lse.ac.uk	R520	Thu 11:30-12:30
	Francesco Nava x6353 f.nava@lse.ac.uk	S482	Thu 11:30-12:30 TBC
Class teachers	Michel Azulai M.D.Azulai@lse.ac.uk		
	Kasia Grabowska K.A.Grabowska@lse.ac.uk		
	Luis Martinez L.R.Martinez@lse.ac.uk		
Secretary	Gisela Ladfico x6674 G.Lafico@lse.ac.uk		
Lectures	Wed 11:00 Hong Kong Theatre [CLM D1] Thu 14:00 Old Theatre		

Course Organisation

- 20 lectures in the first term focusing primarily on price-taking market behaviour by economic agents. Topics include the firm, the consumer, general equilibrium, uncertainty and risk, welfare economics. Where possible the similarity of the economic problems in each topic is exploited in order to highlight the re-use of results.
- 20 lectures in the second term examining problems of strategic interaction among economic agents. This part of the course introduces fundamental concepts in microeconomic theory for strategic environments. The course begins with an introduction to game theory. Fundamental solution concepts are presented and discussed in detail. Static models of complete and incomplete information and dynamic models of complete information are covered in the course. The subgame perfect Folk theorem and its consequences on repeated interactions are detailed in a few lectures. Some of the most relevant applications of such models of behaviour are also introduced. Particular attention will be devoted to: imperfect competition, adverse selection, signalling, moral hazard, externalities and public goods.
- 20 weekly classes beginning in week 3. Class material follows the lectures with about a two-week lag.
- The course text is F. A. Cowell, *Microeconomics: Principles and Analysis* (Oxford University Press 2006) [**C**]. In the second term Most materials covered can be found in the main textbook. Supplementary readings in the second term may be taken from M. J Osborne, *An Introduction to Game Theory* (Oxford University Press, 2004) [**O**] and B. Salanié, *The Economics of Contracts: A Primer* (MIT Press, 2005) [**S**].
- On-line resources are provided at http://darp.lse.ac.uk/EC202. These include answers to weekly class work hand-in exercises and PowerPoint presentation files of lectures.

Lectures, Reading and Classes: First Term

Week	*Lecture Topics*	*Text*	*Exercises*	*Hand-in Work*
1	The firm	C 2		NO
2	The firm and the market	C 2,3		NO
3	The consumer	C 4	2.5	*
4	The consumer and the market	C 4,5	2.9, 3.2	*
5	A simple economy	C 6	4.2, 4.3, 4.6	NO
6	General equilibrium 1	C 7	4.7, 4.8, 4.11	*
7	General equilibrium 2	C 7	5.3, 5.7, 5.8	*
8	Uncertainty and risk	C 8	6.5, 7.3	NO
9	Welfare 1	C 9	7.4, 8.1	*
10	Welfare 2	C 9	8.9, 9.1	*

Lectures, Reading and Classes: Second Term

Week	*Lecture Topics*	*Text*	*Extra*	*Weekly Work*	*Hand-in Work*
1	Static Games	C 10.2	O2.1, O2.6	9.2 , 9.3, 9.5	NO
	Dominance and Nash Equilibrium	C 10.3.1-4	O2.8-9		
2	Mixed Strategy Nash Equilibrium	C 10.3.5	O4.1-4	9.6	NO
	Oligopoly	C 10.4	O3.1-2		
3	Incomplete Information Games	C 10.7	O9.1-3	10.1,10.2, 10.3	*
	Bayes Nash Equilibrium				
4	Dynamic Games	C 10.5-6	O5.1-4	10.4, 10.7.1-3, 10.17, O282.1	NO
	Subgame Perfection				
5	Imperfect Competition			10.7.4-5, 10.12, O183.1-2	*
	Repeated Games: Introduction				
6	Repeated Games: Folk Theorem	C 10.5	O10.1-3, O15.1	10.13, 10.15.1-2	NO
	Adverse Selection: Monopoly		S2, S3.1.3		
7	Adverse Selection: Competition	C 11.2	S3.2.1	10.15.3, 10.16, O429.1, O442.1	*
	Competitive Insurance Markets				
8	Signalling	C 11.3	S4, O10.5-6	11.1, 11.2	NO
9	Moral Hazard	C 11.4	S5.1-3.5	11.5, 11.6	*
10	Externalities	C 13.4	O2.8.4, O9.5	11.8,13.5, 13.6	*
	Public Goods	C 13.6.1-4			

Course Requirements

Classes

Classes begin in the third week of the first term and continue into the third term.

- Teachers assign specific students to prepare short presentations of the exercises in the text chapters. But *all* students should make a reasonable attempt in advance of the class. Team up in small groups if you find this helpful, but make sure that you personally understand *why* the exercise "works."
- Answers to exercises will be posted on the website.
- You should also do the mini problems in the text since they are designed to help you with some of the steps involved in the reasoning. Again, feel free to work together on this. (Answers are in the text, Appendix B).

Hand-in Assignments

- In the starred weeks you have to hand in written assignments which will be posted on the website
- These assignments are of the same scope and difficulty as exam questions.
- These have to be *your own* work. Do *not* work with others on your hand-in assignments.

Examination

There will be a single three-hour, four-question paper. The 2007-2011 papers (in the Library) can be used as general guidance to the style and level of difficulty. The 2012 paper will follow the format of the 2011 paper:

- Question 1 (the compulsory question) is worth 40% of total marks.
- Question 1 requires candidates to answer 5 out of 8 parts.
- The three other questions are worth 20% each.

EC202 - Extra Game Theory Problems
Lent Term 2013

Weekly Course Assignments

Due to popular requests I suggest a few extra practice game theory problems from Osborne's manual. You don't need to practice on these problems, but you may benefit from them. Some are harder than what I would ask on exam day. Solutions are available on the webpage under "Solutions O".

1. Static Complete Information Games
   - 33.1 (Contributing to a Public Good)
   - 48.1 (Voting)
   - 34.3 (Choosing a Route)
   - 80.2 (A Fight)
   - 114.2 (Games with Mixed Strategy Equilibria)
   - 114.4 (Swimming with Sharks)
   - 141.1 (Finding All Mixed Equilibria)
2. Static Incomplete Information Games
   - 282.1 (Fighting an Opponent of Unknown Strength)
   - 282.2 (An Exchange Game)
   - 282.3 (Adverse Selection)
   - 291.1 (reporting a crime with an unknown number of witnesses)
3. Dynamic Complete Information Games: NE vs SPE
   - 163.1 (Nash Equilibria of Extensive Form Games)
   - 163.2 (Voting by Alternating Veto)
   - 173.2 (Finding Subgame Perfect Equilibria)
   - 173.4 (Burning a Bridge)
   - 183.1 (NE of the Ultimatum Game)
   - 183.2 (SPE of the Ultimatum Game)
   - 189.1 (Stackelberg with Quadratic Costs)
4. Repeated Complete Information Games: Subgame Perfect Folk Theorem
   - 428.1 (Strategies in an Infinitely Repeated PD)
   - 429.1 (Grim Trigger Strategy in a General PD)
   - 442.1 (Deviations from Grim Trigger Strategy)
   - 443.1 (Delayed modified Grim Trigger Strategy)
   - 452.3 (a-b) (Minmax Payoffs)

Hand In – Problem Set 1 Francesco Nava
Microeconomic Principles II – EC202 Lent Term 2013

Each question is worth 25 marks. Please give your answers to your class teacher by Friday of week 3 LT. If you do not hand in at your class, make arrangements with your class teacher about where to bring it. Thank you!

1. Consider the following complete information strategic form game:

$1\backslash 2$	L	C	R
T	$3,4$	$1,3$	$6,2$
M	$2,1$	$9,4$	$0,2$
B	$2,2$	$2,3$	$4,2$

   (a) Find the pure strategy Nash equilibria.
   (b) Are there any strictly dominated strategies if players can only play pure strategies?
   (c) Are there any strictly dominated strategies if players can employ a mixed strategies?
   (d) Find the mixed strategy Nash Equilibrium.

2. Consider an auction with two buyers participating and a single object for sale. Suppose that each buyer knows the values of all the other bidders. Order players so that values decrease, $x_1 > x_2$. Consider a 2^{nd} price sealed bid auction. In such auction: all players simultaneously submit a bid b_i; the object is awarded to the highest bidder; the winner pays the second highest submitted bid to the auctioneer; the losers pay nothing. Suppose ties are broken in favor of player 1. That is: if $b_1 = b_2$ then 1 is awarded the object.

   (a) Characterize the best response correspondence of each player.
   (b) Characterize all the Nash equilibria for a given profile (x_1, x_2).
   (c) Consider the Nash equilibrium in which both players bid their value [ie: $b_i = x_i$ for $i \in \{1, 2\}$]. Is this a dominant strategy equilibrium?

Hand In – Problem Set 2 Francesco Nava

Microeconomic Principles II – EC202 Lent Term 2013

Each question is worth 25 marks. Please give your answers to your class teacher by Friday of week 5 LT. If you do not hand in at your class, make arrangements with your class teacher about where to bring it. Thank you!

1. Consider an economy with two producers competing to supply a market. Suppose that the cost function of the first firm displays a constant marginal costs, while the second firm displays increasing marginal costs. In particular assume that:

$$\begin{aligned} c_1(q_1) &= q_1^2 + 2q_1 \\ c_2(q_2) &= 4q_2 \end{aligned}$$

Suppose that the inverse demand in this market is linear and satisfies:

$$p(q) = 10 - 2q$$

Assume that the two firms compete à la Cournot.

(a) Derive the Cournot production levels, profits and the equilibrium price.

(b) Assume that firms form a cartel to sell their output as a monopolist. Derive the cartel production levels, profits and the equilibrium price. Compare them to the competitive and Cournot outcomes.

(c) Assume that firms do not account for their market power, but simply equalize marginal costs to prices. Derive the competitive production levels, profits and the equilibrium price. Compare them to the Cournot outcomes.

2. Four patients have to undergo surgery and rehabilitation in one of two hospitals. Hospital A specializes in surgery. But its elite surgery unit is small. The likelihood of successful surgery, p_A^S, depends on the number of patients treated in the surgery unit, n, as follows:

$$p_A^S(n) = \begin{cases} 17/20 & \text{if} \quad n = 1 \\ 15/20 & \text{if} \quad n = 2 \\ 11/20 & \text{if} \quad n = 3 \\ 7/20 & \text{if} \quad n = 4 \end{cases} .$$

The rehabilitation unit of hospital A is large, but conventional. The likelihood of successful rehabilitation $p_A^R = 1/2$ is independent of the number of patients treated. Hospital B specializes in rehabilitation. But its elite rehabilitation unit is small. The likelihood of successful rehabilitation, p_B^R, depends on the number of patients treated by the unit, n, as follows:

$$p_B^R(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 16/20 & \text{if} \quad n = 2 \\ 14/20 & \text{if} \quad n = 3 \\ 12/20 & \text{if} \quad n = 4 \end{cases}$$

The surgery unit of hospital B is large, but conventional. The likelihood of successful surgery $p_B^S = 1/2$ is independent of the number of patients treated. Surgery outcomes are independent of rehabilitation outcomes. A patient's payoff is 1 if both treatments are successful, and 0 otherwise. A patient is classified as recovered, only if both treatments are successful. [Hint: Recall that if events A and B are independent $\Pr(A \cap B) = \Pr(A)\Pr(B)$].

(a) First suppose that the existing health regulations require all patients to undergo surgery and rehabilitation at the same hospital. Patients can only choose in which of the two hospital to get both treatments. Set this problem up as a strategic-form game. Find a Nash equilibrium of this game. What is the average probability of recovery among the four patients in this equilibrium?

(b) In an attempt to increase the average recovery probability regulators decide to lift the ban on having surgery and rehabilitation at different hospitals. Now patients are free to choose in which hospital to get either treatment. Set this problem up as a strategic-form game. Find a Nash equilibrium of this game. What is the average probability of recovery among the four patients in this equilibrium?

(c) Still unsatisfied about the average recovery probability, regulators decide to try a third policy, in which one of the four patients is randomly selected and sent to the two elite units (surgery in A and rehabilitation in B), while the remaining three are sent to the larger conventional units (surgery in B and rehabilitation in A). What is the average probability of recovery among the four patients with this policy in place?

(d) Compare the average probabilities of recovery under the three regulations. Give an intuitive explanation to the observed change in recovery probabilities.

Hand In – Problem Set 3 Francesco Nava

Microeconomic Principles II – EC202 Lent Term 2013

Each question is worth 25 marks. Please give your answers to your class teacher by Friday of week 7 LT. If you not to hand in at your class, make arrangements with your class teacher about where to bring it. Thank you!

1. Consider a static game of incomplete information with two players and in which player 2 has two possible types. Call them type a and type b. Suppose that the probability of player two being of type a is 0.7 and that payoffs are described by the matrix below:

$1\backslash 2.a$	L	R
T	$4,2$	$0,1$
M	$3,0$	$1,1$
B	$2,4$	$3,3$

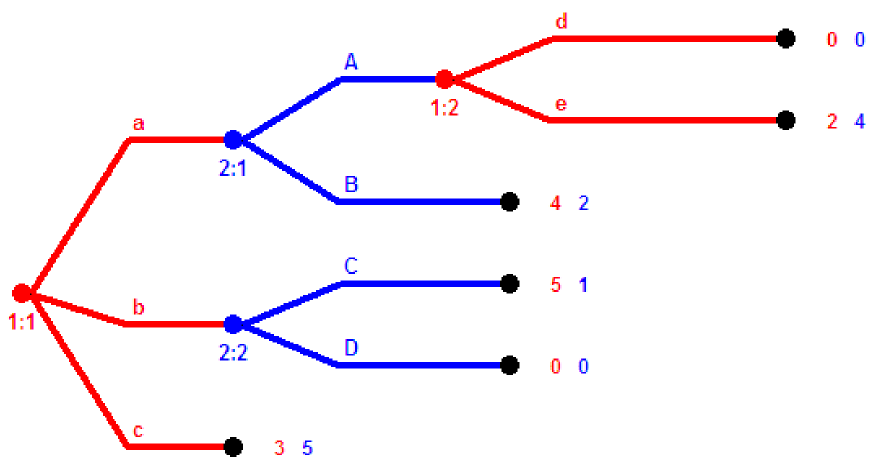
$1\backslash 2.b$	L	R
T	$0,1$	$0,2$
M	$1,1$	$9,1$
B	$3,2$	$4,1$

   (a) What are the possible strategies of each player? Is any one of them dominated. [10 marks]

   (b) Compute a pure strategy Bayes Nash equilibrium. [10 marks]

   (c) Is it a dominant strategy equilibrium? [5 marks]

2. Consider the following extensive form game:



   (a) Find the unique Subgame Perfect equilibrium of this game. [8 marks]

   (b) Find a pure strategy Nash equilibrium with payoffs $(3,5)$. [8 marks]

   (c) Find a pure strategy Nash equilibrium with payoffs $(4,2)$. [9 marks]

3. Two firms compete to sell a good. Firm 1 has higher total costs of production than firm 2, but is the Stackelberg leader and has to produce before firm 2. Firm 2 can observe the output of firm 1 prior to making his output decision. The total costs for the two firms respectively satisfy:

$$C_1(q_1) = 3(q_1)^2$$
$$C_2(q_2) = (q_2)^2 + 4q_2$$

The inverse demand for the output produced by the two firms in this market satisfies:

$$p(q_1 + q_2) = \begin{cases} 10 - 2(q_1 + q_2) & \text{if} \quad q_1 + q_2 \leq 5 \\ 0 & \text{if} \quad q_1 + q_2 > 5 \end{cases}$$

Firms choose how much output to produce in order to maximize their profits.

(a) Compute subgame perfect equilibrium output of each firm and the price for this economy. [12 marks]

(b) Now alter the previous game to allow both firms to make their output decisions simultaneously as in the Cournot model. Compute equilibrium prices and output. [8 marks]

(c) How were prices and output levels affected by such a change? Did the profits of any of the two firms increase? Explain. [5 marks]

4. Consider the following asymmetric Prisoner's Dilemma:

$1\backslash 2$	C	D
C	$3, 4$	$1, 6$
D	$4, 0$	$2, 2$

(a) Find the minmax values of this game. Consider the infinitely repeated version of this game in which all players discount the future at the same rate δ. The following is a "tit for tat" strategy: any player chooses C provided that the other player never chose D; if at any round t a player chooses D, then the other player chooses D in round $t+1$ and continues playing D until the player who first chose D reverts to C; if at any round t a player chooses C, then the other player chooses C in round $t+1$. Write this strategy explicitly. [7 marks]

(b) Find the unique value for the common discount factor δ for which the strategy of part (a) sustains always playing C as a SPE of the infinitely repeated game. [9 marks]

(c) Then, consider the following "trigger" strategy: any player chooses C provided that no player ever played D; otherwise any player chooses D. Write the two incentive constraints that if satisfied would make such a strategy a NE. Then, write the two additional incentive constraints that if satisfied would make such a strategy a SPE. What is the lowest discount rate for which such strategy satisfies all the constraints. [9 marks]

Hand In – Problem Set 4 Francesco Nava

Microeconomic Principles II – EC202 Lent Term 2013

Each question is worth 25 marks. Please give your answers to your class teacher by Friday of week 9 LT. If you do not to hand in at your class, make arrangements with your teacher about where to hand in. Thank you!

1. Consider an economy with a monopolistic electricity supplier. Assume that the costs of producing a unit of electricity are 1$. There are only two goods in this economy namely money, y, and electricity, x. All consumers in this economy are endowed with 100$ in money and no electricity. There are two types of buyers in the economy: type H has high value for electricity, while type L does not. In particular assume that preferences satisfy:

$$\begin{aligned} u(x, y|H) &= 8x^{1/2} + y \\ u(x, y|L) &= (9/2)x^{1/2} + y \end{aligned}$$

   (a) If the monopolist can recognize the type of any individual, find the optimal pricing schedules for both types. Why is this outcome efficient?

   (b) Suppose that 1/8 of all individuals in the population are of type H. If the monopolist cannot recognize the type of any individual, find the separating equilibrium optimal pricing schedules for both types. Why is the outcome inefficient?

   (c) What happens to the economy in part (b) if there is perfect competition? What is the unique price for electricity? Who purchases more electricity than in part (b)?

2. Consider Spence's signalling model. A worker's type is $t \in \{0, 1\}$. The probability that any worker is of type $t = 1$ is equal to 2/3, while the probability that $t = 0$ is equal to 1/3. The productivity of a worker in a job is $(t+1)^2$. Each worker chooses a level of education $e \geq 0$. The total cost of obtaining education level e is $C(e|t) = e^2(2 - t)$. The worker's wage is equal to his expected productivity.

   (a) Characterize all pooling perfect Bayesian equilibrium in which both types of workers choose a strictly positive education level.

   (b) Find all separating perfect Bayesian equilibria.

   (c) Which separating equilibrium survives the intuitive criterion? Is it the one with the lowest education level?

Hand In – Problem Set 5 Francesco Nava

Microeconomic Principles II – EC202 Lent Term 2013

Each question is worth 25 marks. Please give your answers to your class teacher by Friday of week 10 LT. If you do not to hand in at your class, make arrangements with your teacher about where to hand in. Thank you!

1. Consider a "Principal-Agent" model. Suppose that the Agent is a worker who can choose any one of two effort levels, $e \in \{1, 2\}$. Two output levels are feasible for the Principal, namely $q \in \{0, 9\}$. The probability that the Principal achieves high output depends on the effort of the Agent. In particular assume that $\Pr(q = 9|e = 1) = 1/6$ and $\Pr(q = 9|e = 2) = 5/6$. If w denotes the salary of the worker, the expected payoff of the Principal is:

$$\Pi(w|e) = 9 \Pr(q = 9|e) - w$$

The reservation payoff of the worker is 0, while his payoff satisfies:

$$U(e|w) = 2w - e^2$$

   (a) Suppose that the Principal can observe the effort chosen by the worker. Characterize the full information contracts. Do these contracts induce the worker to choose the efficient effort level?

   (b) Suppose that the Principal cannot observe the worker's effort choice, but only output. Thus, he can offer only wage contracts $\{\overline{w}, \underline{w}\}$ which depend on the output produced. If the worker chooses his effort to maximize expected utility, characterize the incentive compatibility constraint that the Principal must satisfy if the worker is to exert high effort $e = 2$.

   (c) Find the optimal wages that a Principal would set in this environment to maximize its profits. Which effort level would be chosen by the Agent in the equilibrium?

2. Consider a Principal-Agent problem with: three exogenous states of nature $\{H, M, L\}$; two effort levels $\{e_a, e_b\}$; and two output levels distributed as follows as a function of the state of nature and the effort level:

	H	M	L
Probability	20%	60%	20%
Output Under e_a	25	25	4
Output Under e_b	25	4	4

The principal is risk neutral, while the agent has a utility function $w^{1/2}$, when receiving a wage w, minus the effort cost which is zero if e_b is chosen, and 1 otherwise. The agent's reservation utility is 0.

   (a) Derive the optimal wage schedule set by the principal when both effort and output are observable.

   (b) Derive the optimal wage schedule set by the principal when only output is observable.

   (c) If the principal cannot observe effort, how much would he be willing to pay for a technology that, prior to the beginning of the game, reveals when the state L is realized?

Microeconomic Principles II EC202

Intro – Lecture 0

Francesco Nava

London School of Economics

January 2013

Notes

Introduction

The Lent Term part of the course:

- introduces several fundamental concepts and results in Game Theory:
    - A model of strategic decision making
    - Static and Dynamic Solution Concepts [NE, DSE, BNE, SPE]
    - Folk Theorems
- provides an introduction to some of its most fruitful applications:
    - Imperfect Competition
    - Adverse Selection
    - Signaling
    - Moral Hazard
    - Public Goods
    - Externalities

Notes

Why Game Theory?

In many relevant economic environments the well being of individuals depends on decisions made by others

Game theory sheds light on behavior in strategic environments

Game theoretic models are commonly used to:

1. study oligopolistic competition [cartels, competition laws]
2. model insurance contracts [public option and socialized healthcare]
3. write incentive contracts [long term incentive contracts]
4. understand voting and political systems
5. design markets and mechanisms [revenue maximizing auctions]
6. bid in auctions [ebay, spectrum auctions]
7. model externalities and public goods
8. make decisions on war strategies

Notes

Objectives of the Course are to:

1. Provide a model of behavior in static strategic environments
2. Understand how private information affects behavior
3. Present various models of dynamic strategic decision-making
4. Understand Folk theorem for repeated games
5. Provide models of imperfect competition
6. Provide a model of monopolistic and competitive insurance markets
7. Understand moral hazard and incentive contracts
8. Understand limitations of efficient markets [public goods, externalities]

Notes

General Information

Lecturer: Francesco Nava

Email: f.nava@lse.ac.uk

Course Website: personal.lse.ac.uk/nava/EC202
darp.lse.ac.uk/EC202

Time and Location: Wed 11-12am & Thur 14-15pm

Office Hours: Wed 130-230pm, Room LIF 3.20

Notes

Weekly Course Program (Weeks 1-5)

1. Game Theory: Dominance and Pure Strategy Nash Equilibrium
   - Readings: C10.2, C10.3.1 to 10.3.4
   - Supplementary: O2.1, O2.6, O2.8 to O2.9
2. Mixed Strategy Nash Equilibrium and Duopoly
   - Readings: C10.3.5, C10.4
   - Supplementary: O4.1 to O4.4, O3.1 to O3.2
3. Incomplete Information and Bayes Nash Equilibrium
   - Readings: C10.7; Supplementary: O9.1 to O9.3
4. Extensive Form Games
   - Readings: C10.5; Supplementary: O5.1 to O5.4
5. Imperfect Competition & Repeated Games
   - Readings: C10.5 to C10.6; Supplementary: O14, O15.1 to O15.2

Notes

Weekly Course Program (Weeks 6-10)

6. Folk Theorem & Adverse Selection (Monopoly)
   - Readings: C11.2; Supplementary: S2
7. Adverse Selection (Competition)
   - Readings: C11.2; Supplementary: S3.1.3, S3.2.1
8. Signaling
   - Readings: C11.3; Supplementary: S4, O10.5, O10.6
9. Moral Hazard
   - Readings: C11.4; Supplementary: S5.1 to S5.3.5
10. Externalities & Public Goods
    - Readings: C13.4, C13.6.1 to C13.6.4
    - Supplementary: O2.8.5, O9.5

Notes

Course Requirements: Classwork

- Classes follow lectures with a one/two weeks lag
- The work for each week consists of one or more exercises
  Answers to exercises will be posted throughout the year
  You are expected to attempt the classwork before the class
- You should try to solve the mini-problems as they are designed to help
  Answers to the mini-problems are in Appendix B of the text
- In weeks 3, 5, 7, 9, and 10 of LT written assignments are required:
  Assignments are of the same scope and difficulty as exam questions
  You are expected to be diligent about deadlines

Notes

Course Requirements: Examination

There will be a single three-hour, four-question paper

The 2005-2012 papers can be used as guidance to the style and difficulty

The 2013 paper will follow the format of the 2011

Principal features are:

- Question 1 (the compulsory question) is worth 40% of total marks
- Question 1 requires candidates to answer 5 out of 8 parts
- The three other questions are worth 20% each

Notes

Materials

Main Textbook

Microeconomics, Cowell, Oxford University Press, 2005 [C]

Slides

Include materials required for the evaluations unless otherwise specified

Slides will be posted before each lecture

Papers

Some papers may be suggested to the interested reader

Supplentary and Alternative Textbook

An Introduction to Game Theory, Osborne, Oxford Press, 2003 [O]

Economics of Contracts, Salanie, MIT Press, 2005 [S]

Notes

Static Complete Information Games

EC 202 – Lectures I & II

Francesco Nava

LSE

January 2013

Notes

Summary

Games of Complete Information:

- Definitions:
  - Game: Players, Actions, Payoffs
  - Strategy
  - Best Response
- Solution Concepts [in pure strategies]:
  - Dominant Strategy Equilibrium
  - Nash Equilibrium
- Properties of Nash Equilibria:
  - Non-Existence and Multiplicity
  - Inefficiency
- Examples

Notes

Introduction to Games

Any environment in which the choices of an individual affect the well being of others can be modeled as a game.

What pins down a specific game:

- Who participates in a game [Players]
- The choices that participants have [Choices]
- The well being of individuals [Payoffs]
- The information that individuals have [Rules of the Game]
- The timing of events and decisions [Rules of the Game]

Notes

Introduction to Games

In most models discussed in MT:

- individual decisions did not affect the well being of others
- any dependence would just hinge from equilibrium prices

Lectures I, II & III discuss **complete information strategic form games**.

In such environments:

- Individuals know everything, but for the decisions made by others
- All decisions take place at once

Notes

Complete Information (Strategic Form) Game

A complete information game G consists of:

- A set of players:
  - N of size n
- An action set for each player in the game:
  - A_i for player i's
  - An action profile $\mathbf{a} = (a_1, a_2, ..., a_n)$ picks an action for each player
- A utility map for each player mapping action profiles to payoffs:
  - $u_i(\mathbf{a})$ denotes player i's payoff of action profile $\mathbf{a}$

$B\backslash G$	s	m
s	5,2	1,2
m	0,0	3,5

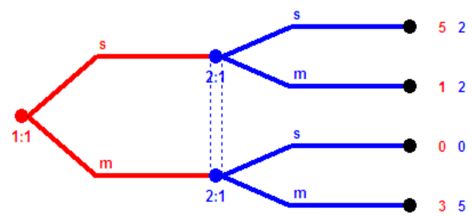
Notes

Representing Simoultaneous Move Complete Info Games

Strategic Form

$1\backslash 2$	s	m
s	5,2	1,2
m	0,0	3,5

Extensive Form



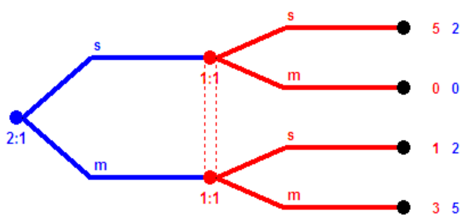
Notes

Representing Simoultaneous Move Complete Info Games

Strategic Form

1\2	s	m
s	5,2	1,2
m	0,0	3,5

Extensive Form



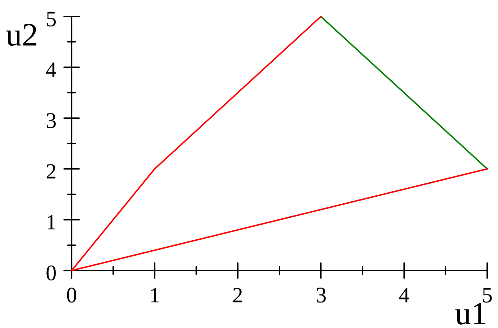
Notes

Feasible Payoffs and Efficiency

Strategic Form

1\2	s	m
s	5,2	1,2
m	0,0	3,5

Feasible Payoffs



Efficient payoffs are on the north-east boundary (in green)

Notes

Information and Pure Strategies

A strategy in a game:

- is a map from information into actions
- it defines a plan of action for a player

In a complete information strategic form game:

- players have no private information
- players act simultaneously

In this context a strategy is any element of the set of actions

For instance a (pure) strategy for player i is simply $a_i \in A_i$

Notes

Best Responses

Define a profile of actions chosen by all players other than i by $\mathbf{a}_{-i}$:

$$\mathbf{a}_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$$

The best response correspondence of player i is defined by:

$$b_i(\mathbf{a}_{-i}) = \arg\max_{a_i \in A_i} u_i(a_i, \mathbf{a}_{-i}) \quad \text{for any } \mathbf{a}_{-i}$$

Thus a_i is a best response to $\mathbf{a}_{-i}$ – i.e. $a_i \in b_i(\mathbf{a}_{-i})$ – if and only if:

$$u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a_i', \mathbf{a}_{-i}) \quad \text{for any } a_i' \in A_i$$

BR identifies the optimal action for a player given choices made by others

For instance:

$B \backslash G$	s	m
s	5,2	1,2
m	0,0	3,5

Notes

Strict Dominance

- Strategy a_i **strictly dominates** a_i' if:

$$u_i(a_i, \mathbf{a}_{-i}) > u_i(a_i', \mathbf{a}_{-i}) \quad \text{for any } \mathbf{a}_{-i}$$

- a_i is **strictly dominant** if it strictly dominates any other a_i'
- a_i is **strictly undominated** if no strategy strictly dominates a_i
- a_i is **strictly dominated** if a strategy strictly dominates a_i

In the following example s is strictly dominant for B:

$B \backslash G$	s	m
s	5,-	2,-
m	0,-	1,-

Notes

Weak Dominance

- Strategy a_i **weakly dominates** a_i' if:

$$u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a_i', \mathbf{a}_{-i}) \quad \text{for any } \mathbf{a}_{-i}$$
$$u_i(a_i, \mathbf{a}_{-i}) > u_i(a_i', \mathbf{a}_{-i}) \quad \text{for some } \mathbf{a}_{-i}$$

- a_i is **weakly dominant** if it weakly dominates any other a_i'
- a_i is **weakly undominated** if no strategy weakly dominates a_i
- a_i is **strictly dominated** if a strategy strictly dominates a_i

In the following example s is weakly dominant for B:

$B \backslash G$	s	m
s	5,-	2,-
m	0,-	2,-

Notes

Dominance Examples

One strictly and one weakly dominated strategy:

1\2	L	C	R
T	-,1	-,2	-,1
B	-,0	-,1	-,3

One strictly dominant strategy:

1\2	L	C	R
T	-,1	-,2	-,3
B	-,0	-,1	-,2

One weakly dominant strategy:

1\2	L	C	R
T	-,1	-,2	-,2
B	-,0	-,0	-,1

Notes

Dominant Strategy Equilibrium

Definitions (Dominant Strategy Equilibrium DSE)

A strict Dominant Strategy equilibrium of a game G consists of a strategy profile $\mathbf{a}$ such that for any $\mathbf{a}'_{-i}$ and $i \in N$:

$$u_i(a_i, \mathbf{a}'_{-i}) > u_i(a'_i, \mathbf{a}'_{-i}) \quad \text{for any } a'_i \in A_i$$

- For weak DSE change $>$ with $\geq$ ...
- a profile $\mathbf{a}$ is a DSE **iff** $a_i \in b_i(\mathbf{a}'_{-i})$ for any $\mathbf{a}'_{-i}$ and $i \in N$
- Example (Prisoner's Dilemma):

$B \backslash S$	N	C
N	5,5	0,6
C	6,0	1,1

Notes

Iterative Elimination of Dominated Strategies

To find dominant strategies eliminate dominated strategies from the game

If necessary repeat the process to possibly rule out more strategies

Consider the following example:

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

⇒

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

At the first instance only D is dominated for player 1

No strategy is dominated a priori for player 2

[Strategies in green in the table are dominated and thus eliminated]

Notes

Iterative Elimination of Dominated Strategies

Once D has been eliminated from the game:

Strategy R is dominated for player 2

No strategy is dominated for player 1

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

⇒

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

Once R has been eliminated from the game:

Strategy T is dominated for player 1

A final iteration yields (M, L) as the only surviving strategies

Notes

Dominance: Final Considerations

Dominance is often considered a benchmark of rationality:

- Rational players never choose dominated strategies
- *Common knowledge of rationality* means:
  players only employ strategies that survive iterative elimination

Dominance is a simple concept buy with important limitations:

- Often there is no dominant strategy even after iteration
- It often leads to inefficient outcomes

Thus a weaker notion of equilibrium needs to be introduced to model behavior especially for richer setups

Notes

Nash Equilibrium: Introduction

Dominance was the appropriate solution concept if players had no information or beliefs about choices made by others

The weaker notion of equilibrium that will be introduced presumes that:

- players have correct beliefs about choices made by others
- players choices are optimal given such beliefs
- the environment is common knowledge among players

Such model allows for tighter predictions when dominance has no bite

Notes

Nash Equilibrium

Definition (Nash Equilibrium NE)

A (pure strategy) Nash equilibrium of a game G consists of a strategy profile $\mathbf{a} = (a_i, \mathbf{a}_{-i})$ such that for any $i \in N$:

$$u_i(\mathbf{a}) \geq u_i(a_i', \mathbf{a}_{-i}) \quad \text{for any } a_i' \in A_i$$

- a profile $\mathbf{a}$ is a NE **iff** $a_i \in b_i(\mathbf{a}_{-i})$ for any $i \in N$

Properties:

- Strategy profiles are independent
- Strategy profiles common knowledge

Notes

More Examples

Games may have more NE's (Battle of the Sexes):

$B\backslash G$	s	m
s	5,2	1,2
m	0,0	3,5

Nash equilibria may not be efficient (Prisoner's Dilemma):

$B\backslash S$	N	C
N	5,5	0,6
C	6,0	1,1

Pure strategy Nash equilibria may not exist (Matching Pennies):

$B\backslash G$	H	T
H	0,2	2,0
T	2,0	0,2

Notes

Examples, Properties and Limitations

Games may have more NE's (Battle of the Sexes):

$B \backslash G$	s	m
s	5,2	1,2
m	0,0	3,5

Nash equilibria may not be efficient (Prisoner's Dilemma):

$B \backslash S$	N	C
N	5,5	0,6
C	6,0	1,1

Pure strategy Nash equilibria may not exist (Matching Pennies):

$B \backslash G$	H	T
H	0,2	2,0
T	2,0	0,2

Notes

Some Nash Equilibria are More Risky

Consider the Stag Hunt game:

$B \backslash G$	Stag	Hare
Stag	9,9	0,8
Hare	8,0	8,8

Best responses for this game are:

$B \backslash G$	Stag	Hare
Stag	9,9	0,8
Hare	8,0	8,8

Both players choosing to go for the stag is NE
Such NE involves greater risks of miscoordination
than the NE in which both go for the hare

Notes

Three Player Example

- A game with more than 2 players:

3	L		R	
1\2	A	B	A	B
T	1,0,1	1,0,0	0,1,1	0,1,1
D	0,1,1	1,2,0	1,0,1	2,1,1

Notes

Three Player Example

- A game with more than 2 players:

3	L		R	
1\2	A	B	A	B
T	1,0,1	1,0,0	0,1,1	0,1,1
D	0,1,1	1,2,0	1,0,1	2,1,1

- To find all PNE check best reply maps:

3	L		R	
1\2	A	B	A	B
T	1,0,1	1,0,0	0,1,1	0,1,1
D	0,1,1	1,2,0	1,0,1	2,1,1

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Notes

First Price Auction Example

An auctioneer sells one object with the following rule:

- Buyers simultaneously submit sealed bids (a_i)
- Highest bid wins the object
- Winner pays his own bid
- Ties are broken in favor of buyer with lowest index

N buyers participate at the auction

Their values for the object are $v_1 > v_2 > ... > v_N$

Payoffs of player i is:

$$(v_i - a_i)\mathbb{I}(a_i \geq \max_{j \neq i} a_j)$$

NE: $a_1 = a_2 = v_2$ and $a_i \leq v_2$ for any $i > 2$

Notes

First Price Auction All Nash Equilibria

Consider the case for $N = \{1, 2\}$ and values $v_1 > v_2$

The best response maps for the two players are:

$$b_1(a_2) = \begin{cases} a_1 < a_2 & \text{if} \quad a_2 > v_1 \\ a_1 \leq a_2 & \text{if} \quad a_2 = v_1 \\ a_1 = a_2 & \text{if} \quad a_2 < v_1 \end{cases}$$

$$b_2(a_1) = \begin{cases} a_2 \leq a_1 & \text{if} \quad a_1 \geq v_2 \\ a_2 = a_1 + \varepsilon & \text{if} \quad a_1 < v_2 \end{cases}$$

The Nash Equilibria of the game satisfy:

- $b_1 = b_2 \in [v_2, v_1]$

Notes

War of Attrition Example

Consider a game with two competitors $N = \{1, 2\}$ involved in a fight:

- Suppose that the value of winning the fight is v_i for $i \in N$
- Competitors choose how much effort to put in a fight $a_i \in [0, \infty)$
- The payoff of competitor i given their effort levels are:

$$v_i \mathbb{I}(a_i > a_j) + (v_i/2)\mathbb{I}(a_i = a_j) - \min_{i\in\{1,2\}} a_i$$

- Best response functions satisfy:

$$b_i(a_j) = \begin{cases} a_i > a_j & \text{if} \quad a_j < v_i \\ a_i = 0 \quad \text{or} \quad a_i > a_j & \text{if} \quad a_j = v_i \\ a_i = 0 & \text{if} \quad a_j > v_i \end{cases}$$

All Nash Equilibria of the game satisfy one of the following:

- $a_1 = 0$ and $a_2 \geq v_1$
- $a_2 = 0$ and $a_1 \geq v_2$

Notes

DSE implies NE

Fact

Any dominant strategy equilibrium is a Nash equilibrium

Proof.

If $\mathbf{a}$ is a DSE then $a_i \in b_i(\mathbf{a}'_{-i})$ for any $\mathbf{a}'_{-i}$ and $i \in N$.
Which implies $\mathbf{a}$ is NE since $a_i \in b_i(\mathbf{a}_{-i})$ for any $i \in N$. $\square$

Notes

Static Complete Information Games

EC 202 – Lecture III: Mixed Strategies

Francesco Nava

LSE

January 2013

Notes

Summary

Games of Complete Information:

- Definitions:
  - Mixed Strategy
  - Dominant Strategy
- Solution Concepts:
  - Nash Equilibrium
- Examples
- NE Existence

Notes

Introduction to Mixed Strategies

A problematic aspect of the solution concepts discussed in the first two lectures was that equilibria did not always exist.

Reasons for the lack of existence were:

- Non-convexities in the choice sets
- Discontinuities of the best response correspondences

Today we introduce mixed strategies which solve both problems and guarantee existence of at least a Nash equilibrium.

Notes

Mixed Strategy Definition

Consider complete information static game $\{N, \{A_i, u_i\}_{i \in N}\}$

A mixed strategy for $i \in N$ is a probability distribution over actions in A_i

Thus σ_i is a mixed strategy if:

- $\sigma_i(a_i) \geq 0$ for any $a_i \in A_i$
- $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$

Intuitively $\sigma_i(a_i)$ is the probability that player i chooses to play a_i

E.G. $\sigma_1(B) = 0.3$ and $\sigma_1(C) = 0.7$ is a mixed strategy for 1 in:

1\2	B	C
B	2,0	0,2
C	0,1	1,0

Notes

Simplex

The set of possible probability distributions on a set B is:

- called simplex
- and denoted by $\Delta(B)$

Such set is:

- Closed
- Bounded
- Convex

Notes

Payoffs from Mixed Strategies

As in the last lectures often we denote:

$$\mathbf{ff}_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N)$$

The payoff to player i from choosing $\sigma_i \in \Delta(A_i)$ when others follow σ_{-i} is:

$$\begin{aligned} u_i(\sigma_i, \mathbf{ff}_{-i}) &= \textstyle\sum_{\mathbf{a}\in\mathbf{A}} \Pr(\mathbf{a}) u_i(\mathbf{a}) = \\ &= \textstyle\sum_{\mathbf{a}\in\mathbf{A}} \prod_{j\in N} \sigma_j(a_j) u_i(\mathbf{a}) \end{aligned}$$

E.G. If players follow $\sigma_1(B) = \sigma_2(B) = 0.3$ in the game:

1\2	B	C
B	2,0	0,2
C	0,1	1,0

The payoff to player 1 is: $u_1(\sigma_1, \sigma_2) = (.09)2 + (.49)1 + (.42)0 = 0.67$

Notes

Best Reply Maps

Denote the best reply correspondence of i by $b_i(\mathbf{ff}_{-i})$

The map is defined by:

$$b_i(\mathbf{ff}_{-i}) = \arg\max_{\sigma_i \in \Delta(A_i)} u_i(\sigma_i, \mathbf{ff}_{-i})$$

For instance consider the game:

1\2	s	m
s	5,2	1,2
m	0,0	3,5

If $\sigma_1(s) = 1$ then any $\sigma_2(s) \in [0,1]$ satisfies $\sigma_2 \in b_2(\sigma_1)$

If $\sigma_1(s) < 1$ then only $\sigma_2(s) = 0$ satisfies $\sigma_2 \in b_2(\sigma_1)$

Notes

Dominated Strategies

- Strategy σ_i **weakly dominates** a_i if:

$$u_i(\sigma_i, \mathbf{a}_{-i}) \geq u_i(a_i, \mathbf{a}_{-i}) \quad \text{for any } \mathbf{a}_{-i}$$
$$u_i(\sigma_i, \mathbf{a}_{-i}) > u_i(a_i, \mathbf{a}_{-i}) \quad \text{for some } \mathbf{a}_{-i}$$

- a_i is **weakly undominated** if no strategy weakly dominates it

- This allows us to rule out more strategies than before, eg:

1\2	L	C	R
T	6,6	0,2	0,0
B	0,0	0,2	6,6

- $\sigma_2(L) = \sigma_2(R) = 0.5$ strictly dominates C since:

$$u_2(\sigma_2, a_1) = 3 > u_2(C, a_1) = 2$$

Notes

Nash Equilibrium

Definition (Nash Equilibrium NE)

A Nash equilibrium of a game consists of a strategy profile $\mathbf{ff} = (\sigma_i, \mathbf{ff}_{-i})$ such that for any $i \in N$:

$$u_i(\mathbf{ff}) \geq u_i(a_i, \mathbf{ff}_{-i}) \quad \text{for any } a_i \in A_i$$

Implicit to the definition of NE are the following assumptions:

- Each agent chooses his mixed strategy independently of others
- Each agent knows and believes which strategies the others adopt
- Each agent chooses his strategy to maximize expected utility given his beliefs

Notes

Nash Equilibrium Computation Help

- A strategy profile σ is a Nash Equilibrium if and only if:

$$\begin{aligned} u_i(\mathbf{ff}) &= u_i(a_i, \mathbf{ff}_{-i}) \quad \text{for any } a_i \text{ such that } \sigma_i(a_i) > 0 \\ u_i(\mathbf{ff}) &\geq u_i(a_i, \mathbf{ff}_{-i}) \quad \text{for any } a_i \text{ such that } \sigma_i(a_i) = 0 \end{aligned}$$

- If a_i is strictly dominated, then $\sigma_i(a_i) = 0$ in any Nash equilibrium
- If a_i is weakly dominated, then $\sigma_i(a_i) > 0$ in a Nash equilibrium only if any profile of actions $\mathbf{a}_{-i}$ for which a_i is strictly worse occurs with zero probability

Notes

Examples

Games may have more NEs (Battle of the Sexes):

1\2	s	m
s	5,2	1,1
m	0,0	2,5

There are 2 PNE & a mixed NE in which $\sigma_1(s) = 5/6$ & $\sigma_2(s) = 1/6$:

$$u_1(s, \sigma_2) = 5\sigma_2(s) + (1 - \sigma_2(s)) = 2(1 - \sigma_2(s)) = u_1(m, \sigma_2)$$
$$u_2(m, \sigma_1) = \sigma_1(s) + 5(1 - \sigma_1(s)) = 2\sigma_1(s) = u_2(s, \sigma_1)$$

Games may have only mixed NE (Matching Pennies):

$B\backslash G$	H	T
H	0,2	2,0
T	2,0	0,2

There is a unique NE in which $\sigma_1(H) = \sigma_2(H) = 1/2$:

$$2(1 - \sigma_i(H)) = 2\sigma_i(H)$$

Notes

More Examples

Games may have a continuum of NEs:

1\2	L	R	C
T	4,1	0,2	3,2
D	0,2	2,0	1,0

The game has 1 PNE & a continuum of NEs, namely:

$$\sigma_1(T) = 2/3 \ \ \& \ \sigma_2(R) = 1 - \sigma_2(L) - \sigma_2(C)$$
$$\sigma_2(L) = 1/3 - (2/3)\sigma_2(C) \ \ \text{for } \sigma_2(C) \in [0, 1/2]$$

These are all NE's since:

$$u_2(L, \sigma_1) = \sigma_1(T) + 2(1 - \sigma_1(T)) = 2\sigma_1(T) = u_2(R, \sigma_1) = u_2(C, \sigma_1)$$
$$u_1(T, \sigma_2) = 4\sigma_2(L) + 3\sigma_2(C) = 2\sigma_2(R) + \sigma_2(C) = u_1(D, \sigma_2)$$

Notes

Three Player Example

A game with more than 2 players:

3	L		R	
1\2	A	B	A	B
T	1,0,1	0,1,1	1,1,0	0,0,0
D	0,1,0	1,0,1	0,0,1	1,1,0

The NE conditions require:

$$
\begin{aligned}
u_1(T, \mathbf{ff}_{-1}) &= \sigma_2(A) = (1 - \sigma_2(A)) = u_1(D, \mathbf{ff}_{-1}) \\
u_2(A, \mathbf{ff}_{-2}) &= (1 - \sigma_1(T))\sigma_3(L) + \sigma_1(T)(1 - \sigma_3(L)) = \\
&= \sigma_1(T)\sigma_3(L) + (1 - \sigma_1(T))(1 - \sigma_3(L)) = u_2(B, \mathbf{ff}_{-2}) \\
u_3(L, \mathbf{ff}_{-3}) &= \sigma_1(T) + (1 - \sigma_1(T))(1 - \sigma_2(A)) = \\
&= (1 - \sigma_1(T))\sigma_2(A) = u_3(R, \mathbf{ff}_{-3})
\end{aligned}
$$

The unique solution requires: $\sigma_2(A) = \sigma_3(L) = 1/2$ and $\sigma_1(T) = 0$

Notes

Nash Equilibrium Existence

Theorem (NE Existence)

Any game with a finite number of actions possesses a Nash equilibrium.

Theorem (PNE Existence)

Any game with convex and compact action sets and with continuous and quasi-concave payoff functions possesses a pure strategy Nash equilibrium.

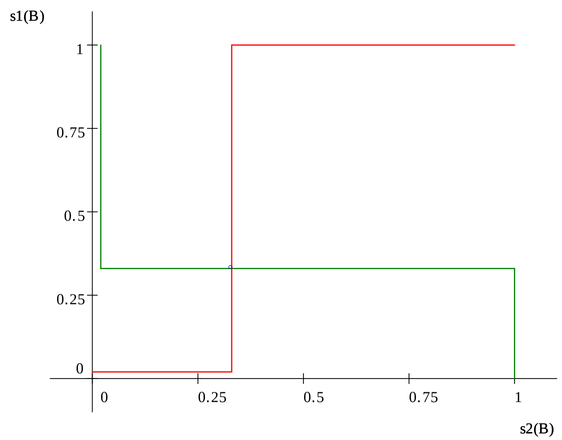
Assumptions in both theorems guarantee that best response maps are continuous (upper-hemicontinuous, closed and convex valued) on convex compact sets and thus existence...

Notes

Continuous Best Responses Imply Existence

A game without PNE and with a single NE:

1\2	B	C
B	2,0	0,2
C	0,1	1,0



Notes

Notes

Oligopoly Pricing
EC 202 – Lecture IV

Francesco Nava

LSE

January 2013

Notes

Summary

The models of competition presented in MT explored the consequences on prices and trade of two extreme assumptions:

- Perfect Competition [Many sellers supplying many buyers]
- Monopoly [One seller supplying many buyers]

Today intermediate assumption is discussed:

- Oligopoly [Few sellers supplying many buyers]

Two models of competition among oligopolists are presented:

- Quantity Competition [aka Cournot Competition]
- Price Competition [aka Bertrand Competition]

Notes

A Duopoly

Consider the following economy:

- There are two firms $N = \{1, 2\}$
- Each firm $i \in N$ produces output q_i with a cost function
  - $c_i(q_i)$ mapping quantities to costs
- Aggregate output in this economy is $q = q_1 + q_2$
- Both firms face an aggregate inverse demand for output
  - $p(q)$ mapping aggregate output to prices
- The payoff of each firm $i \in N$ is its profits:

$$u_i(q_1, q_2) = p(q)q_i - c_i(q_i)$$

Profits depend on the output decisions of both

Notes

Cournot Competition: Duopoly

Competition proceeds as follows:

- All firms simultaneously select their output to maximize profits
- Each firm takes as given the output of its competitors
- Firms account for the effects of their output decision on prices

In particular the decision problem of player $i \in N$ is to:

$$\max_{q_i} u_i(q_i, q_j) = \max_{q_i} p(q_i + q_j)q_i - c_i(q_i)$$

[Historically this is the first known example of Nash Equilibrium – 1838]

[Example: Visa vs Mastercard]

Notes

Cournot Competition: Equilibrium

If standard conditions on primitives of the problem hold:

- a pure strategy Nash equilibrium exists
- the PNE is characterized by the FOC

If so, the problem of any producer $i \in N$ satisfies:

$$\frac{\partial u_i(q_i, q_j)}{\partial q_i} = \underbrace{p(q_i + q_j) + \frac{\partial p(q_i + q_j)}{\partial q_i} q_i}_{\text{Marginal Revenue}} - \underbrace{\frac{\partial c_i(q_i)}{\partial q_i}}_{\text{Marginal Cost}} \begin{cases} \leq 0 & \text{if} \quad q_i = 0 \\ = 0 & \text{if} \quad q_i > 0 \end{cases}$$

Marginal revenue accounts for the distortion in prices

Prices decrease if inverse demand is downward-sloping

FOC defines the best response (aka *reaction function*) of player i:

$$q_i = b_i(q_j)$$

Notes

Cournot Example

Consider the following economy:

- $p(q) = 2 - q$
- $c_1(q_1) = q_1^2$ and $c_2(q_2) = 3q_2^2$

Firm i's problem is to choose production q_i given choice of the other q_j:

$$\max_{q_i} (2 - q_i - q_j) q_i - c_i(q_i)$$

The best reply map of each firm is determined by FOC:

$$2 - 2q_1 - q_2 - 2q_1 = 0 \quad \Rightarrow \quad q_1 = b_1(q_2) = (2 - q_2)/4$$
$$2 - 2q_2 - q_1 - 6q_2 = 0 \quad \Rightarrow \quad q_2 = b_2(q_1) = (2 - q_1)/8$$

Cournot Equilibrium outputs are: $q_1 = 14/31$ and $q_2 = 6/31$

Perfect competition outputs are larger: $q_1^* = 3/5$ and $q_2^* = 1/5$

Notes

Collusion and Cartels

Suppose that the producers collude by forming a *cartel*

A cartel maximizes the joint profits of the two firms:

$$\max_{q_1, q_2} p(q)q - c_1(q_1) - c_2(q_2)$$

First order optimality of this problem requires for any $i \in N$:

$$\underbrace{p(q) + \frac{\partial p(q)}{\partial q} q}_{\text{Marginal Revenue}} - \underbrace{\frac{\partial c_i(q_i)}{\partial q_i}}_{\text{Marginal Cost}} \begin{cases} \leq 0 & \text{if} \quad q_i = 0 \\ = 0 & \text{if} \quad q_i > 0 \end{cases}$$

Aggregate profits are higher in the cartel

Players account for effects of their output choice on others

But the profits of each individual do not necessarily increase

Notes

Collusion Example

Consider the previous duopoly, but suppose that a cartel is in place

If so, FOC for the cartel production satisfy:

$$\begin{aligned} 2 - 2\overline{q}_1 - 2\overline{q}_2 - 2\overline{q}_1 = 0 \quad &\Rightarrow \quad \overline{q}_1 = (1 - \overline{q}_2)/2 \\ 2 - 2\overline{q}_2 - 2\overline{q}_1 - 6\overline{q}_2 = 0 \quad &\Rightarrow \quad \overline{q}_2 = (1 - \overline{q}_1)/4 \end{aligned}$$

Cartel outputs are: $\overline{q}_1 = 3/7$ and $\overline{q}_2 = 1/7$

Cournot outputs are larger: $q_1 = 14/31$ and $q_2 = 6/31$

Cartel profits are: $\overline{u}_1 = 3/7$ and $\overline{u}_2 = 1/7$

Cournot profits are: $u_1 = 392/961$ and $u_2 = 144/961$

Total profits are larger with a cartel in place, but not all firms may benefit

Notes

Defection from a Cartel

Suppose that the firm j produces the cartel output $\overline{q}_j$

If so, firm i may benefit by producing more than the cartel output since:

$$b_i(\overline{q}_j) > \overline{q}_i$$

In this scenario sustaining a cartel may be hard without output monitoring

In the example this was the case as firms preferred to increase output:

$$\begin{aligned} b_1(1/7) &= 13/28 > 3/7 \\ b_2(3/7) &= 11/56 > 1/7 \end{aligned}$$

If so the problem of sustaining the cartel becomes a Prisoner's dilemma

Notes

Bertrand Competition: Duopoly

Competition proceeds as follows:

- All firms simultaneously quote a price to maximize profits
- Each firm takes as given the price quoted by its competitors
- Firms account for the effects of their pricing decision on sales

Consider an economy with:

- Two producers with constant marginal costs c
- Aggregate demand for output given by $q(p) = (b_0 - p)/b$

Given the prices demand of output from firm $i \in N$ is:

$$q_i(p_i, p_j) = \begin{cases} q(p_i) & \text{if} \quad p_i < p_j \\ q(p_i)/2 & \text{if} \quad p_i = p_j \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

Notes

Bertrand Competition: Monopoly

If only one firm operated in such market it would choose the price to:

$$\max_p u(p) = \max_p q(p)(p-c)$$

Thus a profit maximizing monopolist would sell goods at a price:

$$\overline{p} = (b_0 + c)/2$$

With two producers the problem of each firm becomes:

$$\max_{p_i} u_i(p_i, p_j) = \max_{p_i} q_i(p_i, p_j)(p_i - c)$$

Notes

Bertrand Competition: Best Responses

In the Bertrand model, i's optimal pricing is aimed at "maximizing sales"

In particular firm i would set prices as follows (for ε small):

- If $p_j > \overline{p}$, set $p_i = \overline{p}$ and capture all the market at the monopoly price
- If $\overline{p} \geq p_j > c$, set $p_i = p_j - \varepsilon$, undercut j and capture all the market
- If $c \geq p_j$, set $p_i = c$ as there are no benefits by pricing below MC

Such logic requires the best response of each player to satisfy:

$$p_i = b_i(p_j) = \begin{cases} \overline{p} & \text{if} \quad p_j > \overline{p} \\ p_j - \varepsilon & \text{if} \quad \overline{p} \geq p_j > c \\ c & \text{if} \quad c \geq p_j \end{cases}$$

Thus in the unique NE both firms set $p_1 = p_2 = c$ and perfect competition emerges with just 2 firms!

Notes

Cournot vs Bertrand Competition

Bertrand model predicts that duopoly is enough to push down prices to marginal cost (as in perfect competition)

Cournot model instead predicts that few producers do not suffice to eliminate markups (prices above marginal cost)

In both models there are incentives to form a cartel and to charge the monopoly price

Neither model is intrinsically better

Accuracy of either model depends on the fundamentals of the economy:

- Bertrand works better when capacity is easy to adjust
- Cournot works better when capacity is hard to adjust

Notes

Notes

Games of Incomplete Information

EC202 Lectures V & VI

Francesco Nava

LSE

January 2013

Notes

Summary

Games of Incomplete Information:

- Definitions:
  - Incomplete Information Game
  - Information Structure and Beliefs
  - Strategies
  - Best Reply Map
- Solution Concepts in Pure Strategies:
  - Dominant Strategy Equilibrium
  - Bayes Nash Equilibrium
- Examples
- EXTRA: Mixed Strategies & Bayes Nash Equilibria

Notes

Incomplete Information (Strategic Form)

An incomplete information game consists of:

- N the set of players in the game
- A_i player i's action set
- $\mathbb{X}_i$ player i's set of possible signals
  - A profile of signals $x = (x_1, ..., x_n)$ is an element $\mathbb{X} = \times_{j \in N} \mathbb{X}_j$
- f a distribution over the possible signals
- $u_i : A \times \mathbb{X} \to \mathbb{R}$ player i's utility function, $u_i(a|x)$

Notes

Bayesian Game Example

Consider the following Bayesian game:

- Player 1 observes only one possible signal: $\mathbb{X}_1 = \{C\}$
- Player 2's signal takes one of two values: $\mathbb{X}_2 = \{L, R\}$
- Probabilities are such that: $f(C, L) = 0.6$
- Payoffs and action sets are as described in the matrix:

$1\backslash 2.L$	y_2	z_2
y_1	1,2	0,1
z_1	0,4	1,3

$1\backslash 2.R$	y_2	z_2
y_1	1,3	0,4
z_1	0,1	1,2

Notes

Information Structure

Information structure:

- X_i denotes the signal as a random variable
- belongs to the set of possible signals $\mathbb{X}_i$
- x_i denotes the realization of the random variable X_i
- $X_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$ denotes a profile of signals for all players other than i
- Player i observes only X_i
- Player i ignores X_{-i}, but knows f

With such information player i forms beliefs regarding the realization of the signals of the other players x_{-i}

Notes

Beliefs about other Players' Signals [Take 1]

In this course we consider models in which signals are independent:

$$f(x) = \prod_{j \in N} f_j(x_j)$$

This implies that the signal x_i of player i is independent of X_{-i}

Beliefs are a probability distribution over the signals of the other players

Any player forms beliefs about the signals received by the other players by using Bayes Rule

Independence implies that conditional observing $X_i = x_i$ the beliefs of player i are:

$$f_i(x_{-i}|x_i) = \prod_{j \in N \setminus i} f_j(x_j) = f_{-i}(x_{-i})$$

Notes

Extra: Beliefs about other Players' Signals [Take 2]

Also in the general case with interdependence players form beliefs about the signals received by the others by using Bayes Rule

Conditional observing $X_i = x_i$ the beliefs of player i are:

$$f_i(x_{-i}|x_i) = \Pr(X_{-i} = x_{-i}|X_i = x_i) =$$

$$= \frac{\Pr(X_{-i} = x_{-i} \quad \cap \quad X_i = x_i)}{\Pr(X_i = x_i)} =$$

$$= \frac{\Pr(X_{-i} = x_{-i} \quad \cap \quad X_i = x_i)}{\sum_{y_{-i}\in\mathbb{X}_{-i}} \Pr(X_{-i} = y_{-i} \quad \cap \quad X_i = x_i)} =$$

$$= \frac{f(x_{-i}, x_i)}{\sum_{y_{-i}\in\mathbb{X}_{-i}} f(y_{-i}, x_i)}$$

Beliefs are a probability distribution over the signals of the other players

Notes

Strategies

Strategy Profiles:

- A strategy consists of a map from available information to actions:

$$\alpha_i : \mathbb{X}_i \longrightarrow A_i$$

- A strategy profile consists of a strategy for every player:

$$\alpha(X) = (\alpha_1(X_1), ..., \alpha_N(X_N))$$

- We adopt the usual convention:

$$\alpha_{-i}(X_{-i}) = (\alpha_1(X_1), ..., \alpha_{i-1}(X_{i-1}), \alpha_{i+1}(X_{i+1}), ..., \alpha_N(X_N))$$

Notes

Bayesian Game Example Continued

Consider the following game:

- Player 1 observes only one possible signal: $\mathbb{X}_1 = \{C\}$
- Player 2's signal takes one of two values: $\mathbb{X}_2 = \{L, R\}$
- Probabilities are such that: $f(C, L) = 0.6$
- Payoffs and action sets are as described in the matrix:

$1\backslash 2.L$	y_2	z_2
y_1	1,2	0,1
z_1	0,4	1,3

$1\backslash 2.R$	y_2	z_2
y_1	1,3	0,4
z_1	0,1	1,2

- A strategy for player 1 is an element of the set $\alpha_1 \in \{y_1, z_1\}$
- A strategy for player 2 is a map $\alpha_2 : \{L, R\} \to \{y_2, z_2\}$
- Player 1 cannot act upon 2's private information

Notes

Dominant Strategy Equilibrium

- Strategy α_i **weakly dominates** α_i' if for any a_{-i} and $x \in \mathbb{X}$:

$$u_i(\alpha_i(x_i), a_{-i}|x) \geq u_i(\alpha_i'(x_i), a_{-i}|x) \quad \text{[strict somewhere]}$$

- Strategy α_i is **dominant** if it dominates any other strategy α_i'
- Strategy α_i is **undominated** if no strategy dominates it

Definitions (Dominant Strategy Equilibrium DSE)

A Dominant Strategy equilibrium of an incomplete information game is a strategy profile α that for any $i \in N$, $x \in \mathbb{X}$ and $a_{-i} \in A_{-i}$ satisfies:

$$u_i(\alpha_i(x_i), a_{-i}|x) \geq u_i(\alpha_i'(x_i), a_{-i}|x) \quad \text{for any } \alpha_i' : \mathbb{X}_i \to A_i$$

- I.e. α_i is optimal independently of what others know and do

Notes

Interim Expected Utility and Best Reply Maps

The **interim stage** occurs just after a player knows his signal $X_i = x_i$

It is when strategies are chosen in a Bayesian game

The **interim expected** utility of a (pure) strategy profile α is defined by:

$$U_i(\alpha|x_i) = \sum_{\mathbb{X}_{-i}} u_i(\alpha(x)|x) f(x_{-i}|x_i) : \mathbb{X}_i \to \mathbb{R}$$

With such notation in mind notice that:

$$U_i(a_i, \alpha_{-i}|x_i) = \sum_{\mathbb{X}_{-i}} u_i(a_i, \alpha_{-i}(x_{-i})|x) f(x_{-i}|x_i)$$

The **best reply** correspondence of player i is defined by:

$$b_i(\alpha_{-i}|x_i) = \arg\max_{a_i \in A_i} U_i(a_i, \alpha_{-i}|x_i)$$

BR maps identify which actions are optimal given the signal and the strategies followed by others

Notes

Pure Strategy Bayes Nash Equilibrium

Definitions (Bayes Nash Equilibrium BNE)

A pure strategy Bayes Nash equilibrium of an incomplete information game is a strategy profile α such that for any $i \in N$ and $x_i \in \mathbb{X}_i$ satisfies:

$$U_i(\alpha|x_i) \geq U_i(a_i, \alpha_{-i}|x_i) \quad \text{for any } a_i \in A_i$$

BNE requires **interim optimality** (i.e. do your best given what you know)

BNE requires $\alpha_i(x_i) \in b_i(\alpha_{-i}|x_i)$ for any $i \in N$ and $x_i \in \mathbb{X}_i$

Notes

Bayesian Game Example Continued

Consider the following Bayesian game with $f(C, L) = 0.6$:

$1\backslash 2.L$	y_2	z_2
y_1	1,2	0,1
z_1	0,4	1,3

$1\backslash 2.R$	y_2	z_2
y_1	1,3	0,4
z_1	0,1	1,2

The best reply maps for both player are characterized by:

$$b_2(\alpha_1|x_2) = \begin{cases} y_2 & \text{if} \quad x_2 = L \\ z_2 & \text{if} \quad x_2 = R \end{cases} \qquad b_1(\alpha_2) = \begin{cases} y_1 & \text{if} \quad \alpha_2(L) = y_2 \\ z_1 & \text{if} \quad \alpha_2(L) = z_2 \end{cases}$$

The game has a unique (pure strategy) BNE in which:

$$\alpha_1 = y_1,\ \alpha_2(L) = y_2,\ \alpha_2(R) = z_2$$

DO NOT ANALYZE MATRICES SEPARATELY!!!

Notes

Relationships between Equilibrium Concepts

If α is a DSE then it is a BNE. In fact for any action a_i and signal x_i:

$$u_i(\alpha_i(x_i), a_{-i}|x) \geq u_i(a_i, a_{-i}|x) \quad \forall a_{-i}, x_{-i} \quad \Rightarrow$$

$$u_i(\alpha_i(x_i), \alpha_{-i}(x_{-i})|x) \geq u_i(a_i, \alpha_{-i}(x_{-i})|x) \quad \forall \alpha_{-i}, x_{-i} \Rightarrow$$

$$\textstyle\sum_{\mathbb{X}_{-i}} u_i(\alpha(x)|x) f_i(x_{-i}|x_i) \geq \sum_{\mathbb{X}_{-i}} u_i(a_i, \alpha_{-i}(x_{-i})|x) f_i(x_{-i}|x_i) \quad \forall \alpha_{-i} \Rightarrow$$

$$U_i(\alpha|x_i) \geq U_i(a_i, \alpha_{-i}|x_i) \quad \forall \alpha_{-i}$$

Notes

BNE Example I: Exchange

A buyer and a seller want to trade an object:

- Buyer's value for the object is 3$
- Seller's value is either 0$ or 2$ based on the signal, $\mathbb{X}_S = \{L, H\}$
- Buyer can offer either 1$ or 3$ to purchase the object
- Seller choose whether or not to sell

$B \setminus S.L$	sale	no sale
3$	0,3	0,0
1$	2,1	0,0

$B \setminus S.H$	sale	no sale
3$	0,3	0,2
1$	2,1	0,2

- This game for any prior f has a BNE in which:

$$\alpha_S(L) = \text{sale},\ \alpha_S(H) = \text{no sale},\ \alpha_B = 1\$$$

- Selling is strictly dominant for $S.L$
- Offering 1$ is weakly dominant for the buyer

Notes

BNE Example II: Entry Game

Consider the following market game:

- Firm I (the incumbent) is a monopolist in a market
- Firm E (the entrant) is considering whether to enter in the market
- If E stays out of the market, E runs a profit of 1$ and I gets 8$
- If E enters, E incurs a cost of 1$ and profits of both I and E are 3$
- I can deter entry by investing at cost $\{0, 2\}$ depending on type $\{L, H\}$
- If I invests: I's profit increases by 1 if he is alone, decreases by 1 if he competes and E's profit decreases to 0 if he elects to enter

$E \setminus I.L$	Invest	Not Invest
In	0,2	3,3
Out	1,9	1,8

$E \setminus I.H$	Invest	Not Invest
In	0,0	3,3
Out	1,7	1,8

Notes

BNE Example II: Entry Game

Let π denote the probability that firm I is of type L and notice:

- $\alpha_I(H) = \textit{Not Invest}$ is a strictly dominant strategy for $I.H$
- For any value of π, $\alpha_I(L) = \textit{Not Invest}$ and $\alpha_E = \textit{In}$ is BNE:

$$u_I(\textit{Not}, \textit{In}|L) = 3 > 2 = u_I(\textit{Invest}, \textit{In}|L)$$
$$U_E(\textit{In}, \alpha_I(X_I)) = 3 > 1 = U_E(\textit{Out}, \alpha_I(X_I))$$

- For π high enough, $\alpha_I(L) = \textit{Invest}$ and $\alpha_E = \textit{Out}$ is also BNE:

$$u_I(\textit{Invest}, \textit{Out}|L) = 9 > 8 = u_I(\textit{Not}, \textit{Out}|L)$$
$$U_E(\textit{Out}, \alpha_I(X_I)) = 1 > 3(1-\pi) = U_E(\textit{In}, \alpha_I(X_I))$$

$E\backslash I.L$	Invest	Not Invest
In	0,2	3,3
Out	1,9	1,8

$E\backslash I.H$	Invest	Not Invest
In	0,0	3,3
Out	1,7	1,8

Notes

Extra: Mixed Strategies in Bayesian Games

Strategy Profiles:

- A mixed strategy is a map from information to a probability distribution over actions
- In particular $\sigma_i(a_i|x_i)$ denotes the probability that i chooses a_i if his signal is x_i
- A mixed strategy profile consists of a strategy for every player:

$$\sigma(X) = (\sigma_1(X_1), ..., \sigma_N(X_N))$$

- As usual $\sigma_{-i}(X_{-i})$ denotes the profile of strategies of all players, but i
- Mixed strategies are independent (i.e. σ_i cannot depend on σ_j)

Notes

Extra: Interim Payoff & Bayes Nash Equilibrium

The interim expected payoff of mixed strategy profiles σ and (a_i, σ_{-i}) are:

$$U_i(\sigma|x_i) = \sum_{\mathbb{X}_{-i}} \sum_{a \in A} u_i(a|x) \prod_{j \in N} \sigma_j(a_j|x_j) f(x_{-i}|x_i)$$

$$U_i(a_i, \sigma_{-i}|x_i) = \sum_{\mathbb{X}_{-i}} \sum_{a_{-i} \in A_{-i}} u_i(a|x) \prod_{j \neq i} \sigma_j(a_j|x_j) f(x_{-i}|x_i)$$

Definitions (Bayes Nash Equilibrium BNE)

A Bayes Nash equilibrium of a game Γ is a strategy profile σ such that for any $i \in N$ and $x_i \in \mathbb{X}_i$ satisfies:

$$U_i(\sigma|x_i) \geq U_i(a_i, \sigma_{-i}|x_i) \quad \text{for any } a_i \in A_i$$

BNE requires **interim optimality** (i.e. do your best given what you know)

Notes

Extra: Computing Bayes Nash Equilibria

Testing for BNE behavior:

- σ is BNE if only if:

$$U_i(\sigma|x_i) = U_i(a_i, \sigma_{-i}|x_i) \quad \text{for any } a_i \text{ s.t. } \sigma_i(a_i|x_i) > 0$$

$$U_i(\sigma|x_i) \geq U_i(a_i, \sigma_{-i}|x_i) \quad \text{for any } a_i \text{ s.t. } \sigma_i(a_i|x_i) = 0$$

- Strictly dominated strategies are never chosen in a BNE
- Weakly dominated strategies are chosen only if they are dominated with probability zero in equilibrium
- This conditions can be used to compute BNE (see examples)

Notes

Extra: Example I

Consider the following example for $f(1, L) = 1/2$:

$1\backslash 2.L$	X	Y
T	1,0	0,1
D	0,1	1,0

$1\backslash 2.R$	W	Z
T	0,0	1,1
D	1,1	0,0

All BNEs for this game satisfy:

$$\sigma_1(T) = 1/2 \text{ and } \sigma_2(X|L) = \sigma_2(W|R)$$

Such games satisfy all BNE conditions since:

$$\begin{aligned} U_1(T, \sigma_2) &= (1/2)\sigma_2(X|L) + (1/2)(1 - \sigma_2(W|R)) = \\ &= (1/2)(1 - \sigma_2(X|L)) + (1/2)\sigma_2(W|R) = U_1(D, \sigma_2) \\ u_2(X, \sigma_1|L) &= \sigma_1(T) = 1 - \sigma_1(T) = u_2(Y, \sigma_1|L) \\ u_2(W, \sigma_1|R) &= (1 - \sigma_1(T)) = \sigma_1(T) = u_2(Z, \sigma_1|R) \end{aligned}$$

Notes

Extra: Example II

Consider the following example for $f(1, L) = q \leq 2/3$:

$1\backslash 2.L$	X	Y
T	0,0	0,2
D	2,0	1,1

$1\backslash 2.R$	W	Z
T	2,2	0,1
D	0,0	3,2

All BNEs for this game satisfy $\sigma_1(T) = 2/3$ and:

$$\sigma_2(X|L) = 0 \text{ (dominance) and } \sigma_2(W|R) = \frac{3 - 2q}{5 - 5q}$$

Such games satisfy all BNE conditions since:

$$\begin{aligned} U_1(T, \sigma_2) &= 2(1 - q)\sigma_2(W|R) = \\ &= q + 3(1 - q)(1 - \sigma_2(W|R)) = U_1(D, \sigma_2) \\ u_2(X, \sigma_1|L) &= 0 < 2\sigma_1(T) + (1 - \sigma_1(T)) = u_2(Y, \sigma_1|L) \\ u_2(W, \sigma_1|R) &= 2\sigma_1(T) = \sigma_1(T) + 2(1 - \sigma_1(T)) = u_2(Z, \sigma_1|R) \end{aligned}$$

Notes

Dynamic Games
EC202 Lectures VII & VIII

Francesco Nava

LSE

January 2013

Notes

Summary

Dynamic Games:

- Definitions:
  - Extensive Form Game
  - Information Sets and Beliefs
  - Behavioral Strategy
  - Subgame
- Solution Concepts:
  - Nash Equilibrium
  - Subgame Perfect Equilibrium
  - Perfect Bayesian Equilibrium
- Examples: Imperfect Competition

Notes

Dynamic Games

- All games discussed in previous lectures were static. That is:
  - A set of players taking decisions simultaneously
  - or not being able to observe the choices made by others
- Today we relax such assumption by modeling the timing of decisions
- In common instances the rules of the game explicitly define:
  - the order in which players move
  - the information available to them when they take their decisions
- A way of representing such dynamic games is in their Extensive Form
- The following definitions are helpful to define such notion

Notes

Basic Graph Theory

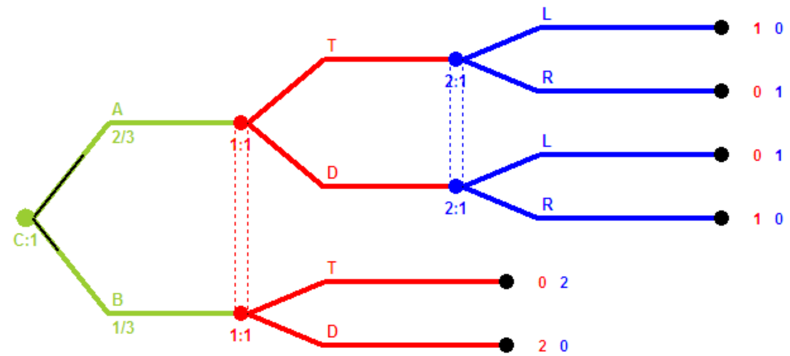
- A graph consists of a set of *nodes* and of a set of *branches*
- Each branch connects a pair of nodes
- A branch is identified by the two nodes it connects
- A path is a set of branches:

$$\{\{x_k, x_{k+1}\} \,|\, k = 1, ..., m\}$$

  where $m > 1$ and every x_k is a different node of the graph.
- A *tree* is a graph in which any pair of nodes is connected by exactly one path
- A *rooted tree* is a tree in which a special nodes designated as the *root*
- A *terminal node* is a node connected by only one branch

Notes

An Extensive Form Game



Notes

Extensive Form Games

An extensive form game is a rooted tree together with functions assigning labels to nodes and branches such that:

1. Each non-terminal node has a player-label in $\{C, 1, ..., n\}$
   - $\{1, ..., n\}$ are the players in the game
   - Nodes assigned to label C are *chance nodes*
   - Nodes assigned to label $i \neq C$ are *decision nodes* controlled by i
2. Each alternative at a chance node has a label specifying its probability:
   - *Chance probabilities* are nonnegative and add to 1
3. Each node controlled by player $i > 0$ has a second label specifying i's *information state*:
   - Thus nodes labeled $i.s$ are controlled by i with information s
   - Two nodes belong to $i.s$ **iff** i cannot distinguish them

Notes

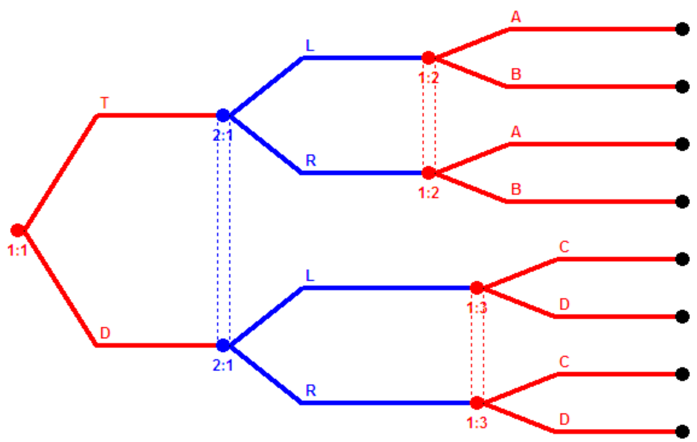
Extensive Form Games

4. Each alternative at a decision node has *move* label:
   - If two nodes x, y belong to the same information set, for any alternative at x there must be exactly one alternative at y with the same move label
5. Each terminal node y has a label that specifies a vector of n numbers $\{u_i(y)\}_{i\in\{1,...,n\}}$ such that:
   - The number $u_i(y)$ specifies the *payoff* to i if the game ends at node y
6. All players have *perfect recall* of the moves they chose

Notes

Perfect Recall

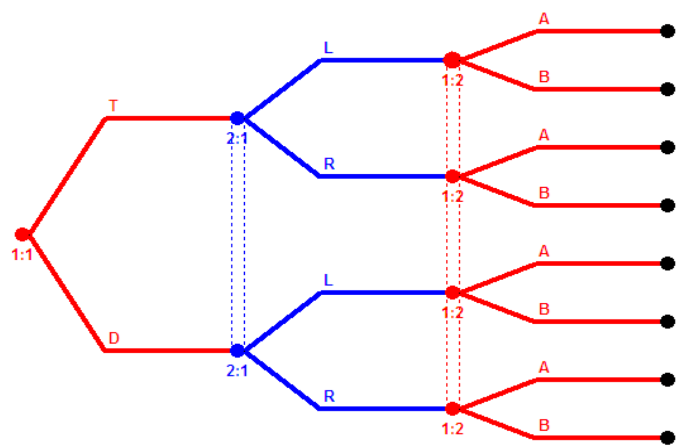
With perfect recall information sets 1.2 and 1.3 cannot coincide:



Notes

Without Perfect Recall

Without perfect recall assumption this is possible:

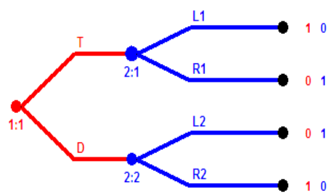


Notes

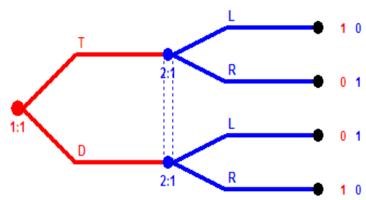
Perfect Information

An extensive form game has *perfect information* if no two nodes belong to the same information state

With Perfect Information



Without Perfect Information



Notes

Behavioral Strategies

Throughout let:

- S_i be the set information states of player $i \in N$
- $A_{i.s}$ be the action set of player i at info state $s \in S_i$

A *behavioral strategy* for player i maps information states to probability distributions over actions

In particular $\sigma_{i.s}(a_{i.s})$ is the probability that player i at information stage s chooses action $a_{i.s} \in A_{i.s}$

Throughout denote:

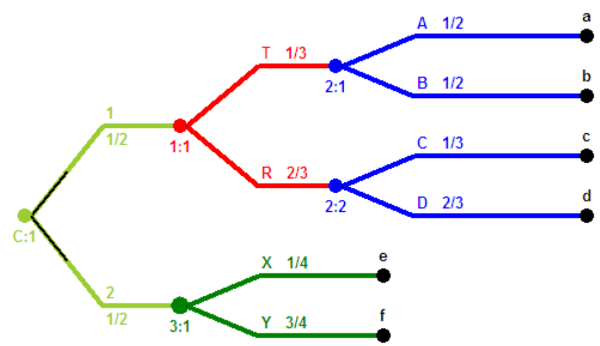
- a behavioral strategy of player i by $\sigma_i = \{\sigma_{i.s}\}_{s \in S_i}$
- a profile of behavioral strategy by $\sigma = \{\sigma_i\}_{i \in N}$
- the chance probabilities by $\pi = \{\pi_{0.s}\}_{s \in S_0}$

Notes

Probabilities over Terminal Nodes

For any terminal node y and any behavioral strategy profile σ, let $P(y|\sigma)$ denote the probability that the game ends at node y

E.g. in the following game $P(c|\sigma) = \pi_{0.1}(1)\sigma_{1.1}(R)\sigma_{2.2}(C) = 1/9$:



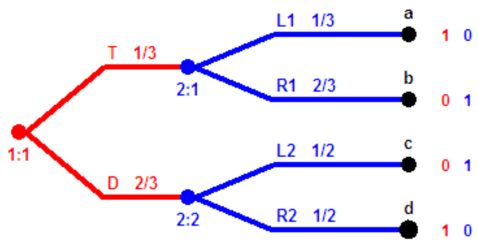
Notes

Expected Payoffs

If Ω denotes the set of end notes, the expected payoff of player i is:

$$U_i(\sigma) = U_i(\sigma_i, \sigma_{-i}) = \sum_{y \in \Omega} P(y|\sigma) u_i(y)$$

E.g. in the following game $U_1(\sigma) = 4/9$ and $U_2(\sigma) = 5/9$



Notes

Nash Equilibrium

Definition (Nash Equilibrium – NE)

A Nash Equilibrium of an extensive form game is any profile of behavioral strategies such that:

$$U_i(\sigma) \geq U_i(\sigma_i', \sigma_{-i}) \quad \text{for any } \sigma_i' \in \times_{s \in S_i} \Delta(A_{i.s})$$

Recall that σ_i' is any mapping from information sets to probability distributions over available actions

The definition of NE is as in strategic form games

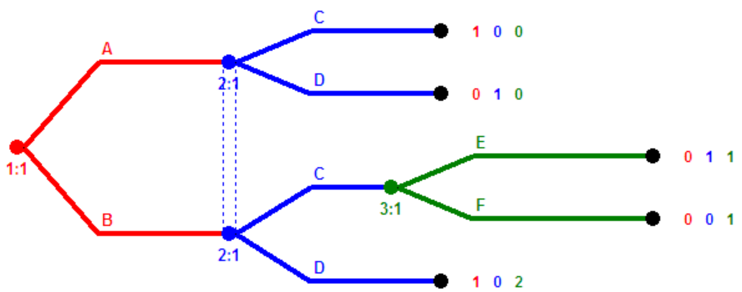
What differs is the strategy (behavioral) that is expressed at every single decision stage and not on profiles of decisions for every individual

Notes

NE Example

A game may have many NEs:

- $\sigma_1(A) = 0$, $\sigma_2(C) = 0$, $\sigma_3(E) = 0$
- $\sigma_1(A) = \sigma_3(E)/(1+\sigma_3(E))$, $\sigma_2(C) = 1/2$, $\sigma_3(E) \in [0,1]$

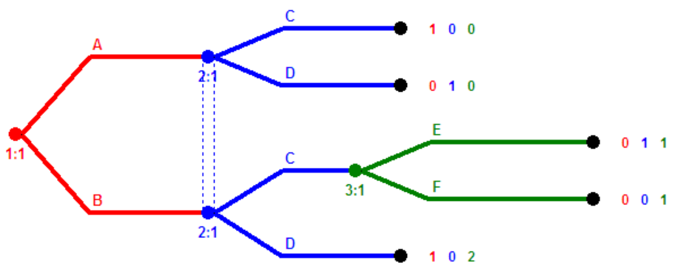


Notes

NE Example Continued I

Profile $\sigma_1(A) = 0$, $\sigma_2(C) = 0$, $\sigma_3(E) = 0$ is NE since:

$$U_1(B, \sigma_{-1}) = 1 > 0 = U_1(A, \sigma_{-1})$$
$$U_2(D, \sigma_{-2}) = 0 \geq 0 = U_2(C, \sigma_{-2})$$
$$U_3(F, \sigma_{-3}) = 2 \geq 2 = U_3(E, \sigma_{-3})$$

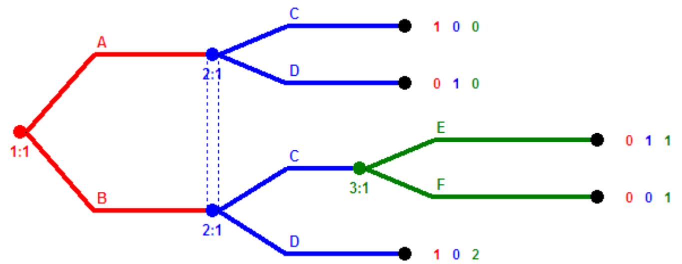


Notes

NE Example Continued II

Profile $\sigma_1(A) = \frac{\sigma_3(E)}{(1+\sigma_3(E))}$, $\sigma_2(C) = \frac{1}{2}$, $\sigma_3(E) \in [0,1]$ is NE since:

$$U_1(A,\sigma_{-1}) = \sigma_2(C) = (1-\sigma_2(C)) = U_1(B,\sigma_{-1})$$
$$U_2(D,\sigma_{-2}) = \sigma_1(A) = (1-\sigma_1(A))\sigma_3(E) = U_2(C,\sigma_{-2})$$
$$U_3(F,\sigma_{-3}) = (1-\sigma_1(A))(2-\sigma_2(C)) = U_3(E,\sigma_{-3})$$



Notes

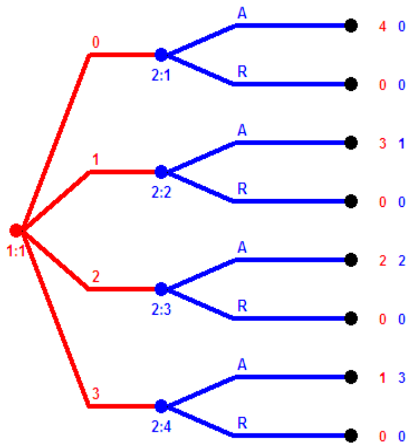
Ultimatum Game

NE1:
$\sigma_{1.1} = [0]$ $\sigma_{2.1} = [A]$

NE2:
$\sigma_{1.1} = [1]$
$\sigma_{2.2} = [A]$ $\sigma_{2.1} = [R]$

NE3:
$\sigma_{1.1} = [2]$
$\sigma_{2.3} = [A]$ $\sigma_{2.2} = [R]$...

NE4:
$\sigma_{1.1} = [3]$
$\sigma_{2.4} = [A]$ $\sigma_{2.3} = [R]$...



Notes

Ultimatum Game

Strategy $\sigma_{1.1} = [0]$, $\sigma_{2.1} = [A]$ is NE for any $(\sigma_{2.2}, \sigma_{2.3}, \sigma_{2.4})$ since:

$$U_1(0, \sigma_2) = 4 > U_1(a_1, \sigma_2) \quad \text{for any } a_1 \in \{1, 2, 3\}$$
$$U_2(\sigma_2, 0) = 0 = U_2(a_2, 0) \quad \text{for any } a_2 \in \{A, R\}^4$$

Strategy $\sigma_{1.1} = [0]$, $\sigma_{2.2} = [A]$, $\sigma_{2.1} = [R]$ is NE for any $(\sigma_{2.3}, \sigma_{2.4})$ since:

$$U_1(1, \sigma_2) = 3 > U_1(a_1, \sigma_2) \quad \text{for any } a_1 \in \{0, 2, 3\}$$
$$U_2(\sigma_2, 1) = 1 \geq U_2(a_2, 1) \quad \text{for any } a_2 \in \{A, R\}^4$$

A similar argument works for the other two proposed equilibria

Only the first two equilibria, however, involve threats that are credible, since player 2 would never want to refuse and offer worth at least 1$

Notes

Subgame Perfect Equilibrium

A *successor* of a node x is a node that can be reached from x for an appropriate profile of actions

Definition (Subgame)

A *subgame* is a subset of an extensive form game such that:

1. It begins at a single node
2. It contains all successors
3. If a game contains an information set with multiple nodes then either all of these nodes belong to the subset or none does

Definition (Subgame Perfect Equilibrium – SPE)

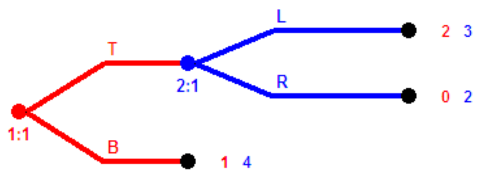
A *subgame perfect equilibrium* is any NE such that for every subgame the restriction of strategies to this subgame is also a NE of the subgame.

Notes

NE but not SPE

The following game has a continuum of NE but only one SPE:

- $\sigma_{1.1}(T) = 1$ and $\sigma_{2.1}(L) = 1$ is unique SPE
- $\sigma_{1.1}(T) = 0$ and $\sigma_{2.1}(L) \leq 1/2$ are all NE, but none SPE



Notes

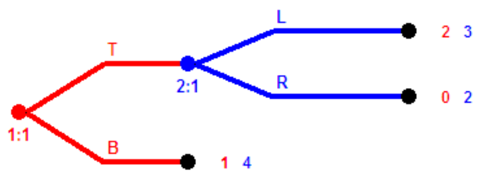
NE but not SPE

Strategy $\sigma_{1.1}(T) = 1$ and $\sigma_{2.1}(L) = 1$ is the unique SPE since:

$$U_{1.1}(T, L) = 2 > 1 = U_{1.1}(B, L)$$
$$U_{2.1}(L) = 3 > 2 = U_{2.1}(R)$$

Any strategy $\sigma_{1.1}(T) = 0$ and $\sigma_{2.1}(L) = q \leq 1/2$ is NE since:

$$U_1(B, \sigma_2) = 1 \geq 2q = U_1(T, \sigma_2)$$
$$U_2(L, B) = 4 = 4 = U_2(R, B)$$



Notes

Computing SPE – Backward Induction

Definition of SPE is demanding because it imposes discipline on behavior even in subgames that one expects not to be reached

SPE however is easy to compute in perfect information games

Backward-induction algorithm provides a simple way:

- At every node leading only to terminal nodes players pick actions that are optimal for them if that node is reached
- At all preceding nodes players pick an actions that optimal for them if that node is reached knowing how all their successors behave
- And so on until the root of the tree is reached

A pure strategy SPE exists in any perfect information game

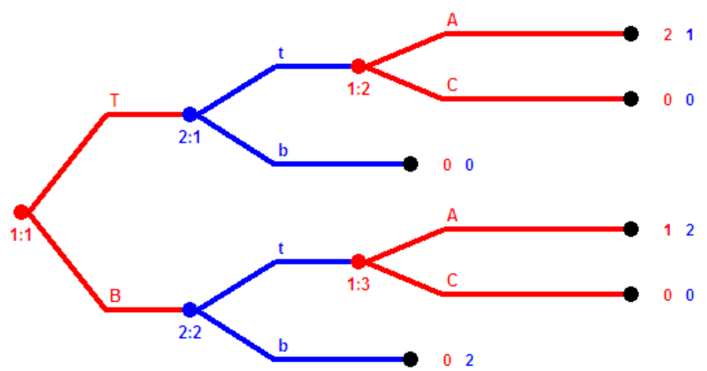
Notes

Backward Induction Example

SPE: $\sigma_{1.2} = \sigma_{1.3} = [A]$, $\sigma_{2.1} = [t]$, any $\sigma_{2.2}$ and $\sigma_{1.1} = [T]$

NE but not SPE: $\sigma_{1.2} = \sigma_{1.3} = [A]$, $\sigma_{2.1} = [b]$, any $\sigma_{2.2}$ and $\sigma_{1.1} = [B]$

Again NE may support empty threats



Notes

Duopoly: Stackelberg Competition

Implicit to both Cournot and Bertrand models was the assumption that no producer could observe actions chosen by others before making a decision

In the Stackelberg duopoly model however:

- Players choose how many goods to supply to the market (as Cournot)
- One player moves first (the *leader*)
- While the other player moves after having observed the decision of the leader (the *follower*)
- Both players account for the distortions that their output choices have on equilibrium prices

To avoid empty threats restrict attention to the SPE of the dynamic game

Notes

Duopoly: Stackelberg Competition

The game is solved by backward induction:

- Consider the subgame in which the leader has produced q_L units
- In this subgame (as in Cournot) the decision problem of follower is to:

$$\max_{q_F} p(q_L + q_F)q_F - c_F(q_F)$$

- Solving such problem defines the best response to the follower $b_F(q_L)$
- By SPE the leader takes the follower's strategy into account when choosing his output
- Thus the decision problem of the leader is as follows:

$$\max_{q_L} p(q_L + b_F(q_L))q_L - c_L(q_L)$$

Notes

Stackelberg Example

Consider the following economy:

- $d(q) = 2 - q$
- $c_F(q) = q^2$ and $c_L(q) = 3q^2$

The problems of both players are respectively defined by:

$$\max_{q_F} (2 - q_L - q_F) q_F - c_F(q_F)$$
$$\max_{q_L} (2 - q_L - b_F(q_L)) q_L - c_L(q_L)$$

Optimality of each firm is determined by FOC:

$$2 - 2q_F - q_L - 2q_F = 0 \quad \Rightarrow \quad q_F = b_F(q_L) = (2 - q_L)/4$$
$$3/2 - (3/2)q_L - 6q_L = 0 \quad \Rightarrow \quad q_L = 1/5$$

Stackelberg Equilibrium outputs are: $q_F = 9/20$ and $q_L = 1/5$

Cournot Equilibrium outputs are: $\overline{q}_F = 14/31$ and $\overline{q}_L = 6/31$

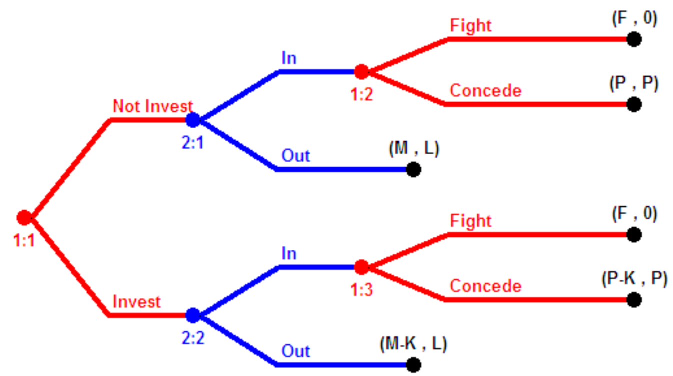
Notes

Duopoly: Market Entry

Consider the following game between two producers

Firm 1 is the incumbent and firm 2 is the potential entrant

Assume $P > L > 0$ and $M > P > F$



Notes

Duopoly: Market Entry

To find an SPE with successful deterrence, notice that:

- If 1 does not invest, it prefers to concede if entry takes place as $P > F$
- Thus firm 2 prefers to enter if 1 does not invest as $P > L$
- If 1 does invest it prefers to fight if entry takes place, provided that:

$$F > P - K$$

- If so firm 2 prefers to stay out if 1 has invested as $L > 0$
- Thus firm 1 prefers to invest and deter entry if:

$$M - K > P$$

An SPE exists in which entry is effectively deterred if the cost satisfies:

$$M - P > K > P - F$$

Notes

Extra: Dynamics and Uncertainty

If an extensive form game does not display perfect information, subgame perfection cannot be imposed at every information set, but only on subgames

In such games a further equilibrium refinement may help to highlight the relevant equilibria of the game by selecting those which Bayes rule

Definition (Perfect Bayesian Equilibrium – PBE)

A *perfect Bayesian equilibrium* of an extensive form game consists of a profile of behavioral strategies and of beliefs at each information set of the game such that:

1. strategies form an SPE given the beliefs
2. beliefs are updated using Bayes rule at each information set reached with positive probability

Notes

Repeated Games

EC202 Lectures IX & X

Francesco Nava

LSE

January 2013

Notes

Summary

Repeated Games:

- Definitions:
  - Feasible Payoffs
  - Minmax
  - Repeated Game
  - Stage Game
  - Trigger Strategy
- Main Result:
  - Folk Theorem
- Examples: Prisoner's Dilemma

Notes

Feasible Payoffs

Q: What payoffs are feasible in a strategic form game?

A: A profile of payoffs is feasible in a strategic form game if can be expressed as a weighed average of payoffs in the game.

Definition (Feasible Payoffs)

A profile of payoffs $\{w_i\}_{i \in N}$ is feasible in a strategic form game $\{N, \{A_i, u_i\}_{i \in N}\}$ if there exists a distribution over profiles of actions π such that:

$$w_i = \sum_{a \in A} \pi(a) u_i(a) \quad \text{for any} \quad i \in N$$

Unfeasible payoffs cannot be outcomes of the game

Points on the north-east boundary of the feasible set are Pareto efficient

Notes

Minmax

Q: What's the worst possible payoff that a player can achieve if he chooses according to his best response function?

A: The minmax payoff.

Definition (Minmax)

The (pure strategy) *minmax* payoff of player $i \in N$ in a strategic form game $\{N, \{A_i, u_i\}_{i \in N}\}$ is:

$$\underline{u}_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i})$$

Mixed strategy minmax payoffs satisfy:

$$\underline{v}_i = \min_{\sigma_{-i}} \max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})$$

The mixed strategy minmax is not higher than the pure strategy minmax.

Notes

Example: Prisoner's Dilemma

Minmax Payoff: $(1, 1)$

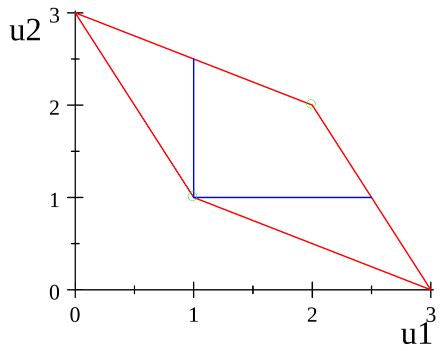
Feasible Payoff: contained in red boundaries

Pareto Efficient Payoffs: $(2, 2; 3, 0)$ and $(2, 2; 0, 3)$

Stage Game

1\2	w	s
w	2,2	0,3
s	3,0	1,1

Payoffs



Notes

Example: Battle of the Sexes

Minmax Payoff: $(2, 2)$

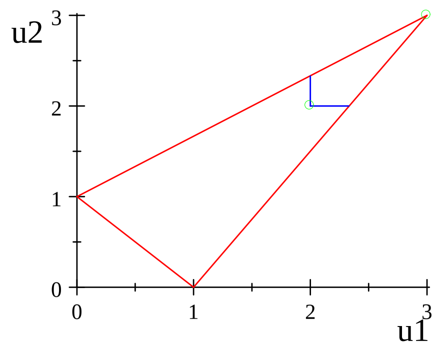
Feasible Payoff: contained in red boundaries

Pareto Efficient Payoffs: $(3, 3)$

Stage Game

1\2	w	s
w	3,3	1,0
s	0,1	2,2

Payoffs



Notes

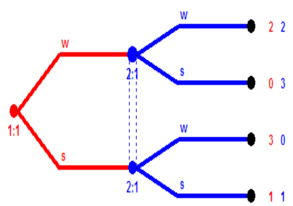
Repeated Game: Timing

Consider any strategic form game $G = \{N, \{A_i, u_i\}_{i \in N}\}$

Call G the *stage game*

An infinitely *repeated game* describes a strategic environment in which the stage game is played repeatedly by the same players infinitely many times

Round 1 ... Round t ...



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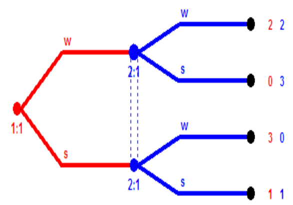

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Notes

Repeated Games: Payoffs and Discounting

The value to player $i \in N$ of a payoff stream $\{u_i(1), u_i(2), ..., u_i(t), ...\}$ is:

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(t)$$

The term $(1-\delta)$ amounts to a simple normalization

... and guarantees that a constant stream $\{v, v, ...\}$ has value v

Future payoffs are discounted at rate δ

An infinitely repeated game can be used to describe strategic environments in which there is no certainty of a final stage

In such view δ describes the probability that the game does not end at the next round which would result in a payoff of 0 thereafter

Notes

Repeated Games: Perfect Information and Strategies

Today we restrict attention to perfect information repeated games

In such games all players prior to each round observe the actions chosen by all other players at previous rounds

Let $a(s) = \{a_1(s), ..., a_n(s)\}$ denote the action profile played at round s

A *history* of play up to stage t thus consists of:

$$h(t) = \{a(1), a(2), ..., a(t-1)\}$$

In this context strategies map histories (ie information) to actions:

$$\alpha_i(h(t)) \in A_i$$

Notes

Dilemma Folk Theorem

Prisoner's
Consider the prisoner's dilemma discussed earlier:

1\2	w	s
w	2,2	0,3
s	3,0	1,1

To understand how equilibrium behavior is affected by repetition, let's show why $(2, 2)$ is SPE of the infinitely repeated prisoner's dilemma

Folk theorem shows that any feasible payoff that yields to both players at least their minmax value is a SPE of the infinitely repeated game if the discount factor is sufficiently high

Notes

Grim Trigger Strategies [Draw Paths...]

Consider the following strategy (known as *grim trigger strategy*):

$$a_i(t) = \begin{cases} w & \text{if either } a(t-1) = (w, w) \text{ or } t = 0 \\ s & \text{otherwise} \end{cases}$$

If all players follow such strategy, no player can deviate and benefit at any given round provided that $\delta \geq 1/2$

In subgames following $a(t-1) = (w, w)$ no player benefits from a deviation if:

$$(1-\delta)\left(3 + \delta + \delta^2 + \delta^3 + ...\right) = 3 - 2\delta \leq 2 \quad \Leftrightarrow \quad \delta \geq 1/2$$

In subgames following $a(t-1) \neq (w, w)$ no player benefits from a deviation since:

$$(1-\delta)\left(0 + \delta + \delta^2 + \delta^3 + ...\right) = \delta \leq 1 \quad \Leftrightarrow \quad \delta \leq 1$$

Notes

Folk Theorem

Theorem (SPE Folk Theorem)

In any two-person infinitely repeated game:

1. *For any discount factor δ, the discounted average payoff of each player in any SPE is at least his minmax value in the stage game*
2. *Any feasible payoff profile that yields to all players at least their minmax value is the discounted average payoff of a SPE if the discount factor δ is sufficiently close to 1*
3. *If the stage game has a NE in which each players' payoff is his minmax value, then the infinitely repeated game has a SPE in which every players' discounted average payoff is his minmax value*

Notes

Testing SPE in Repeated Games

Definition (One-Deviation Property)

A strategy satisfies the *one-deviation property* if no player can increase his payoff by changing his action at the start of any subgame in which he is the first-mover given other players' strategies and the rest of his own strategy

Fact

A strategy profile in an extensive game with perfect information and infinite horizon is a SPE if and only if it satisfies the one-deviation property

This observation can be used to test whether a strategy profile is a SPE of an infinitely repeated game as we did in the Prisoner's dilemma example

Notes

Example

To practice let's show why $(1.5, 1.5)$ is also and SPE of the repeated PD:

$$a_i(t+1) = \begin{cases} w & \begin{array}{c} \text{if } [a_i(t) = s \ \textbf{ and } \ a_j(t) = w \textbf{ and} \\ a(k) \notin \{(s,s),(w,w)\} \ \text{ for } k < t] \\ \textbf{or} \ \text{ if } \ [t = 0 \ \textbf{ and } \ i = 1] \end{array} \\ s & \text{otherwise} \end{cases}$$

If both follow such strategy, no player can deviate and benefit at any given round if $\delta \geq 1/2$

After $a(t-1) = (s,w), (w,s)$, no player benefits from a deviation if $\delta \geq 1/2$:

$$1 \leq (1-\delta)\left(3\delta + 3\delta^3 + 3\delta^5 + ...\right) = 3\delta/(1+\delta)$$

$$(1-\delta)\left(2 + \delta + \delta^2 ...\right) = 2 - \delta \leq (1-\delta)\left(3 + 3\delta^2 + 3\delta^4 ...\right) = 3/(1+\delta)$$

After $a(t-1) = (w,w), (s,s)$, no player benefits from a deviation since:

$$(1-\delta)\left(0 + \delta + \delta^2 + \delta^3 + ...\right) = \delta \leq 1 \quad \Leftrightarrow \quad \delta \leq 1$$

Notes

Last Example

Consider the following game – with minmax payoffs of $(1, 1)$:

1\2	A	B
A	0,0	4,1
B	1,4	3,3

Two PNE: (A, B) and (B, A) with payoffs $(1, 4)$ and $(4, 1)$

Always playing (B, B) is SPE of the repeated game for δ high enough

Consider the following grim trigger strategy:

$$a(t) = \begin{cases} (B,B) & \text{if } \; a(s) = (B,B) \; \text{ for any } \; s < t \quad \textbf{or} \quad t = 0 \\ (B,A) & \text{if } \; a(s) = \begin{cases} (B,B) & \text{for } \; s < z \\ (A,B) & \text{for } \; s = z \end{cases} \quad \text{for } \; z \in \{0, .., t-1\} \\ (A,B) & \text{if } \; a(s) = \begin{cases} (B,B) & \text{for } \; s < z \\ (B,A) & \text{for } \; s = z \end{cases} \quad \text{for } \; z \in \{0, .., t-1\} \end{cases}$$

Notes

No Profitable Deviation

1\2	A	B
A	0,0	4,1
B	1,4	3,3

If all follow such strategy, no player can deviate and benefit if $\delta \geq 1/3$

Following $a(s) = (B, B)$ for $\forall s < t$ no player deviates since:

$$(1-\delta)\left(4 + \delta + \delta^2 + \delta^3 + ...\right) = 4 - 3\delta \leq 3 \quad \Leftrightarrow \quad \delta \geq 1/3$$

Following $a(s) = (B, B)$ for $\forall s < z$ and $a(z) = (A, B)$ no one deviates as:

$$(1-\delta)\left(0 + \delta + \delta^2 + \delta^3 + ...\right) = \delta \leq 1 \quad \Leftrightarrow \quad \delta \leq 1$$
$$(1-\delta)\left(3 + 4\delta + 4\delta^2 + 4\delta^3 + ...\right) = 3 + \delta \leq 4 \quad \Leftrightarrow \quad \delta \leq 1$$

Following $a(s) = (B, B)$ for $\forall s < z$ and $a(z) = (B, A)$ no one deviates for symmetric reasons.

Notes

Adverse Selection

EC202 Lectures XI & XII

Francesco Nava

LSE

January 2013

Notes

Summary

Adverse Selection:

1. Hidden Characteristics
2. Uninformed party moves first

- Monopoly:
  - One type of consumer
  - Multiple types of consumer
- Competition
- Definitions:
  - Pooling Equilibrium
  - Separating Equilibrium
- Insurance Markets

Notes

Adverse Selection: Monopoly Setup

- Two goods economy: x and y
- One firm produces good x using y
- Constant marginal cost c
- Firm chooses a pricing schedule $P(x)$, eg:
  - Uniform price:
$$P(x) = px$$
  - Two-part tariff:
$$P(x) = \quad p_0 + p_1 x \quad \text{if} \quad x > 0$$
  - Multi-part tariff:
$$P(x) = \begin{cases} p_0 + p_1 x & \text{if} \quad 0 < x \leq z \\ p_0 + p_1 z + p_2(x - z) & \text{if} \quad x > z \end{cases}$$

Notes

Complete Information: One cosumer type

Begin by looking at the complete information benchmark:

- All consumers are all identical
- Endowments given by $(e_x, e_y) = (0, Y)$
- For a fee schedule F, the budget constraint of an individual is:
$$y(x) = \begin{cases} Y - P(x) & \text{if} \quad x > 0 \\ Y & \text{if} \quad x = 0 \end{cases}$$
- Preferences over the two goods are given by:
$$U(x, y(x)) = \psi(x) + y(x) = \begin{cases} Y + \psi(x) - P(x) & \text{if} \quad x > 0 \\ Y & \text{if} \quad x = 0 \end{cases}$$
- Assume: $\psi(0) = 0$, $\psi_x > 0$, $\psi_{xx} < 0$

Notes

Consumer's Problem & Participation Constraints

Consider the decision problem of a consumer facing schedule P:

- A consumer purchases some good x only if:

$$U(x, y) - U(0, Y) = \psi(x) - P(x) \geq 0 \qquad \text{(PC)}$$

- Such constraint is known as *Participation Constraint (PC)*
- If PC, holds a consumer chooses $x > 0$ in order to:

$$\max_x U(x, y(x)) \Rightarrow P_x(x) = \psi_x(x)$$

- For $p = P_x$ & $\varphi = \psi_x^{-1}$, the demand associated to P is:

$$x^*(P) = \begin{cases} \varphi(p(x^*)) & \text{if} \quad \psi(x^*) - P(x^*) \geq 0 \\ 0 & \text{if} \quad \psi(x^*) - P(x^*) < 0 \end{cases} \qquad \text{(FOC)}$$

Notes

Firms's Decision Problem

Given such demand consider the decision problem of the firm:

- A firm chooses P to maximize profits:

$$\max_P P(x) - cx \text{ subject to PC and } x = x^*(P)$$

- PC must hold with equality at $x^*(P)$ or else the firm could increase profits by raising prices by a lump sum until PC holds, thus:

$$P(x^*) = \psi(x^*)$$

- The firm can in effect choose x^* by changing P (exploiting FOC)
- Using these two facts the problem of the monopolist's becomes:

$$\max_x \psi(x) - cx \Rightarrow \psi_x(x) = c = p(x)$$

- The resulting equilibrium demand is $x^*(P) = \varphi(c)$

Notes

Firms's Decision Problem

A few more comments on the solution of the firm's problem:

- The optimal pricing schedule is a two-part tariff:

$$P(x) = p_0 + p_1 x \quad \text{with} \quad p_1 = c \;\; \& \;\; p_0 \quad \text{such that PC holds}$$
$$\Rightarrow p_0 = \psi(\varphi(c)) - p_1 \varphi(c)$$

- Unlike in the standard monopolist problem, the solution of this problem is efficient as prices equal marginal costs
- It is still exploitative however because buyers are left at their reservation utility:

$$U(x, y) - U(0, Y) = 0$$

- There are other ways of implementing the same outcome such as a take it or leave it offer:

$$[x, P] = [\varphi(c), \psi(\varphi(c))]$$

Notes

Complete Information: Multiple cosumer types

Suppose that consumers have multiple types:

- Let $t \in \{L, H\}$ denote the type of a consumer with $H > L$
- Let $\pi(t)$ denote the proportion of types t in the population
- The monopolist knows the type of every consumer
- Preferences of a consumer of type t are:

$$U(x, y) = t\psi(x) + y = \begin{cases} Y + t\psi(x) - P(x) & \text{if} \quad x > 0 \\ Y & \text{if} \quad x = 0 \end{cases}$$

- Setup meets the regularity condition known as *single crossing condition* (it requires indifference curves of two types to cross only once)
- Consumers cannot resell the units purchased

Notes

Complete Information: Multiple cosumer types

The with more types is similar to the single type scenario:

- The firm price discriminates both types of costumers $P(t)$
- The participation constraint of type t becomes:

$$U(x,y|t) - U(0,Y|t) = t\psi(x) - P(x|t) \geq 0 \qquad \text{(PC(t))}$$

- If PC(t), holds a consumer chooses $x(t) > 0$ in order to:

$$\max_x U(x, y(x)|t) \Rightarrow t\psi_x(x) = P_x(x|t) \equiv p(x|t)$$

- The demand by type t associated to $P(t)$ is:

$$x^*(P|t) = \begin{cases} \varphi\left(\frac{p(x^*(t)|t)}{t}\right) & \text{if} \quad t\psi(x^*(t)) - P(x^*(t)|t) \geq 0 \\ 0 & \text{if} \quad t\psi(x^*(t)) - P(x^*(t)|t) < 0 \end{cases}$$

Notes

Firms's Decision Problem

Given such demand consider the decision problem of the firm:

- A firm chooses P to maximize profits:

$$\max_P \sum_t \pi(t)[P(x(t)|t) - cx(t)] \text{ subject to PC(t) and FOC(t)}$$

- PC(t) holds with equality at $x^*(P|t)$, thus $P(x(t)|t) = t\psi(x(t))$
- The firm can effectively choose $x(t)$ by changing P
- Using these two facts the problem of the monopolist's becomes:

$$\max_{x(t)} \sum_t \pi(t)[t\psi(x(t)) - cx(t)] \Rightarrow t\psi_x(x(t)) = c = p(x(t)|t)$$

- The resulting equilibrium demand is $x^*(t) = \varphi(c/t)$
- To a type t consumer the firm optimally offers a two-part tariff:

$$P(x|t) = p_0(t) + p_1 x \text{ such that :}$$
$$p_1 = c \;\;\&\;\; p_0(t) = t\psi(x^*(t)) - cx^*(t)$$

Notes

Incomplete Information: Multiple cosumer type

If the firm cannot recognize the two types and knows only $\pi(t)$:

- Firm may still offer several pricing schedules $P(t)$...
  ... but cannot guarantee that type t purchases only $P(t)$
- Each consumer decides which type he reports to be...
  ... and pays according to $P(s)$ if he reports to be type s
- The net-payoff of a consumer of type t claiming to be s is:

$$V(s|t) = t\psi(x^*(s)) - P(x^*(s)|s)$$

- If the firm keeps offering the complete information $P(t)$...
  ... both types of consumers purchase $P(L)$ since:

$$V(L|H) = (H-L)\psi(x^*(L)) > 0 = V(H|H)$$
$$V(L|L) = 0 > (L-H)\psi(x^*(H)) = V(H|L)$$

Notes

Incomplete Information: No Pooling

Offering the same contracts however is not optimal for the firm:

- Consider decreasing $p_0(H)$ to $\overline{p}_0(H) > p_0(L)$ so that:

$$V(H|H) = H\psi(x^*(H)) - [\overline{p}_0(H) + p_1 x^*(H)] = V(L|H)$$

- Such a change would increase the firm's profits as:

$$\pi(H)\overline{p}_0(H) + \pi(L)p_0(L) \geq p_0(L)$$

Theorem (No Pooling)

It is not optimal for the firm to offer contracts that lead consumers to pool

Notes

Incomplete Info: Participation & Incentive Constraints

The previous remark implies that the firm wants to satisfy both:

- The *participation constraint* for any type $t \in \{L, H\}$:

$$V(t|t) \geq 0 \qquad \text{(PC(t))}$$

- The *incentive constraint* for any type $t \neq s \in \{L, H\}$:

$$V(t|t) \geq V(s|t) \qquad \text{(IC(t))}$$

- The firm chooses $P(t)$ and in effect also $x^*(t)$ by exploiting FOC(t):

$$P_x(x^*(t)|t) = t\psi_x(x^*(t)) \qquad \text{(FOC(t))}$$

For $P(t) = P(x(t)|t)$, the problem of the firm can be written as:

$$\max_{x(t),P(t)} \sum_{t\in\{L,H\}} \pi(t)[P(t) - cx(t)] \text{ subject to}$$
$$V(t|t) \geq V(s|t) \text{ for any } t \in \{L, H\}$$
$$V(t|t) \geq 0 \text{ for any } t \in \{L, H\}$$

Notes

Incomplete Information: Optimal Pricing

Prior to solving the problem, notice that:

- PC(L) holds with equality (otw firm can increase profits raising $P(L)$):

$$V(L|L) = L\psi(x(L)) - P(L) = 0$$

- IC(H) holds with equality (otw firm can increase profits raising $P(H)$):

$$V(H|H) = H\psi(x(H)) - P(H) = H\psi(x(L)) - P(L) = V(L|H)$$

- PC(H) is strict (by the previous two equalities and $H > L$):

$$V(H|H) = H\psi(x(H)) - P(H) > 0$$

- IC(L) is strict (by no pooling theorem as otw $x(H) = x(L)$):

$$V(L|L) = L\psi(x(L)) - P(L) > L\psi(x(H)) - P(H) = V(H|L)$$

Notes

Incomplete Information: Optimal Pricing

The previous remarks simplify the firm's problem to:

$$\max_{x(t),P(t)} \left[\sum_{t\in\{L,H\}} \pi(t)[P(t) - cx(t)] \right] + \lambda V(L|L) + \mu[V(H|H) - V(L|H)]$$

First order optimality requires:

$$\begin{aligned} -\pi(H)c + \mu H\psi_x(x(H)) &= 0 & \text{(x(H))} \\ -\pi(L)c + \lambda L\psi_x(x(L)) - \mu H\psi_x(x(L)) &= 0 & \text{(x(L))} \\ \pi(H) - \mu &= 0 & \text{(P(H))} \\ \pi(L) - \lambda + \mu &= 0 & \text{(P(L))} \end{aligned}$$

Notice that $\mu = \pi(H)$, $\lambda = 1$ and thus:

$$H\psi_x(x(H)) = c$$

$$L\psi_x(x(L)) = \frac{c}{1 - (\pi(H)/\pi(L))\,[(H/L) - 1]}$$

$P(H)$ and $P(L)$ are pinned down by the two binding constraints

Notes

Incomplete Information: No Distortion at the Top

Notice that the optimality conditions for $x(t)$ require that:

$$MRS_{xy}(H) = MRT_{xy} = c$$
$$MRS_{xy}(L) > MRT_{xy} = c$$

This principle carries over to more general setups and requires:

Theorem (No Distortion at the Top)

In the second-best pricing optimum for the firm the high valuation types are offered a non distortionary (efficient) contract

In general (if the single-crossing condition is met) second best-optimum $x_{SB}(t)$ if compared to full-information optimum $x_{FB}(t)$ satisfies:

$$x_{SB}(H) = x_{FB}(H)$$
$$x_{SB}(L) < x_{FB}(L)$$
$$x_{SB}(L) < x_{SB}(H)$$

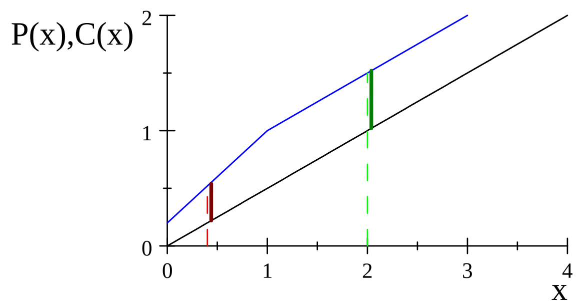
Notes

Incomplete Information: Competition

With competition and free entry firms do not run positive profits

Or else entering firms would profit by offering $P'(x|t) \in [C(x), P(x|t))$

As they would sell to all buyers $\Longrightarrow$ competition requires $P(x|t) = C(x)$



In blue P(x), in black C(x), dashed in light red x(L), in dark red Profits(L), dashed in light green x(H), in dark red Profits(H)

Notes

Example: Competition in Insurance Markets

Consider the following economy:

- Individuals have two types $\{H, L\}$
- The fraction of individuals of type t is π_t
- Any individual can be healthy or sick
- The probability of type t being sick is σ_t
- Assume that $\sigma_H > \sigma_L$
- The income of an individual is Y if healthy and $Y - K$ if sick
- Let y_t denote the consumption of type t if healthy & x_t if sick
- Preference of type t satisfy:

$$\sigma_t u(x_t) + (1 - \sigma_t) u(y_t)$$

Notes

Example: Competition in Insurance Markets

- The insurance market is competitive (free entry)
- Consumers can buy insurance coverage $z_t \in [0, K]$...
  ... at a unit price p_t [ie total premium $p_t z_t$]
- If they do so their consumption in the two states becomes:

$$\begin{aligned} y_t &= Y - p_t z_t \\ x_t &= Y - K - p_t z_t + z_t = Y - K + (1 - p_t) z_t \end{aligned}$$

- If so the problem of a consumer becomes:

$$\max_{z_t \in [0,K]} \sigma_t u(x_t) + (1 - \sigma_t) u(y_t)$$

- Thus, FOC with respect to z_t requires for type t:

$$\sigma_t (1 - p_t) u'(x_t) = (1 - \sigma_t) p_t u'(y_t)$$

Notes

Example: Competition in Insurance Markets

- FOC can be written in terms of MRS as:

$$\frac{u'(x_t)}{u'(y_t)} = \frac{1 - \sigma_t}{\sigma_t} \frac{p_t}{1 - p_t}$$

- Thus a consumer of type t wants:

$$\begin{array}{rlll} \text{Full Insurance:} & z_t = K & \text{if} & p_t = \sigma_t \\ \text{Under Insurance:} & z_t < K & \text{if} & p_t > \sigma_t \\ \text{Over Insurance:} & z_t > K & \text{if} & p_t < \sigma_t \end{array}$$

- The profits of an insurance company are given by:

$$\textstyle\sum_t \pi_t z_t (p_t - \sigma_t)$$

  thus a company does not run a loss provided that $p_t \geq \sigma_t$

Notes

Competition in Insurance Markets: Full Info

Assume that insurance companies can distinguish the two types

If so, the companies set a different price for each type

Since the markets are perfectly competitive insurance companies:

- Offer price $p_t = \sigma_t$ to type $t \in \{H, L\}$
- At such prices all consumers fully insure
- And each firm makes zero profits

Thus no entrant could benefit from offering competing policies

Notes

Competition in Insurance Markets: Incomplete Info

If insurance companies cannot distinguish the two type:

- Offering the complete information contracts is suboptimal...
  ... as all players claim to be of type L to pay $p_L = \sigma_L < p_H$
- This cannot be optimal for a firm since it would run a loss:

$$\pi_H K(p_L - \sigma_H) + \pi_L K(p_L - \sigma_L) < 0$$

- Alternatively a firm may not attempt to distinguish consumers...
  but may offer a price that would lead to break even if all fully insure:

$$p = \pi_H \sigma_H + \pi_L \sigma_L$$

- If so, low risk type L wants to under insure as $\sigma_L < p$ and...
  high risk type H wants over insurance as $\sigma_H > p$ and...
  would pick $z_H = K$

Notes

Competition in Insurance Markets: No Pooling

If, however, different types respond to p as detailed above:

- A company can tell types apart as only type H buys full insurance...
- And prefers to raise prices to those individuals to σ_H

A consumer of type H thus prefers to mimic type L:

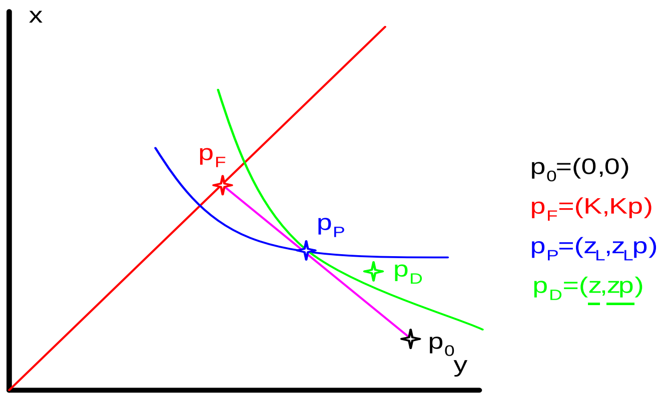
- Buying z_L units (defined by FOC(L)) at price p

If so, the firm benefits by offering a policy $\left(\underline{p}, \underline{z}\right)$:

- that is preferred by type L consumers but not by type H
- it entails a lower price $\underline{p} \in (\sigma_L, p)$ and a lower $\underline{z} < z_L$ to discourage type H from purchase and to signal them out
- moreover such policy runs a profit as $\underline{p} > \sigma_L$

Notes

Competition in Insurance Markets: No Pooling



Theorem (No Pooling)

There is no pooling equilibrium in a competitive insurance market

Notes

Insurance Markets: Separating Equilibria may not Exist

Thus firms have to offer separating contracts if an equilibrium is to exist:

- Consider $(p_H, z_H) = (\sigma_H, K)$ and $(p_L, z_L) = (\sigma_L, w)$
- For players of type H to choose (p_H, z_H) requires IC:

$$u(Y - \sigma_H K) \geq \sigma_H u(Y - K + (1 - \sigma_L)w) + (1 - \sigma_H)u(Y - \sigma_L w)$$

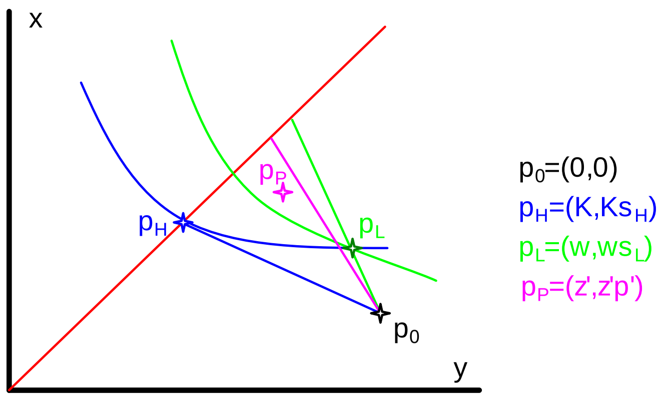
- Similarly players of type Lwould choose (p_L, z_L) since:

$$\sigma_L u(Y - K + (1 - \sigma_L)w) + (1 - \sigma_L)u(Y - \sigma_L w) \geq u(Y - \sigma_H K)$$

- PROBLEM: if π_L is high enough both contracts are dominated...
  ... by pooling contract $(p', z') = (p + \varepsilon, K - \varepsilon)$
- If so a competitive insurance market may have no equilibrium
- Cause: Profits from each type depend directly on hidden info!

Notes

Insurance Markets: Separating Equilibria may not Exist



Theorem (No Equilibrium)

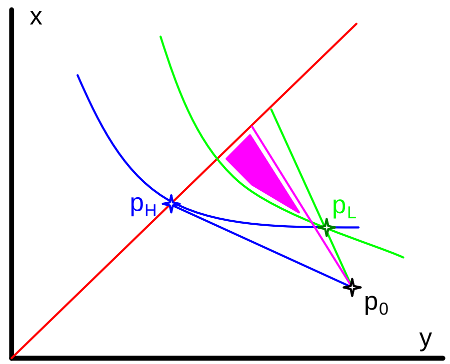
No equilibrium may exist in a competitive insurance market

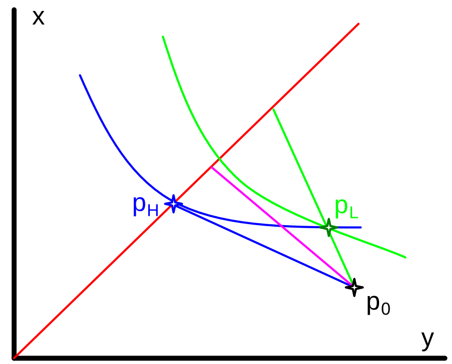
Notes

Insurance Markets: Separating Equilibria may not Exist

The magenta region (left plot) identifies the pooling contracts that are profitable if purchased by both types and that are accepted by both types:

- if such region is non-empty (left plot) no equilibrium exists
- if the region is empty (right plot) a separating equilibrium exists





Notes

Notes

Signaling
EC202 Lectures XIII & XIV

Francesco Nava

LSE

February 2013

Notes

Summary

Signaling:

1. Hidden Characteristics
2. Informed party moves first

- Costly Signals:
  - Educational Choice
  - Pooling Equilibria
  - Separating Equilibria
- Costless Signals

Notes

A Signaling Model

Consider the following educational choice model:

- There are two types of workers $\{g, b\}$
- Type t having probability π_t
- Workers can signal their type by acquiring education
- Different types have different costs to acquire education
- Firms can distinguish workers only by their education and ...
  ... compete on wages to hire workers given such information

Timing:

1. Nature determines the type of each worker
2. Workers decide how much education to acquire
3. Firms simultaneously make wage offers (Bertrand)
4. Workers decide whether or not to accept an offer

Notes

Signaling: Educational Choice Model

In particular consider the following model:

- Individuals can acquire any level $e \in [0, 1]$ education
- The cost of acquiring level e education for type t is $c(e|t)$
- Suppose that costs satisfy:

$$c(0|t) = 0 \ \ \& \ \ c_e(e|t) > 0 \ \ \& \ \ c_{ee}(e|t) > 0$$
$$c_e(e|g) < c_e(e|b)$$

- Suppose that firms offer a wage schedule $w(e)$
- If so the payoff of a worker of type t with education z is:

$$u(e|t) = w(e) - c(e|t)$$

- Assumptions on costs and preference imply that the single crossing condition is met (ie indifference curves cross once)

Notes

Educational Choice Model: Firms

In this economy firms are modeled as follows:

- Firms know that the productivity of a worker of type t is $f(t)$
- If firms knew the type of a worker, they'd pay type t exactly $f(t)$
  Bertrand competition implies that wages equal productivity
  Recall that Bertrand $\Rightarrow$ price equals marginal cost
- Initially firms know only the probability that a worker is of type t, π_t
- Firms can observe the educational decisions of workers
- After workers have made their educational decision, firms:
  1. update their beliefs on the basis of this new information
  2. pick a wage schedule $w(e)$ that depends on education

Notes

Educational Choice Model: Types of Equilibria

This model has two different types of equilibria:

- Separating equilibria in which the two types of worker:
  1. choose different education levels
  2. are paid different wages
  3. prefer not to mimic the other type
- Pooling equilibria in which the two types of worker:
  1. choose the same education level
  2. are paid the same wage
  3. prefer not to be separated from the other type

Notes

Educational Choice Model: Separating Equilibria I

In a separating equilibrium:

- Workers of different type choose different education levels e_t
- Firms recognize either type t by his education e_t
- Firms pay each type its marginal productivity $w(e_t) = f(t)$
  [Bertrand competition $\Rightarrow$ wages equal productivity]
- For types to reveal themselves at wage $w(e_t)$ it must be that:

$$f(g) - c(e_g|g) \geq f(b) - c(e_b|g) \qquad \text{(IC(g))}$$
$$f(b) - c(e_b|b) \geq f(g) - c(e_g|b) \qquad \text{(IC(b))}$$

- Since education has no effect on productivity and because bad workers are identified by their education anyway $\Rightarrow e_b = 0$
  [In a model with more types only the lowest acquires no education]

Notes

Educational Choice Model: Separating Equilibria II

- The last observation $\Rightarrow$ incentive constraints can be written as:

$$c(e_g|g) \leq f(g) - f(b)$$
$$c(e_g|b) \geq f(g) - f(b)$$

- IC conditions require that $e_g \in [\underline{e}, \overline{e}]$ with boundaries defined by:

$$c(\overline{e}|g) = f(g) - f(b) = c(\underline{e}|b)$$

- If the two types of workers chose in equilibrium education levels $e_g \in [\underline{e}, \overline{e}]$ & $e_b = 0$ the ex-post beliefs of a firm are:

$$\pi_g(e) = \begin{cases} 1 & \text{if} \quad e = e_g \\ 0 & \text{if} \quad e = e_b \end{cases}$$

- Beliefs don't have to be pinned down for $e \neq e_g, e_b$

Notes

Educational Choice Model: Separating Equilibria III

- To guarantee that no worker chooses $e \neq e_g, e_b$ suppose that:

$$w(e) = \begin{cases} f(b) & \text{if} \quad e < e_g \\ f(g) & \text{if} \quad e \geq e_g \end{cases}$$

- If such are the wages no good worker prefers to deviate since:

$$f(g) - c(e_g|g) \geq f(g) - c(e|g) \text{ for } e > e_g$$
$$f(g) - c(e_g|g) \geq f(b) - c(e|g) \text{ for } e < e_g$$

- Moreover no bad worker prefers to deviate since:

$$f(b) \geq f(g) - c(e|b) \text{ for } e > e_g$$
$$f(b) \geq f(b) - c(e|b) \text{ for } e < e_g$$

- More general off equilibrium wages schedules achieve the same result, but complicate the analysis unnecessarily

Notes

Educational Choice Model: Separating Equilibria IV

There is a multiplicity of separating Perfect Bayesian equilibria:

- They are characterized by the education levels satisfying:

$$e_g \in [\underline{e}, \overline{e}] \;\; \& \;\; e_b = 0$$

- Workers receive the efficient wage, namely their productivity
- But no equilibrium is efficient since good workers lose resources to signal their type by investing in education
- The Pareto dominant equilibrium is the one in which $e_g = \underline{e}$ since the cost of acquiring education is the lowest
- The multiplicity is mainly due to the unspecified off equilibrium beliefs

Notes

Educational Choice Model: Separating Equilibria V

The multiplicity disappears for appropriately chosen beliefs:

- The **intuitive criterion** for out of equilibrium beliefs says:

$$e > \underline{e} \implies \pi_g(e) = 1$$

- This criterion is reasonable bad workers prefer $e = 0$ to $e > \underline{e}$:

$$f(b) > f(g) - c(e|b)$$

  which holds by definition of $\underline{e}$
- If firms' beliefs meet the intuitive criterion the only PBE that survives is the one in which $e_g = \underline{e}$ and $e_b = 0$
- Indeed if $e_g > \underline{e}$, good workers prefer to switch to $\underline{e}$ since:

$$f(g) - c(e_g|g) < f(g) - c(\underline{e}|g)$$

- The Pareto dominant equilibrium is the only PBE that survives the intuitive criterion and involves the lowest education levels

Notes

Educational Choice Model: Pooling Equilibria I

In a pooling equilibrium:

- All workers choose same education levels e^*
- Firms cannot recognize workers by their education e^* and pay all workers their expected productivity:

$$w(e^*) = \pi_g f(g) + \pi_b f(b)$$

- To guarantee that no worker chooses $e \neq e^*$ suppose that:

$$w(e) = \begin{cases} f(b) & \text{if} \quad e < e^* \\ \pi_g f(g) + \pi_b f(b) & \text{if} \quad e \geq e^* \end{cases}$$

- For types not to reveal themselves at wage $w(e^*)$ it must be that:

$$w(e^*) - c(e^*|b) \geq f(b)$$

- Such condition requires $e^* \leq \hat{e}$ where $\hat{e}$ is defined by:

$$c(\hat{e}|b) = \pi_g[f(g) - f(b)]$$

Notes

Educational Choice Model: Pooling Equilibria II

There is a multiplicity of pooling Perfect Bayesian equilibria:

- They are characterized by an education level $e^* \in [0, \hat{e}]$
- Workers wages are inefficient and differ from their productivity
- Again the multiplicity is due to the unspecified off equilibrium beliefs
- No pooling equilibrium meets the intuitive criterion
- For convenience let e^+ be defined by:

$$w(e^*) - c(e^*|b) = f(g) - c(e^+|b)$$

- If firms believe that $\pi_g(e) = 1$ if $e > e^+$ and set wage $w(e) = f(g)$
- Then good workers choose $e = e^+ + \varepsilon$ while bad ones do not since:

$$f(g) - c(e|g) > w(e^*) - c(e^*|g) > f(g) - c(e|b)$$

Notes

Comparison: Signaling vs Adverse Selection

The results on signaling **differ** from those on adverse selection since:

- There is a multiplicity of pooling equilibria
- There is a multiplicity of separating equilibrium
- The incentive constraint neither type may bind in a pooling equilibrium

However most result **coincide** when the intuitive criterion is applied:

- There are no pooling equilibria
- There is a unique separating equilibrium
- The incentive constraint of the bad type binds
- The incentive constraint of the good type does not bind
- Inefficiencies arise to provide incentives to the good type

Notes

Costless Signals I

The theory on costless signals is more involved. Some more result can be understood from the following example:

- There are N risk-neutral individuals
- All of them can participate in the production of a public good
- In particular player i chooses his effort $e_i \in \{0, 1\}$
- The public good is produced only if all exert $e_i = 1$
- The cost of exerting effort c_i is private information and costs are uniformly distributed on $[0, 1]$ – ie $\Pr(c_i < b) = b$
- The preferences of player i with cost c_i are – for $a < 1$:

$$u_i(e|c_i) = a\prod_{j\in N} e_j - c_i e_i$$

- The unique BNE of this game has all players exerting no effort
- This follows since there is positive probability that $c_i > a$ for some i

Notes

Costless Signals II

If a signaling stage is introduced prior to effort decision such that:

- Each agent can announce his willingness to exert effort
- In particular each agent can say $\{Yes, No\}$ to him exerting effort

When such signaling stage is added, then there is a BNE in which:

- Any agent announces *Yes* if and only if $c_i \leq a$
- Each agent chooses $e_i = 1$ if and only if all said *Yes*
- This is a BNE since individuals no longer risk wasting their effort

However, many other BNE are possible in which no useful information is exchanged at the communication stage. Such BNE are known as babbling equilibria.

Notes

Moral Hazard

EC202 Lectures XV & XVI

Francesco Nava

LSE

February 2013

Notes

Summary

Hidden Action Problem aka:

1. Moral Hazard Problem
2. Principal Agent Model

- Outline
- Simplified Model:
  - Complete Information Benchmark
  - Hidden Effort
  - Agency Cost
- General Principal Agent Model
  - Complete Information Benchmark
  - Hidden Effort
  - Agency Cost

Notes

Outline: Moral Hazard Problem

The basic ingredients of a moral hazard model are as follows:

- A **principal** and an **agent**, are involved in bilateral relationship
- Principal wants Agent to perform some task
- Agent can choose how much effort to devote to the task
- The outcome of the task is pinned down by a mix of effort and luck
- Principal cannot observe effort and can only motivate Agent by paying him based on the outcome of the task

Timing:

1. Principal chooses a wage schedule which depends on outcome
2. Agent chooses how much effort to devote the task
3. Agent's effort and chance determine the outcome
4. Payments are made according to the proposed wage schedule

Notes

A simple Principal-Agent Model

Consider the following simplified model:

- A task has two possible monetary outcomes: $\{\underline{q}, \overline{q}\}$ with $\underline{q} < \overline{q}$
- Agent can choose one of two effort levels: $\{\underline{e}, \overline{e}\}$ with $\underline{e} < \overline{e}$
- The probability of the high output given effort e is:

$$\overline{\pi}(e) = \Pr(q = \overline{q}|e)$$

- Assume that $\overline{\pi}(\underline{e}) < \overline{\pi}(\overline{e})$ – ie more effort $\Rightarrow$ better outcomes
- Principal chooses a wage schedule w
- Agent is risk averse and his preferences are:

$$U(w, e) = E[u(w, e)]$$

- Principal is risk neutral and his preferences are:

$$V(w) = E[q - w]$$

Notes

Simple Principal-Agent Model: Complete Info I

Begin by looking at the complete information benchmark:

- Principal can observe the effort chosen by Agent
- Principal picks a wage schedule $w(e)$ that depend on Agent's effort
- Agent's reservation utility is $\underline{u}$ – ie what he gets if he resigns
- Thus the participation constraint of Agent is:

$$U(w(e), e) = u(w(e), e) \geq \underline{u}$$

- By picking wages appropriately Principal *de facto* chooses e and w
- The problem of Principal thus is to:

$$\max_{e,w} E[q|e] - w(e) + \lambda[u(w(e), e) - \underline{u}]$$

- Recall that $E[q|e] = \overline{\pi}(e)\overline{q} + \underline{\pi}(e)\underline{q}$

Notes

Simple Principal-Agent Model: Complete Info II

- Recall the problem of Principal:

$$\max_{e,w} E[q|e] - w(e) + \lambda[u(w(e), e) - \underline{u}]$$

- The lowest wage $\widehat{w}(e)$ that induces effort e from Agent is:

$$u(\widehat{w}(e), e) = \underline{u}$$

- Thus Principal chooses to induce effort e^* if and only if:

$$e^* \in \arg\max_{e \in \{\overline{e}, \underline{e}\}} E[q|e] - \widehat{w}(e)$$

- Principal then induces such effort choice by offering wages:

$$w^*(e) = \begin{cases} \widehat{w}(e) & \text{if} \quad e = e^* \\ \widehat{w}(e) - \varepsilon & \text{if} \quad e \neq e^* \end{cases}$$

- Complete info implies that FOC for the wage requires $MC = Price$:

$$1/u_w(w, e) = \lambda$$

Notes

Simple Principal-Agent Model: Incomplete Info I

Now consider the case in which effort is unobservable for Principal:

- Suppose that Principal prefers Agent to exert high effort $\overline{e}$
- Principal can only condition wage $w(q)$ on outcome q:

$$w(q) = \begin{cases} \underline{w} & \text{if} \quad q = \underline{q} \\ \overline{w} & \text{if} \quad q = \overline{q} \end{cases}$$

- Agent's **participation constraint** at e requires:

$$U(w(q), e) = \overline{\pi}(e)u(\overline{w}, e) + \underline{\pi}(e)u(\underline{w}, e) \geq \underline{u} \qquad \text{(PC(e))}$$

- Agent's **incentive constraint** guarantees that he picks high effort:

$$U(w(q), \overline{e}) \geq U(w(q), \underline{e}) \qquad \text{(IC)}$$

Notes

Simple Principal-Agent Model: Incomplete Info II

The problem of a principal who wants the agent to exert $\overline{e}$ is to:

- Maximize his profits by choosing $w(q)$ subject to:
  1. Agent's participation constraint at $\overline{e}$
     ie the agent prefers to exert high effort than to resign
  2. Agent's incentive constraint
     ie the agent prefers to exert high effort than low effort
- Thus the Lagrangian of this problem is:

$$\max_{\underline{w}, \overline{w}} E[q - w(q)|\overline{e}] + \lambda[U(w(q), \overline{e}) - \underline{u}] + \mu[U(w(q), \overline{e}) - U(w(q), \underline{e})]$$

- Recall that:

$$U(w(q), e) = \overline{\pi}(e)u(\overline{w}, e) + \underline{\pi}(e)u(\underline{w}, e)$$
$$E[q - w(q)|e] = \overline{\pi}(e)[\overline{q} - \overline{w}] + \underline{\pi}(e)[\underline{q} - \underline{w}]$$

Notes

Simple Principal-Agent Model: Incomplete Info III

Writing out Lagrangian explicitly the Principal's problem becomes:

$$\max_{\underline{w},\overline{w}} \overline{\pi}(\overline{e})[\overline{q}-\overline{w}] + \underline{\pi}(\overline{e})[\underline{q}-\underline{w}] +$$
$$+\lambda[\overline{\pi}(\overline{e})u(\overline{w},\overline{e}) + \underline{\pi}(\overline{e})u(\underline{w},\overline{e}) - \underline{u}] +$$
$$+\mu[\overline{\pi}(\overline{e})u(\overline{w},\overline{e}) + \underline{\pi}(\overline{e})u(\underline{w},\overline{e}) - \overline{\pi}(\underline{e})u(\overline{w},\underline{e}) - \underline{\pi}(\underline{e})u(\underline{w},\underline{e})]$$

- First order conditions for this problem are:

$$\overline{\pi}(\overline{e})[-1+\lambda u_w(\overline{w},\overline{e}) + \mu u_w(\overline{w},\overline{e})] - \overline{\pi}(\underline{e})\mu u_w(\overline{w},\underline{e}) = 0$$
$$\underline{\pi}(\overline{e})[-1+\lambda u_w(\underline{w},\overline{e}) + \mu u_w(\underline{w},\overline{e})] - \underline{\pi}(\underline{e})\mu u_w(\underline{w},\underline{e}) = 0$$

- By rearranging it is possible to show that:
  1. Both μ and λ are positive if u is increasing and concave
  2. Incentive Constraint binds since $\mu > 0$
  3. Participation constraint at high effort binds since $\lambda > 0$
  4. Wages $\overline{w}, \underline{w}$ are found by solving the two constraints IC & PC

Notes

Simple Principal-Agent Model: Incomplete Info IV

For an explicit characterization let u be additively separable in w and e:

$$u(w,e) = v(w) + \eta(e)$$

First order conditions in this scenario become:

$$\overline{\pi}(\overline{e})[-1+\lambda v_w(\overline{w}) + \mu v_w(\overline{w})] - \overline{\pi}(\underline{e})\mu v_w(\overline{w}) = 0$$
$$\underline{\pi}(\overline{e})[-1+\lambda v_w(\underline{w}) + \mu v_w(\underline{w})] - \underline{\pi}(\underline{e})\mu v_w(\underline{w}) = 0$$

Solving we find that $\lambda, \mu > 0$ (condition (L) parallels the complete info):

$$\lambda = \frac{\overline{\pi}(\overline{e})}{v_w(\overline{w})} + \frac{\underline{\pi}(\overline{e})}{v_w(\underline{w})} > 0 \qquad \text{(L)}$$

$$\mu\left[1 - \frac{\overline{\pi}(\underline{e})}{\overline{\pi}(\overline{e})}\right] = \underline{\pi}(\overline{e})\left[\frac{1}{v_w(\overline{w})} - \frac{1}{v_w(\underline{w})}\right] > 0 \qquad \text{(M)}$$

$$v(\underline{w}) = \underline{u} + \eta(\overline{e})\frac{\overline{\pi}(\underline{e})}{\overline{\pi}(\overline{e}) - \overline{\pi}(\underline{e})} - \eta(\underline{e})\frac{\overline{\pi}(\overline{e})}{\overline{\pi}(\overline{e}) - \overline{\pi}(\underline{e})}$$

$$v(\overline{w}) = \underline{u} - \eta(\overline{e})\frac{\underline{\pi}(\underline{e})}{\overline{\pi}(\overline{e}) - \overline{\pi}(\underline{e})} + \eta(\underline{e})\frac{\underline{\pi}(\overline{e})}{\overline{\pi}(\overline{e}) - \overline{\pi}(\underline{e})}$$

Notes

Simple Principal-Agent Model: Example

Example: $e \in \{0,1\}$, $u(w,e) = w - e^2$, $\underline{u} = 1$,
$\overline{q} = 4$, $\underline{q} = 0$, $\overline{\pi}(1) = 3/4$, $\overline{\pi}(0) = 1/4$

Complete Info: what are $w(0)$, $w(1)$, e^*?

- Wages $w(1) = 2$ and $w(0) = 1$ are found by PC(e):

$$w(1) - 1 = 1 \;\text{ and }\; w(0) = 1$$

- Optimal effort $e^* = 1$ is found by comparing profits:

$$(3/4)(4-2) + (1/4)(-2) = 1 > (1/4)(4-1) + (3/4)(-1) = 0$$

Incomplete Info: what are $\overline{w}$, $\underline{w}$, if firm wants $e^* = 1$?

- Wages $\overline{w} = 5/2$ and $\underline{w} = 1/2$ are found by solving PC(1) and IC:

$$(3/4)(\overline{w}-1) + (1/4)(\underline{w}-1) = 1$$
$$(3/4)(\overline{w}-1) + (1/4)(\underline{w}-1) = (1/4)\overline{w} + (3/4)\underline{w}$$

Notes

Principal-Agent Model

Consider a general setup in which:

- Agent chooses any effort level $e \in [0,1]$
- Agent's reservation utility is still $\underline{u}$
- The state of the world ω lives in some interval Ω
- Output is produced according to a production function

$$q = q(e,\omega)$$

- Principal is risk neutral or risk averse and his preferences are:

$$V(w) = E[v(q-w)]$$

- Agent is risk averse and his preferences are:

$$U(w,e) = E[u(w,e)]$$

- Principal moves first and takes Agent's response as given

Notes

Principal-Agent Model: Complete Info I

Let's begin by analyzing the complete info model:

- Principal can observe Agent's effort e and output q...
  ... thus he can infer ω because he knows $q = f(e, \omega)$
- Agent's participation constraint remains

$$U(w, e) \geq \underline{u}$$

- Principal can choose wages that depend on both e and ω...
  ...this is equivalent to Principal picking both e and $w(\omega)$...
  ... since Principal could choose a wage schedule such that:

$$\overline{w}(e, \omega) = \begin{cases} w(\omega) & \text{if} \quad e \text{ is optimal for Principal} \\ w' \text{ st } U(w', e) < \underline{u} & \text{if} \quad \text{Otherwise} \end{cases}$$

Notes

Principal-Agent Model: Complete Info II

Thus the problem of Principal becomes:

$$\max_{e, w(\cdot)} E[v(q(e, \omega) - w(\omega))] + \lambda[E(u(w(\omega), e)) - \underline{u}]$$

First order conditions require that (note $v_w = v_e \equiv v_x$):

$$E[v_x(q(e, \omega) - w(\omega))q_e(e, \omega)] + \lambda E(u_e(w(\omega), e)) = 0$$
$$-v_x(q(e, \omega) - w(\omega)) + \lambda u_w(w(\omega), e) = 0$$

Combining the two equations one gets that:

$$E[v_x(q(e, \omega) - w(\omega))q_e(e, \omega)] = -E\left[v_x(q(e, \omega) - w(\omega))\frac{u_e(w(\omega), e)}{u_w(w(\omega), e)}\right]$$

If Principal is risk neutral this condition requires (efficiency):

$$MRT_{e,w} = E[q_e(e, \omega)] = -E\left[\frac{u_e(w(\omega), e)}{u_w(w(\omega), e)}\right] = MRS_{e,w}$$

Solving FOC & PC yields the optimal effort e and wage schedule $w(\omega)$

Notes

Principal-Agent Model: Incomplete Info I

Principal observes output q, but not effort e and is thus unable to infer ω:

- Let $f(q|e)$ denote the probability of output q given effort e
- Let $F(q|e)$ denote the cumulative distribution associated to $f(q|e)$
- Assume that $f(q|e)$ satisfies:
  1. Pdf of output has bounded support $[\underline{q}, \overline{q}]$
  2. The support $[\underline{q}, \overline{q}]$ is publicly known
  3. The support $[\underline{q}, \overline{q}]$ does not depend on e
  4. If $e > e'$ then $F(q|e) < F(q|e')$
- Define a proportionate shift in output $\beta_z(q|z)$ by:

$$\beta_e(q|e) = f_e(q|e)/f(q|e)$$

- Since $F(\overline{q}|e) = 1$ implies $F_e(\overline{q}|e) = 0$ we get that:

$$E[\beta_e(q|e)] = \int_{\underline{q}}^{\overline{q}} \beta_e(q|e) f(q|e) dq = F_e(\overline{q}|e) = 0$$

Notes

Principal-Agent Model: Incomplete Info II

- Suppose that Principal offers wage schedule $w(q)$
- If Agent participates, he chooses effort to maximize his wellbeing:

$$\max_{e \in [0,1]} U(w(q), e) = \max_{e \in [0,1]} \int_{\underline{q}}^{\overline{q}} u(w(q), e) f(q|e) dq$$

- First order condition requires:

$$\int_{\underline{q}}^{\overline{q}} [u_e(w(q), e) f(q|e) + u(w(q), e) f_e(q|e)] dq = 0$$

$$\Rightarrow E[u_e(w(q), e)] + E[u(w(q), e)\beta_e(q|e)] = 0$$

- Reduction in wellbeing due to extra effort is exactly compensated ...
  ... by the increase in expected income due higher effort
- Principal takes Agent's FOC as a constraint on his program
  [as was the case with IC in the simplified model]

Notes

Principal-Agent Model: Incomplete Info III

- Principal chooses the wages to maximize his wellbeing subject to:
  1. Agent's participation constraint (PC)
  2. Agent's first order condition (FOC)
- In particular the problem of Principal is:

$$\max_{w(q),e} V(w(q)) + \lambda[U(w(q),e) - \underline{u}] + \mu[\partial U(w(q),e)/\partial e] =$$

$$\max_{w(q),e} E[v(q - w(q))] + \lambda[E[u(w(q),e)] - \underline{u}] +$$

$$+ \mu[E[u_e(w(q),e)] + E[u(w(q),e)\beta_e(q|e)]]$$

- For convenience assume that Agent's utility satisfies $u_{we} = 0$
- More than with complete info if e is high since $\beta_e(q|e) > 0$
- First order conditions require:

$$-v_x(q - w(q)) + \lambda u_w(w(q),e) + \mu u_w(w(q),e)\beta_e(q|e) = 0 \quad \text{[w(q)]}$$

$$E[v(q - w(q))\beta_e(q|e)] + \mu\left[\frac{\partial^2 E[u(w(q),e)]}{\partial e^2}\right] = 0 \quad \text{[e]}$$

Notes

Principal-Agent Model: Incomplete Info IV

- By rearranging terms FOC become:

$$\frac{v_x(q - w(q))}{u_w(w(q),e)} = \lambda + \mu\beta_e(q|e) \quad \text{for any } q$$

$$E[v(q - w(q))\beta_e(q|e)] + \mu\left[\frac{\partial^2 E[u(w(q),e)]}{\partial e^2}\right] = 0$$

- The second condition implies $\mu > 0$ since:
  1. $E[\beta_e(q|e)] = 0$ & $v_x > 0$ imply that $E[v(q - w(q))\beta_e(q|e)] > 0$
  2. Agent's second order conditions imply $\partial^2 E[u(w(q),e)]/\partial e^2 < 0$
- Therefore Agent's FOC holds with equality
- The first condition requires that Principal pays Agent:
  1. More than with complete info if q is high since $\beta_e(q|e) > 0$
  2. Less than with complete info if q is low since $\beta_e(q|e) < 0$

Notes

Principal-Agent Model: Incomplete Info V

Main Conclusions with Moral Hazard:

- Compared to compete info Principal:
  1. pays Agent more when output is high
  2. pays Agent less when output is bad
  3. no longer provides full insurance to Agent on the variable output
- He does so to provide incentives for Agent to exert effort...
  ... since a fully insured Agent would have no motives to exert effort
- This conclusions rely on the information problem of Principal and
  ... would hold even if Principal were risk neutral
- They always hold so long as Agent is risk averse

Notes

Notes

Externalities

EC202 Lectures XVII & XVIII

Francesco Nava

LSE

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Notes

Summary

A common cause of Market Failures are **Externalities**:

1. Production Externalities
   Eg: Pollution (negative) & Research (positive)
2. Consumption Externalities
   Eg: Tobacco (negative) & Deodorant (positive)

- Competitive Outcome is not Pareto Optimal
- Solutions to the Problem
  - Taxes & Subsidies
  - Private Solutions: Reorganization
  - Private Solutions: Pseudo-Markets
- Coase Theorem (Take 1)

Notes

Production & Consumption Externalities

Definition (Externality)

There is an **externality** when an agent's actions **directly** influence the choice possibilities (production set or consumption set) of another agent.

Definition (Consumption Exernality)

There is a **consumption externality** when an agent's actions **directly** influence the consumption set of another agent.

Definition (Production Exernality)

There is a **production externality** when an agent's actions **directly** influence the production set of another agent.

Classical example by Meade: beekeeper and nearby orchard, both increase the other agent's productivity and production possibilities.

Notes

Positive & Negative Externalities

Definition (Positive Exernality)

There is a **positive externality** when an agent's actions **increase** the choice possibilities of another agent.

Definition (Negative Exernality)

There is a **negative externality** when an agent's actions **decrease** the choice possibilities of another agent.

Common causes of externalities are:

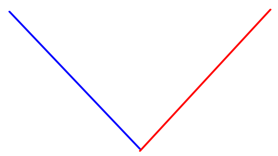
- Networking Effects (investment in assets that facilitate cooperation)
- Civic Action (good norms of behavior that benefit others)
- Undefined Ownership of Resources (excessive use of resources)

Notes

A Simple Model with Externalities I

Topic is discussed with a simple example of production externalities:

- Two goods $\{1, 2\}$, two firms $\{a, b\}$ and one consumer c
- Good 1 is polluting while good 2 is not
- Firm a is situated on river A
- Consumer c is situated on river B
- Firm b is situated after the confluence of the two rivers



Notes

A Simple Model with Externalities II

- Firm a produces good 1 using good 2:

$$y_1 = f(x_2^a)$$

- Consumer c has preferences for the two goods defined by:

$$U(x_1^c, x_2^c)$$

- Firm b produces good 2 using good 1:

$$y_2 = g(x_1^b, y_1, x_1^c)$$

  its output decreases with river pollution which depends on the quantity of good 1 produced and consumed upstream

- Let (e_1, e_2) denote the initial resources of the economy

Notes

Pareto Optimum I

The Pareto Optima of this economy are solutions of the following program:

$$\max_{x_1^c, x_2^c, y_1, y_2, x_1^b, x_2^a} U(x_1^c, x_2^c) \quad \text{subject to}$$

$$\begin{aligned} x_1^c + x_1^b &\leq e_1 + y_1 && (\lambda_1) \\ x_2^c + x_2^a &\leq e_2 + y_2 && (\lambda_2) \\ y_1 &\leq f(x_2^a) && (\mu_1) \\ y_2 &\leq g(x_1^b, y_1, x_1^c) && (\mu_2) \end{aligned}$$

As production constraints bind this corresponds to:

$$\max_{x_1^c, x_2^c, x_1^b, x_2^a} U(x_1^c, x_2^c) \quad \text{subject to}$$

$$\begin{aligned} x_1^c + x_1^b &\leq e_1 + f(x_2^a) && (\lambda_1) \\ x_2^c + x_2^a &\leq e_2 + g(x_1^b, f(x_2^a), x_1^c) && (\lambda_2) \end{aligned}$$

Notes

Pareto Optimum II

The Pareto Optima of this economy are solutions of:

$$\max_{x_1^c, x_2^c, x_1^b, x_2^a} U(x_1^c, x_2^c) \quad \text{subject to}$$

$$\begin{aligned} e_1 + f(x_2^a) - x_1^c - x_1^b &\geq 0 && [\lambda_1] \\ e_2 + g(x_1^b, f(x_2^a), x_1^c) - x_2^c - x_2^a &\geq 0 && [\lambda_2] \end{aligned}$$

Taking first order conditions we get that:

$$\begin{aligned} U_1 - \lambda_1 + \lambda_2 g_3 &= 0 && [x_1^c] \\ U_2 - \lambda_2 &= 0 && [x_2^c] \\ \lambda_2 g_1 - \lambda_1 &= 0 && [x_1^b] \\ \lambda_1 f_1 - \lambda_2 + \lambda_2 f_1 g_2 &= 0 && [x_2^a] \end{aligned}$$

Solving the system of FOC requires:

$$\frac{U_1}{U_2} + g_3 = g_1 = \frac{1}{f_1} - g_2$$

Notes

Pareto Optimum III

Thus efficiency in this economy requires:

$$\frac{U_1}{U_2} + g_3 = g_1 = \frac{1}{f_1} - g_2 \qquad \text{(PO)}$$

What does the PO condition require?

- The LHS is the **Social MRS** of consumer c
  It accounts for the externality of consuming x_1^c units of good 1
- The RHS is the **Social MRT** of firm a
  It accounts for the externality of producing y_1 units of good 1
- The central term is simply the **MRT** of firm b

If externalities are present, PO requires MRS and MRT to be adjusted by their social value to account for the external effects of each agent's decisions have on the rest of the economy (Pigou 1920)

Notes

Competitive Equilibrium I

Optimality in a competitive equilibrium agents requires:

$$\frac{U_1}{U_2} = g_1 = \frac{1}{f_1} \qquad \text{(CE)}$$

What does the CE condition require?

- The LHS is the **Private MRS** of consumer c
  It doesn't account for the externality of consuming x_1^c (over-consumption)
- The RHS is the **Private MRT** of firm a
  It doesn't account for the externality of producing y_1 (over-production)
- The central term is the **Private MRT** of firm b

If externalities are present, CE is not PO because agents only consider for their private MRS and MRT and neglect the social consequences of their behavior

Notes

Competitive Equilibrium II

CE condition can be derived by solving the problems of the 3 players:

$$\max_{x_1^b} p_2 g(x_1^b, y_1, x_1^c) - p_1 x_1^b$$
$$\Rightarrow \quad p_2 g_1 = p_1$$

$$\max_{x_2^a} p_1 f(x_2^a) - p_2 x_2^a$$
$$\Rightarrow \quad p_1 f_1 = p_2$$

$$\max_{x_1^c, x_2^c} U(x_1^c, x_2^c) \quad \text{st} \quad p_1 x_1^c + p_2 x_2^c < y$$
$$\Rightarrow \quad \frac{U_1}{U_2} = \frac{p_1}{p_2}$$

Collecting the three FOC, one gets the desired condition:

$$\frac{U_1}{U_2} = g_1 = \frac{1}{f_1} \qquad \text{(CE)}$$

Notes

Externalities: Remedies

Several remedies have been proposed to fix **Market Failures** ($CE \neq PO$) due to to externalities:

- Quotas
- Subsidies for Depollution
- Rights to Pollute
- Pigovian Taxes
- Integration of Firms
- Compensation Mechanisms

We say that a mechanism **internalizes** an externality if it implements the Pareto Optimum in the economy (ie if $CE = PO$)

Notes

Quotas

Quotas are the simplest way to implement PO consumption of good 1:

- Compute PO consumption levels of good 1 $(\overline{x}_1^c, \overline{y}_1)$
- Forbid firm a from producing more than $\overline{y}_1$
- Forbid consumer c from consuming more than $\overline{x}_1^c$

Problems with quotas are:

- Computing PO requires a detailed knowledge of the economy
- It's an authoritarian solution

It's a commonly used solution (though in a less brutal form), eg:

- Limiting quantities of pollutants emitted by firms and consumers
- Limits in CO_2 emissions of automobiles

Notes

Subsidies for Depollution I

Another way to relax the externality is to subsidize firm for depollution:

- Assume that firm a can invest z_2 units of good 2 in depollution
- If so, its pollution drops from y_1 to $y_1 - d(z_2)$
- The resource constraint for good 2 consumption becomes:

$$x_2^c + x_2^a + z_2 \leq e_2 + y_2$$

- The production constraint for firm b becomes:

$$y_2 \leq g(x_1^b, y_1 - d(z_2), x_1^c)$$

- In which case the PO conditions become

$$\frac{U_1}{U_2} + g_3 = g_1 = \frac{1}{f_1} - g_2 = \frac{1}{f_1} + \frac{1}{d_1} \qquad \text{(PO)}$$

since FOC with respect to PO $\overline{z}_2$ imply that $-g_2 d_1(\overline{z}_2) = 1$

Notes

Subsidies for Depollution II

Consider the CE of this economy if the government subsidizes depollution:

- Let $s(z_2)$ denote the subsidy of the government
- With the subsidies in place the program of firm a becomes:

$$\max_{x_2^a, z_2} p_1 f(x_2^a) + s(z_2) - p_2(x_2^a + z_2)$$

- While the problem of firm b becomes:

$$\max_{x_1^b} p_2 g(x_1^b, y_1 - d(z_2), x_1^c) - p_1 x_1^b$$

- Government induces the socially optimal level of depollution by choosing $s(\cdot)$ so that $s_1(\overline{z}_2) = p_2$
- However, the CE for this economy still requires:

$$\frac{U_1}{U_2} = g_1 = \frac{1}{f_1}$$

and is therefore not PO

Notes

Rights to Pollute I

This is the preferred solution by economist (but not by policy makers):

In particular consider the following remedy:

- Firm b (pollutee) sells rights to pollute to firm a and consumer c (the polluters)
- It receives a price r for any pollution right it sells to consumer c
- It receives a price q for any pollution right it sells to firm a

If so, the solution to consumer c's problem becomes:

$$\frac{U_1}{U_2} = \frac{p_1}{p_2} + \frac{r}{p_2}$$

While the solution to firm a's problem becomes:

$$\frac{1}{f_1} = \frac{p_1}{p_2} - \frac{q}{p_2}$$

Notes

Rights to Pollute II

The problem of firm b is more complex as it needs to decide:

- on how much output to produce
- on how many pollution rights to sell to the polluter

In particular firm b solves the following program:

$$\max_{x_1^b, y_1, x_1^c} p_2 g(x_1^b, y_1, x_1^c) - p_1 x_1^b + r x_1^c + q y_1$$

Optimality conditions for this program require:

$$g_1 = \frac{p_1}{p_2} \quad \& \quad -g_2 = \frac{q}{p_2} \quad \& \quad -g_3 = \frac{r}{p_2}$$

Solving FOC for all three players implies efficiency since:

$$\frac{U_1}{U_2} + g_3 = g_1 = \frac{1}{f_1} - g_2$$

Notes

Rights to Pollute III

The creation of markets for rights to pollute therefore:

- implements the Pareto Optimum
- requires less information than quotas as the government does not need to know preferences and technologies of all individuals and firms

Further consideration:

- Not all individuals pay the same price for the same right to pollute
  In our example $q = r$ only if $g(x_1^b, y_1, x_1^c) = G(x_1^b, y_1 + x_1^c)$
- In our example there is only one supplier and buyer in each open pollution market. To avoid strategic considerations it would be better if there were more.
- We have discussed a "polluters pays" scheme.
  Similar arguments work if "depollution rights" markets are opened where pollutees buy from polluters. Of course the distribution of equilibrium utilities would differ.

Notes

Pigovian Taxes

A different way to solve the externality problem is through taxes:

- One could tax production of good 1 at rate T
- One could tax consumption of good 1 at rate t
- Where $T = q$ and $t = r$ (from the previous remedy)

Such tax rates would:

- solve the externality problem since $PO = CE$
- require a lot of information to be computed exactly

These tax levels are often called Pigovian taxes in honor of Pigou who first wrote about them in 1928.

Notes

Integration of Firms

For convenience assume that consumer c does not pollute: $g_3 = 0$.

Another remedy to the externality problem is to have both firms merge.

In which case the merged firm solves the following problem:

$$\max_{x_1^b, x_2^a} p_1 f(x_2^a) + p_2 g(x_1^b, f(x_2^a)) - p_1 x_1^b - p_2 x_2^a$$

The solution to this problem implies PO since FOC require:

$$\frac{p_1}{p_2} = g_1 = \frac{1}{f_1} - g_2$$

This is not the preferred solution since:

- It disregards considerations of market power
  Big firms usually extract higher rents
- It disregards property rights
  Firms may prefer not to merge

Notes

A Compensation Mechanism

Compensating Mechanisms have been designed to internalize externalities whenever:

- the firms and consumers know all fundamentals of the economy
- while the government does not

Such mechanism guarantee that the government can:

- elicit the Pigovian tax rates from producers and consumers
- set such effective tax rates so that $PO = CE$

The limitations to this approach (suggested in Varian 1994) are that:

- it requires a lot of knowledge on the part of consumers and producers
- it does no better than markets for pollution rights

Notes

Coase Theorem I

In a seminal paper of 1960, Coase doubted the necessity of any government intervention in presence of externalities.

His argument proceeded as follows:

- Let $b(q)$ denote the benefit to the polluter of q units of pollution
- Let $c(q)$ denote the cost to the pollutee of q units of pollution
- Assume that $b' > 0$, $b'' < 0$, $c' > 0$, $c'' > 0$
- Efficient pollution q_* would require $b'(q_*) = c'(q_*)$
- If pollution q_0 is inefficient it must be that $b'(q_0) < c'(q_0)$
- If so, the pollutee can ask the polluter:
  1. to reduce pollution by some small number ε to $q_0 - \varepsilon$
  2. in exchange of a transfer $t\varepsilon$ where $t \in (b'(q_0), c'(q_0))$
- Such trade benefits the polluter since $t > b'(q_0)$
- Such trade also benefits the pollutee since $t < c'(q_0)$

Notes

Coase Theorem II

The previous argument can be repeated so long as $b' < c'$

Moreover a similar argument works for the case in which $b' > c'$

Thus Coase concluded that the following result had to hold:

Theorem

If property rights are clearly defined and transaction costs are negligible, the parties affected by an externality succeed in eliminating any inefficiency through the simple recourse of negotiation.

The two essential ingredients for his claim are:

1. Negligible transaction costs
2. Well defined property rights

Notes

Coase Theorem III

Limitations of the Coase Theorem are due:

1. Non-negligible transaction costs
   In fact the result fails:
   1. If a lawyer is needed and if he charges more than $\varepsilon(c' - b')$
   2. If information about costs and benefits is private [Myerson et al 1983]
2. Well defined property rights
   In fact the result fails when rights are not well defined:
   1. As with open water fishing
   2. As for pollution

But several examples have been reported in which such bargaining occurs

Cheung 1973 shows that in US arrangement with side payments between beekeepers and orchards are common.

Notes

Public Goods
EC202 Lectures XIX & XX

Francesco Nava

LSE

February 2013

Notes

Summary

Market Failures – Public Goods:

- The Free Rider Problem: $PO \neq CE$
  Private benefits are smaller than Social benefits
- Pareto Optima with public goods (BLS condition)
- Private contributions are not PO
  - Subscription Equilibrium
- Solutions to the Problem
  - Lindahl Equilibrium
  - Personalized Taxation
  - Planning Procedure
- The Importance of the Free Rider Problem
- Local Public Goods

Notes

What is a Public Good?

Definitions (Private Goods)

A good is **rival** if consumption by an agent reduces the possibilities of consumption by the other agents.

A good is subject to **exclusion** if you have to pay to consume the good.

A **private good** is both rival and subject to exclusion.

Definitions (Public Goods)

A **public good** is nonrival.

A **pure public good** is both nonrival and not subject to exclusion.

- A pure public good example is national defense
- A good that is subject to exclusion, but nonrival is patented research
- A good that is not subject to exclusion, but rival is free parking

Notes

The Free Rider Problem

Definition (Free Rider Problem)

Free riders are those who consume more than their fair share of a public resource, or bear less than a fair share of the costs of its production.

Free riding is considered to be a "**problem**" only when it leads to the under-production of a public good (and thus to Pareto inefficiency), or when it leads to the excessive use of a common property resource.

- This problem was first formalized by Wicksell in 1896
- Though discussions about the under-provision of public goods date back to Adam Smith's "Wealth of Nations" in 1776
- The 1^{st} solution to the problem was proposed by Lindahl in 1919

Notes

A Simple Economy with Pure Public Goods

Consider the following two goods economy:

- Let n denote the number of consumers (i denotes any one of them)
- Let x denote the private good (an aggregate of all private goods)
- Let z denote the pure public good
- Good z is produced using good x according to a production function:

$$z = f(x)$$

- The production function f can also be expressed as a cost function g:

$$x = g(z) = f^{-1}(z)$$

- Endowments of the two goods are $(e_x, e_z) = (X, 0)$
- Preferences of consumer i are given by $U^i(x_i, z_i)$

Notes

Pareto Optimality with Pure Public Goods I

- The feasibility constraint for the private good requires that:

$$\sum_{i=1}^n x_i \leq X - x$$

- Since good z is nonrival, feasibility only requires that:

$$z_i \leq z \quad \text{for any } i \in \{1, ..., n\}$$

- Let α_i denote the Pareto weight on the utility of player i
- The Pareto optima are found by solving the following problem:

$$\max_{x, x_1, ..., x_n, z, z_1, ..., z_n} \sum_{i=1}^n \alpha_i U^i(x_i, z_i) \quad \text{subject to}$$
$$z \leq f(x)$$
$$\sum_{i=1}^n x_i \leq X - x$$
$$z_i \leq z \quad \text{for any } i \in \{1, ..., n\}$$

Notes

Pareto Optimality with Pure Public Goods II

- Eliminating constraints that certainly bind, the problem becomes:

$$\max_{x_1,\ldots,x_n,z} \textstyle\sum_{i=1}^n \alpha_i U^i(x_i, z) \quad \text{subject to}$$
$$\textstyle\sum_{i=1}^n x_i \leq X - g(z) \qquad (\mu)$$

- First order conditions require that:

$$\alpha_i U_x^i(x_i, z) = \mu \qquad (x_i)$$
$$\textstyle\sum_{i=1}^n \alpha_i U_z^i(x_i, z) = \mu g'(z) \qquad (z)$$

- Solving the two equations yield the following PO condition:

$$\sum_{i=1}^n \frac{U_z^i(x_i, z)}{U_x^i(x_i, z)} = g'(z) = \frac{1}{f'(x)} \qquad \text{(BLS)}$$

  which is known as the Bowen-Lindhal-Samuelson condition
- Thus BLS implies that **Social MRS** equals **MRT**

Notes

Subscription Equilibrium I

Consider an economy in which individuals decide how much to contribute to the production of the public good:

- Denote by X_i the resources of player i
- Denote by s_i the resources that i devotes to the production of z
- The resource constraint for player i requires that:

$$x_i + s_i = X_i$$

- The amount of public good produced z hence satisfies:

$$z = f\left(\textstyle\sum_{i=1}^n s_i\right)$$

- The problem of consumer i is to choose his contribution s_i given s_{-i}:

$$\max_{x_i, s_i} U^i(x_i, z) \quad \text{subject to}$$
$$z = f\left(\textstyle\sum_{i=1}^n s_i\right)$$
$$x_i + s_i = X_i$$

Notes

Subscription Equilibrium II

- Since the constraints bind, the problem of consumer i simplifies to:

$$\max_{s_i} U^i\left(X_i - s_i, f\left(\sum_{i=1}^n s_i\right)\right)$$

- First order conditions thus require that:

$$\frac{U_z^i(x_i, z)}{U_x^i(x_i, z)} = \frac{1}{f'\left(\sum_{i=1}^n s_i\right)}$$

- Thus FOC implies that **Private MRS** equals **MRT**
- FOC differs from BLS because individuals only care about their private benefits from investment
- Under general conditions subscription equilibrium implies that too little public good is produced (ie there is free rider problem)

Notes

Lindahl Equilibrium I

In 1919 Lindahl (a Swedish economist) proposed the following solution to the aforementioned free rider problem:

- Assume that personal prices can be established
- Let p_i be the price paid by consumer i
- The producer of the public good thus receives a price:

$$p = \sum_{i=1}^n p_i$$

and chooses his output given the price to maximize profits:

$$\max_z pz - g(z)$$

- Thus he produces until marginal costs are equal to the price p:

$$p = g'(z) \tag{O1}$$

Notes

Lindahl Equilibrium II

- Consumer i chooses x_i & z_i to maximize utility given his price p_i:

$$\max_{x_i, z_i} U^i(x_i, z_i) \quad \text{subject to}$$
$$x_i + p_i z_i = X_i$$

- FOC therefore require that he equalizes his MRS to the price p_i:

$$\frac{U_z^i(x_i, z_i)}{U_x^i(x_i, z_i)} = p_i \tag{O2}$$

- Market clearing in the public good sector requires that the individual demand of any consumer be equal to the supply:

$$z_i(\mathbf{p}) = z(\mathbf{p}) \quad \text{for any } i \in \{1, ..., n\} \tag{MC}$$

Notes

Lindahl Equilibrium III

Conditions O1 and O2 added up for any i together with MC imply that:

$$\sum_{i=1}^{n} \frac{U_z^i(x_i, z_i)}{U_x^i(x_i, z_i)} = \sum_{i=1}^{n} p_i = p = g'(z)$$

which is the BLS condition for Pareto optimality.

The **advantage** of such implementation is that it restores PO.

The **disadvantage** is that it requires the existence of n "micromarkets" in which a sole consumer buys the public good at his personalized price.

Thus it is hard to implement such solution if consumers valuations for the public good are private information, since everyone has an incentive to underestimate his demand in order to pay a lower price,

Notes

Personalized Taxation

If consumers can be taxed for their consumption of the public good:

- The budget constraint of any consumer i becomes:

$$x_i + t_i(z_i) = X_i$$

- Consumers equalize MRS to the marginal cost of the public good:

$$\frac{U_z^i(x_i, z_i)}{U_x^i(x_i, z_i)} = t_i'(z_i)$$

- Setting tax rates equal to the Lindahl prices yields PO:

$$t_i(z_i) = p_i z_i$$

- This approach has the same advantages and disadvantages than the Lindahl equilibrium (it works if the government has detailed information about the preferences in the population)

Notes

Planning Procedures & Pivot Mechanisms

More recently implementation mechanisms have been devised that do not require the regulator to be informed about the tastes of consumers and that still restore Pareto optimality:

1. Planning Procedures (Malinnvaud, Dreze & Poussin 1971):
   - a dynamic revelation mechanism that converges to PO
   - planning office is uninformed about preferences in the economy
2. Pivot Mechanism (Vickrey, Clarke, Groves 1971)
   - a static revelation mechanism that implements any PO
   - the designer is uninformed about preferences in the economy
   - it implements PO as a dominant strategy equilibrium

Notes

Property of Public Goods

It is often asserted that because of the free rider problem public goods should be provided by the public sector (eg: police, defense, justice system)

Such claim was first made by Adam Smith in the "Wealth of Nations"

Several classical authors (Mill & Samuelson) illustrated the principle through the lighthouse example (a pure public good)

In 1974 Coase contested such argument pointing out that:

- the free rider problem was simply a problem of positive externalities
- thus it could be solved by private bargaining (Coase Theorem)

Coase argued: that British lighthouses were traditionally a responsibility of a private national company that perceived a fixed right which was discharged by any ship landing in a British port; and that this was not detrimental to British naval commerce since shipowners were more conscious of paying for this service and had more incentives to monitor that the service was rendered

Notes

The Importance of the Free Rider Problem

In large economies without public goods individuals have no incentive to game prices by altering their demand (Roberts-Postlewaite 1976)

With public goods, however, individuals have incentives to underestimate their demand since it affects their contribution significantly, but it affects the provision of the public good only marginally

Several empirical studies however have pointed out that the importance of the free rider problem has been exaggerated by theory since individuals have a tendency towards honesty (Bohm 1972, Ledyard 1995 – survey)

The conclusions of these studies are that individuals:

- game prices, but less than in the NE subscription game
- contribute less if the game is repeated
- contribute more if allowed to communicate

Notes

Local Public Goods

Definition (Local Public Good)

Local public goods are public goods that apply only to the inhabitants of a particular geographical area (garbage collection, public transport, parks)

Tiebout was the first to study their theory in 1956

The main feature of these markets is that individuals are free to decide in which community to live

Thus individuals will move away from communities with too few or poorly financed public goods

Tiebout showed that this process had an efficient equilibrium if there are perfect mobility and perfect information

Notes

Notes

Choice Under Uncertainty

Review

Francesco Nava

LSE

January 2013

Notes

Summary

Choice Under Uncertainty:

- Statistics Review
- Definitions:
  - Lottery & Fair Lottery
  - Expected Utility
  - Risk Attitudes (Aversion, Neutrality, Loving)
  - Certainty Equivalent
- Measures of Risk Aversion:
  - Relative Risk Aversion
  - Absolute Risk Aversion
- Insurance, a first take:
  - Actuarially Fair Insurance
  - Under-insurance at Unfair Prices

Notes

Statistics Review

A **random variable** X is a variable that records the possible outcomes x of a random event

Any random variable X is characterized by:

- the set of possible outcomes that can occur ($\mathbb{X}$) and by
- a probability distribution over the possible outcomes ($f : \mathbb{X} \to [0,1]$)

Given a numerical random variable $\{\mathbb{X}, f\}$ (that is $\mathbb{X} \subseteq \mathbb{R}$):

- The probability of $X = x$ is denoted by $f(x)$
- The **expected value** of the RV X is denoted and defined by:

$$E(X) = \sum_{x \in \mathbb{X}} x f(x)$$

- The **variance** of the RV X is denoted and defined by:

$$V(X) = \sum_{x \in \mathbb{X}} (x - E(x))^2 f(x)$$

Notes

Lotteries and Fair Lotteries

A **lottery** X is a random variable over monetary outcomes

- Any lottery is characterized by an outcome set and a probability distribution over monetary outcomes $\{\mathbb{X}, f\}$
- In general monetary outcomes can also be negative

A lottery X is said to be **fair** if $E(X) = 0$

Examples:

- $\mathbb{X} = \{2, -1\}$, $f(2) = 1/3$, $f(-1) = 2/3$ is fair since:

$$E(X) = 2 * (1/3) - 1 * (2/3) = 0$$

- $\mathbb{X} = \{2, -1\}$, $f(2) = 1/2$, $f(-1) = 1/2$ is unfair since:

$$E(X) = 2 * (1/2) - 1 * (1/2) = 1/2$$

- Such a lottery would be fair if an entry fee of $1/2$ were charged

Notes

Expected Utility & Risk Preferences

Consider a decision maker that has preferences over monetary outcomes defined by a strictly increasing utility function $u : \mathbb{X} \to \mathbb{R}$

The **expected utility** of a lottery X is defined by:

$$E(u(X)) = \sum_{x \in \mathbb{X}} u(x) f(x)$$

The expected utility may differ from the utility of the expected value!!!

Preferences over lotteries:

- An individual is **risk averse** if $u'' < 0$
- An individual is **risk neutral** if $u'' = 0$
- An individual is **risk loving** if $u'' > 0$

Notes

Risk Preferences

Preferences over lotteries:

- A risk averse individual prefers $E(X)$ to the lottery X:
$$E(u(X)) < u(E(X))$$
- A risk neutral individual is indifferent between a lottery X and $E(X)$:
$$E(u(X)) = u(E(X))$$
- A risk loving individual prefers a lottery X to $E(X)$:
$$E(u(X)) > u(E(X))$$

Example, consider $\mathbb{X} = \{1, 9\}$ and $f(1) = f(9) = 1/2$:

- If preferences are concave, say $u(x) = x^{1/2}$, we get that:
$$E(u(X)) = 2 < \sqrt{5} = u(E(X))$$
- If preference are convex, say $u(x) = x^2$, we get that:
$$E(u(X)) = 41 > 25 = u(E(X))$$

Notes

Certainty Equivalent

The **certainty equivalent** x_{CE} of a lottery X is defined by:

$$u(x_{CE}) = E(u(X)) \quad \Leftrightarrow \quad x_{CE} = u^{-1}(E(u(X)))$$

The certainty equivalent is the amount of money x_{CE} that leaves the individual indifferent between the lottery X and the certain outcome x_{CE}

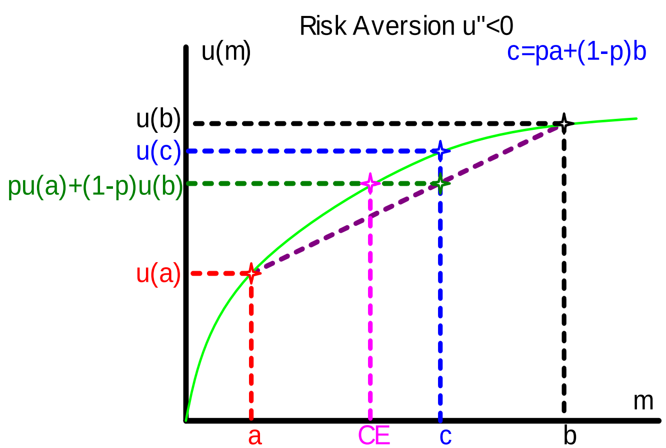
For any given lottery X we have that if:

- an individual is risk averse then $x_{CE} < E(X)$
- an individual is risk neutral then $x_{CE} = E(X)$
- an individual is risk loving then $x_{CE} > E(X)$

Notes

Risk Aversion

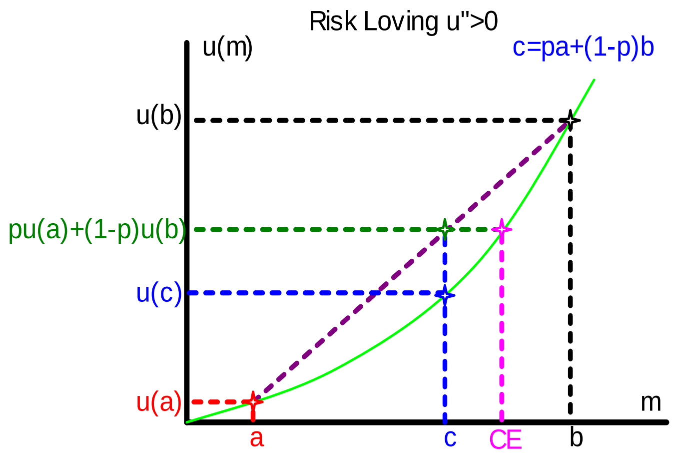
Consider a lottery $\mathbb{X} = \{a, b\}$, $f(a) = p$, $f(b) = 1 - p$ and a risk averse individual:



Notes

Risk Loving

Consider a lottery $\mathbb{X} = \{a, b\}$, $f(a) = p$, $f(b) = 1 - p$ and a risk loving individual:



Notes

Relative Risk Aversion (Pratt)

The **coefficient of relative risk aversion** is defined by:

$$R(x) = -x\frac{u''(x)}{u'(x)}$$

It is a measure of risk aversion of individuals

Preferences u display constant relative risk aversion CRRA if $R' = 0$

Any CRRA preference takes the form:

$$u(x) = \alpha x^{\gamma} + \beta \text{ for } \alpha > 0, \gamma \in (0, 1) \ \& \ \forall \beta$$

If preferences are CRRA then for any $k > 0$:

$$E(u(X)) = u(x_{CE}) \quad \Leftrightarrow \quad E(u(kX)) = u(kx_{CE})$$

Risk aversion does not change with proportional changes in the stakes

Notes

Absolute Risk Aversion

The **coefficient of absolute risk aversion** is defined by:

$$A(x) = -\frac{u''(x)}{u'(x)}$$

It is a measure of risk aversion of individuals

Preferences u display constant relative risk aversion CARA if $A' = 0$

Any CARA preference takes the form:

$$u(x) = -\alpha e^{-\gamma x} + \beta \text{ for } \alpha > 0, \gamma > 0 \ \& \ \forall \beta$$

If preferences are CARA then for any $k > 0$:

$$E(u(X)) = u(x_{CE}) \quad \Leftrightarrow \quad E(u(k+X)) = u(k + x_{CE})$$

Risk aversion does not change with additive changes in the stakes

Notes

A Simple Insurance Model I

Consider the following decision problem faced by a risk averse individual:

- There are two possible states of the world $\{H, S\}$
- The individual can be healthy H or sick S
- The probability of being sick is p
- The income of an individual is:
  - Y if healthy
  - $Y - L$ if sick
- Let y denote the consumption if healthy and x if sick
- Preference satisfy $u'' \in (-\infty, 0)$ and:

$$pu(x) + (1-p)u(y)$$

Notes

A Simple Insurance Model II

- Consumers can buy insurance coverage $z \in [0, L]$
- The unit price of insurance is q
- Therefore the total premium is qz
- If they do so, their consumption in the two states becomes:

$$\begin{aligned} y &= Y - qz \\ x &= Y - L - qz + z = Y - L + (1-q)z \end{aligned}$$

- If so the problem of a consumer becomes:

$$\max_z pu(x) + (1-p)u(y)$$

- FOC with respect to z requires:

$$p(1-q)u'(x) = (1-p)qu'(y)$$

Notes

A Simple Insurance Model III

- FOC can be written in terms of MRS as:

$$\frac{u'(x)}{u'(y)} = \frac{1-p}{p}\frac{q}{1-q}$$

- Thus a consumer of type t wants:

$$\begin{array}{rlll} \text{Full Insurance:} & z = L & \text{if} & q = p \\ \text{Under Insurance:} & z < L & \text{if} & q > p \\ \text{Over Insurance:} & z > L & \text{if} & q < p \end{array}$$

- This is the case because $u'' < 0$ implies:

$$q \left\{ \begin{array}{c} = \\ > \\ < \end{array} \right\} p \;\Leftrightarrow\; \frac{u'(x)}{u'(y)} \left\{ \begin{array}{c} = \\ > \\ < \end{array} \right\} 1 \;\Leftrightarrow\; x \left\{ \begin{array}{c} = \\ < \\ > \end{array} \right\} y$$

Notes

A Simple Insurance Model IV

- Suppose that an insurance company is selling the contract
- Its profits on the contract sold are given by:

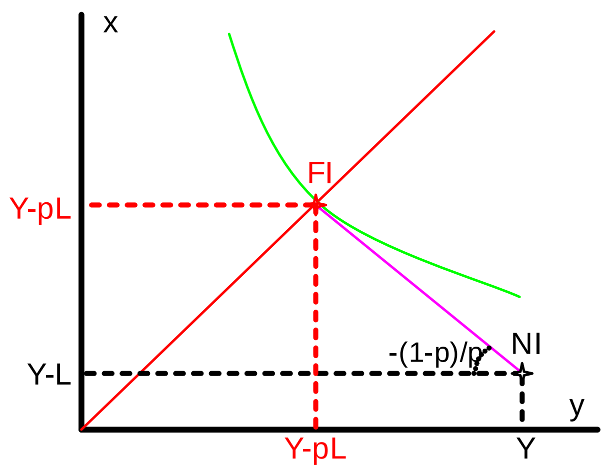
$$(1-p)qz - p(1-q)z = (q-p)z$$

- The company' profits are:
  - positive if $q > p$
  - negative if $q < p$
  - zero if $q = p$
- The insurance price is **actuarially fair** if $q = p$

Notes

Full Insurance at Fair Prices

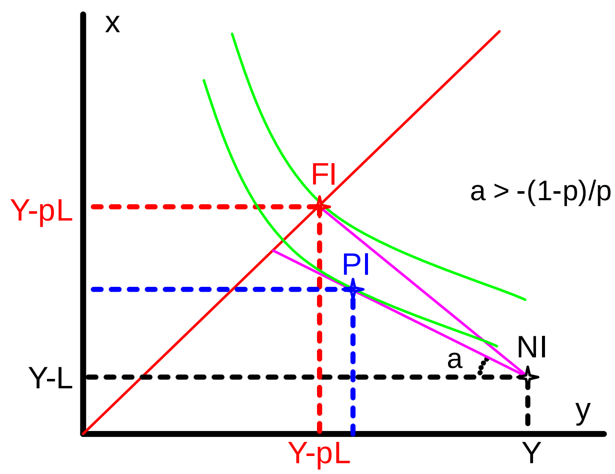
Graphically full insurance at actuarially fair prices occurs since:



Notes

Under-Insurance at Unfair Prices

Graphically under-insurance occurs if $q > p$ since:



Notes

Notes

Moral Hazard EZ
Lectures XV & XVII (Easy)

Francesco Nava

LSE

January 2013

Notes

Summary

Hidden Action Problem aka:

1. Moral Hazard Problem
2. Principal Agent Model

- Outline
- Simplified Model:
  - Complete Information Benchmark
  - Hidden Effort
  - Agency Cost
- General Principal Agent Model
  - Complete Information Benchmark
  - Hidden Effort
  - Agency Cost

Notes

Outline: Moral Hazard Problem

The basic ingredients of a moral hazard model are as follows:

- A **principal** and an **agent**, are involved in bilateral relationship
- Principal wants Agent to perform some task
- Agent can choose how much effort to devote to the task
- The outcome of the task is pinned down by a mix of effort and luck
- Principal cannot observe effort and can only motivate Agent by paying him based on the outcome of the task

Timing:

1. Principal chooses a wage schedule which depends on outcome
2. Agent chooses how much effort to devote the task
3. Agent's effort and chance determine the outcome
4. Payments are made according to the proposed wage schedule

Notes

A simple Principal-Agent Model

Consider the following simplified model:

- A task has two possible monetary outcomes: $\{\underline{q}, \overline{q}\}$ with $\underline{q} < \overline{q}$
- Agent can choose one of two effort levels: $\{e_1, e_2\}$ with $e_1 < e_2$
- The probability of the high output given effort e_i is:

$$p_i = \Pr(q = \overline{q} | e_i)$$

- Assume that $p_1 < p_2$ – ie more effort $\Rightarrow$ better outcomes
- Principal chooses a wage schedule w
- Agent is risk averse and his preferences are:

$$U(w, e) = E[u(w, e)]$$

- Principal is risk neutral and his preferences are:

$$V(w) = E[q - w]$$

Notes

Simple Principal-Agent Model: Complete Info I

Begin by looking at the complete information benchmark:

- Principal can observe the effort chosen by Agent
- Principal picks a wage schedule w_i that depend on Agent's effort
- Agent's reservation utility is $\underline{u}$ – ie what he gets if he resigns
- Thus the participation constraint of Agent is:

$$U(w_i, e_i) = u(w_i, e_i) \geq \underline{u}$$

- By picking wages appropriately Principal *de facto* chooses e_i and w_i
- The problem of Principal thus is to:

$$\max_{e_i, w_i} E[q|e_i] - w_i + \lambda[u(w_i, e_i) - \underline{u}]$$

- Recall that $E[q|e_i] = p_i\overline{q} + (1 - p_i)\underline{q}$

Notes

Simple Principal-Agent Model: Complete Info II

- Recall the problem of Principal:

$$\max_{e_i, w_i} E[q|e_i] - w_i + \lambda[u(w_i, e_i) - \underline{u}]$$

- The lowest wage v_i that induces effort e_i from Agent is:

$$u(v_i, e_i) = \underline{u}$$

- Thus Principal chooses to induce effort e_* if and only if:

$$e_* \in \arg\max_{e_i \in \{e_1, e_2\}} E[q|e_i] - v_i$$

- Principal then induces such effort choice by offering wages:

$$w_{i*} = \begin{cases} v_i & \text{if} \quad e_i = e_* \\ v_i - \varepsilon & \text{if} \quad e_i \neq e_* \end{cases}$$

- Complete info implies that FOC for the wage requires $MC = Price$:

$$1/u_w(w_i, e_i) = \lambda$$

Notes

Simple Principal-Agent Model: Incomplete Info I

Now consider the case in which effort is unobservable for Principal:

- Suppose that Principal prefers Agent to exert high effort e_2
- Principal can only condition wage $w(q)$ on outcome q:

$$w(q) = \begin{cases} \underline{w} & \text{if} \quad q = \underline{q} \\ \overline{w} & \text{if} \quad q = \overline{q} \end{cases}$$

- Agent's **participation constraint** at e_i requires:

$$U(w(q), e_i) = p_i u(\overline{w}, e_i) + (1 - p_i) u(\underline{w}, e_i) \geq \underline{u} \qquad \text{(PC(e))}$$

- Agent's **incentive constraint** guarantees that he picks high effort:

$$U(w(q), e_2) \geq U(w(q), e_1) \qquad \text{(IC)}$$

Notes

Simple Principal-Agent Model: Incomplete Info II

The problem of a principal who wants the agent to exert e_2 is to:

- Maximize his profits by choosing $w(q)$ subject to:
  1. Agent's participation constraint at e_2
     ie the agent prefers to exert high effort than to resign
  2. Agent's incentive constraint
     ie the agent prefers to exert high effort than low effort
- Thus the Lagrangian of this problem is:

$$\max_{\underline{w}, \overline{w}} E[q - w(q)|e_2] + \lambda[U(w(q), e_2) - \underline{u}] +$$
$$+ \mu[U(w(q), e_2) - U(w(q), e_1)]$$

- Recall that:

$$U(w(q), e_i) = p_i u(\overline{w}, e_i) + (1 - p_i) u(\underline{w}, e_i)$$
$$E[q - w(q)|e_i] = p_i[\overline{q} - \overline{w}] + (1 - p_i)[\underline{q} - \underline{w}]$$

Notes

Simple Principal-Agent Model: Incomplete Info III

Writing out Lagrangian explicitly the Principal's problem becomes:

$$\begin{aligned}
&\max_{\underline{w},\overline{w}} p_2[\overline{q} - \overline{w}] + (1-p_2)[\underline{q} - \underline{w}] + \\
&\quad + \lambda[p_2 u(\overline{w}, e_2) + (1-p_2)u(\underline{w}, e_2) - \underline{u}] + \\
&\quad + \mu[p_2 u(\overline{w}, e_2) + (1-p_2)u(\underline{w}, e_2) - p_1 u(\overline{w}, e_1) - (1-p_1)u(\underline{w}, e_1)]
\end{aligned}$$

- First order conditions for this problem are:

$$\begin{aligned}
p_2[-1 + \lambda u_w(\overline{w}, e_2) + \mu u_w(\overline{w}, e_2)] - p_1 \mu u_w(\overline{w}, e_1) &= 0 \\
(1-p_2)[-1 + \lambda u_w(\underline{w}, e_2) + \mu u_w(\underline{w}, e_2)] - (1-p_1)\mu u_w(\underline{w}, e_1) &= 0
\end{aligned}$$

- By rearranging it is possible to show that:
  1. Both μ and λ are positive if u is increasing and concave
  2. Incentive Constraint binds since $\mu > 0$
  3. Participation constraint at high effort binds since $\lambda > 0$
  4. Wages $\overline{w}, \underline{w}$ are found by solving the two constraints IC & PC

Notes

Simple Principal-Agent Model: Incomplete Info IV

For an explicit characterization let u be additively separable in w and e:

$$u(w, e) = v(w) + \eta(e)$$

First order conditions in this scenario become:

$$\begin{aligned}
p_2[-1 + \lambda v_w(\overline{w}) + \mu v_w(\overline{w})] - p_1 \mu v_w(\overline{w}) &= 0 \\
(1-p_2)[-1 + \lambda v_w(\underline{w}) + \mu v_w(\underline{w})] - (1-p_1)\mu v_w(\underline{w}) &= 0
\end{aligned}$$

Solving we find that $\lambda, \mu > 0$ (condition (L) parallels the complete info):

$$\lambda = \frac{p_2}{v_w(\overline{w})} + \frac{1-p_2}{v_w(\underline{w})} > 0 \qquad \text{(L)}$$

$$\mu\left[1 - \frac{p_1}{p_2}\right] = (1-p_2)\left[\frac{1}{v_w(\overline{w})} - \frac{1}{v_w(\underline{w})}\right] > 0 \qquad \text{(M)}$$

$$\begin{aligned}
v(\underline{w}) &= \underline{u} + \eta(e_2)\frac{p_1}{p_2 - p_1} - \eta(e_1)\frac{p_2}{p_2 - p_1} \\
v(\overline{w}) &= \underline{u} - \eta(e_2)\frac{1-p_1}{p_2 - p_1} + \eta(e_1)\frac{1-p_2}{p_2 - p_1}
\end{aligned}$$

Notes

Simple Principal-Agent Model: Example I

Example: $e \in \{0,1\}$, $u(w,e) = 2w^{1/2} - e$, $\underline{u} = 1$,
$\overline{q} = 4$, $\underline{q} = 0$, $p_1 = 3/4$, $p_0 = 1/4$

Complete Info: what are w_1, w_0, e^*?

- Wages w_1 and w_0 are found by PC(e):

$$2w_1^{1/2} - 1 = 1 \quad \Rightarrow \quad w_1 = 1$$
$$2w_0^{1/2} - 0 = 1 \quad \Rightarrow \quad w_0 = 1/4$$

- Optimal effort $e_* = 1$ is found by comparing profits:

$$\frac{3}{4}\overline{q} + \frac{1}{4}\underline{q} - w_1 > \frac{1}{4}\overline{q} + \frac{3}{4}\underline{q} - w_0$$
$$\frac{3}{4}4 - 1 = 2 > \frac{3}{4} = \frac{1}{4}4 - \frac{1}{4}$$

- Thus the agent is fully insured by the principal

Notes

Simple Principal-Agent Model: Example II

Incomplete Info: what are $\overline{w}$, $\underline{w}$, if principal wants $e_* = 1$?

- Wages $\overline{w} = 25/16$ and $\underline{w} = 1/16$ are found by solving PC(1) and IC:

$$\frac{3}{4}(2\overline{w}^{1/2} - 1) + \frac{1}{4}(2\underline{w}^{1/2} - 1) = 1$$
$$\frac{3}{4}(2\overline{w}^{1/2} - 1) + \frac{1}{4}(2\underline{w}^{1/2} - 1) = \frac{1}{4}(2\overline{w}^{1/2}) + \frac{3}{4}(2\underline{w}^{1/2})$$

- If principal wants $e_* = 0$, a wage $w_* = 1/4$ satisfying PC(0) suffices:

$$2w_*^{1/2} - 0 = 1$$

- The principal, however, prefers $e_* = 1$ since:

$$\frac{3}{4}(\overline{q} - \overline{w}) + \frac{1}{4}(\underline{q} - \underline{w}) > \frac{1}{4}\overline{q} + \frac{3}{4}\underline{q} - w_*$$
$$\frac{3}{4}(4 - \frac{25}{16}) + \frac{1}{4}(-\frac{1}{16}) = \frac{29}{16} > \frac{3}{4} = \frac{1}{4}4 - \frac{1}{4}$$

- The principal cannot fully insure the agent with incomplete information since it would undermine the incentives to exert effort

Notes

Principal-Agent Model

Consider a somewhat more general setup in which:

- Agent chooses any effort level $e \in \{e_1, ..., e_n\}$
- Agent's reservation utility is still $\underline{u}$
- Output q can take one of m values $\{q_1, ..., q_m\}$
- The probability of output takes value q_j given effort e_i is $p_{ij} > 0$
- Principal is risk neutral and his preferences are:

$$V(w) = E[q - w]$$

- Agent is risk averse and his preferences are:

$$U(w, e) = E[u(w) - e]$$

- Principal moves first and takes Agent's response as given

Notes

Principal-Agent Model: Complete Info I

Let's begin by analyzing the complete info model:

- Principal can observe Agent's effort e and output q
- Agent's participation constraint remains:

$$U(w, e) \geq \underline{u}$$

- Principal can choose wages w_{ij} that depend on both e_i and q_j...

  ...this is equivalent to Principal picking both e_i and w_j...

  ... since Principal could choose a wage schedule such that:

$$w_{ij} = \begin{cases} w_j & \text{if} \quad e_i \text{ is optimal for Principal} \\ w \text{ st } U(w, e_i) < \underline{u} & \text{if} \quad e_i \text{ is not optimal for Principal} \end{cases}$$

Notes

Principal-Agent Model: Complete Info II

Thus the problem of Principal becomes:

$$\max_{i,\mathbf{w}} \sum_{j=1}^{n} p_{ij}[q_j - w_{ij}] + \lambda \sum_{j=1}^{n} p_{ij}[u(w_{ij}) - e_i - \underline{u}]$$

First order conditions with respect to wages require that:

$$\frac{1}{u'(w_{ij})} = \lambda \quad \Rightarrow \quad w_{ij} \text{ is independent of } i \text{ and } j$$

Principal insures the risk averse Agent

Since PC binds at the optimal effort let $\overline{w}_i = u^{-1}(e_i + \underline{u})$

Principal chooses the effort level e_* to maximize profits:

$$\max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} p_{ij}q_j - \overline{w}_i = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} p_{ij}q_j - u^{-1}(e_i + \underline{u})$$

In fact, effort e_* can be sustained by the wage schedule:

$$w_{ij} = \begin{cases} u^{-1}(e_* + \underline{u}) & \text{if } e_i = e_* \\ w < u^{-1}(e_i + \underline{u}) & \text{if } e_i \neq e_* \end{cases}$$

This is efficient since Agent is fully insured against risk

Notes

Principal-Agent Model: Incomplete Info I

Principal observes output q, but not effort e:

- Wages can only depend on output w_j
- Agent chooses his effort in private

Given a wage schedule w_j Agent's problem becomes:

$$\max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} p_{ij}[u(w_j) - e_i]$$

Thus, if Agent chooses effort e_i, then $(n-1)$ incentive constraints:

$$\sum_{j=1}^{n} p_{ij}[u(w_j) - e_i] \geq \sum_{j=1}^{n} p_{kj}[u(w_j) - e_k] \qquad \text{(IC(k))}$$

must bind for any $k \neq i$.

Moreover, Agent's PC must still bind at the optimal effort

Notes

Principal-Agent Model: Incomplete Info II

- Principal chooses the wages w_j to maximize his wellbeing subject to:
  1. Agent's participation constraint (PC)
  2. Agent's incentive constraints (IC)
- In particular the problem of Principal is:

$$\max_{\mathbf{w},i} \sum_{j=1}^{n} p_{ij}[q_j - w_j] + \lambda \sum_{j=1}^{n} p_{ij}[u(w_j) - e_i - \underline{u}] + \\ + \sum_{k \neq i} \mu_k \left[\sum_{j=1}^{n} [p_{ij}u(w_j) - p_{kj}u(w_j)] - e_i + e_k\right]$$

- First order conditions with respect to wage w_j requires:

$$\frac{1}{u'(w_j)} = \lambda + \sum_{k \neq i} \mu_k \left[1 - \frac{p_{kj}}{p_{ij}}\right]$$

- The wages are no longer constant $\Rightarrow$ Agent is not fully insured
- Principal pays more when output reveals that an action more favorable to him is likely to have been chosen by Agent

Notes

Principal-Agent Model: Incomplete Info III

Main Conclusions with Moral Hazard:

- Compared to compete info Principal:
  1. pays Agent more when output is high
  2. pays Agent less when output is bad
  3. no longer provides full insurance to Agent on the variable output
- He does so to provide incentives for Agent to exert effort...

  ... since a fully insured Agent would have no motives to exert effort
- This conclusions rely on the information problem of Principal and

  ... would hold even if Principal were risk averse
- They always hold so long as Agent is risk averse

Notes

Principal-Agent Model: Example I

Example: $e \in \{0, 1/2, 1\}$, $u(w,e) = 2w^{1/2} - e$, $\underline{u} = 1$,
$\overline{q} = 4$, $\underline{q} = 0$, $\overline{p}_1 = 3/4$, $\overline{p}_{1/2} = 1/2$, $\overline{p}_0 = 1/4$

Complete Info: what are w_0, $w_{1/2}$, w_1, e^*?

- Wages w_0, $w_{1/2}$ and w_1 are found by PC(e):

$$\begin{aligned} 2w_1^{1/2} - 1 = 1 \quad &\Rightarrow \quad w_1 = 1 \\ 2w_{1/2}^{1/2} - 1/2 = 1 \quad &\Rightarrow \quad w_{1/2} = 9/16 \\ 2w_0^{1/2} - 0 = 1 \quad &\Rightarrow \quad w_0 = 1/4 \end{aligned}$$

- Optimal effort $e^* = 1$ is found by comparing profits:

$$\frac{3}{4}\overline{q} + \frac{1}{4}\underline{q} - w_1 > \frac{1}{2}\overline{q} + \frac{1}{2}\underline{q} - w_{1/2} > \frac{1}{4}\overline{q} + \frac{3}{4}\underline{q} - w_0$$
$$\frac{3}{4}4 - 1 = 2 > \frac{1}{2}4 - \frac{9}{16} = \frac{23}{16} > \frac{1}{4}4 - \frac{1}{4} = \frac{3}{4}$$

- Thus the agent is fully insured by the principal at w_1

Notes

Principal-Agent Model: Example II

Incomplete Info: what are $\overline{w}$, $\underline{w}$, if P wants $e^* = 1$?

- Wages $\overline{w} = 25/16$ and $\underline{w} = 1/16$ are found by solving PC(1) and IC:

$$\begin{aligned} &\frac{3}{4}(2\overline{w}^{1/2}) + \frac{1}{4}(2\underline{w}^{1/2}) - 1 = 1 \\ &\frac{3}{4}(2\overline{w}^{1/2}) + \frac{1}{4}(2\underline{w}^{1/2}) - 1 = \frac{1}{2}(2\overline{w}^{1/2}) + \frac{1}{2}(2\underline{w}^{1/2}) - \frac{1}{2} \\ &\qquad \geq \frac{1}{4}(2\overline{w}^{1/2}) + \frac{3}{4}(2\underline{w}^{1/2}) \end{aligned}$$

- If P wants $e^* = 1/2$, same constraints bind in this example, thus:

$$\overline{w} = 25/16 \;\;\&\;\; \underline{w} = 1/16$$

- If P wants $e^* = 0$, a wage $w = 1/4$ satisfying PC(0) suffices:

$$2w^{1/2} - 0 = 1$$

Notes

Principal-Agent Model: Example III

- The principal in this example, however, prefers $e^* = 1$ since:

$$\frac{3}{4}(\overline{q} - \overline{w}) + \frac{1}{4}(\underline{q} - \underline{w}) > \frac{1}{2}(\overline{q} - \overline{w}) + \frac{1}{2}(\underline{q} - \underline{w}) > \frac{1}{4}\overline{q} + \frac{3}{4}\underline{q} - w$$

$$\frac{3}{4}(4 - \frac{25}{16}) + \frac{1}{4}(-\frac{1}{16}) > \frac{1}{2}(4 - \frac{25}{16}) + \frac{1}{2}(-\frac{1}{16}) > \frac{1}{4}4 - \frac{1}{4}$$

- In general the wages that support $e^* = 1/2$ and $e^* = 1$ do not need to be the same!!!
- The binding constraints could change leading to changes in wages
- In this example this does not happen since all the IC bind at once

Notes

Notes