EC319: Auction Theory
MICHAELMAS TERM

The first part of the course consists of an introduction to the game theoretical analysis of auctions. It presents standard auction formats and strategic behavior in such environments. Auctions will be analyzed both in private and interdependent value environments. Fundamental topics such as the revenue equivalence theorem, the optimal auction design problem and the linkage principle will be discussed extensively. Throughout the course we will consider departures from the standard model allowing for heterogeneity amongst players, budget constraints and resale. Time permitting a model of bidding rings will be presented. The focus of the course is mainly theoretical, but when possible empirical and experimental evidence supporting the formal models will be discussed with references to relevant work in the field.

Lecturer: Francesco Nava, f.nava@lse.ac.uk

Office Hours: Wed 1.30pm-3.00pm, Building 32L, Office 3.20

Course Website: moodle.lse.ac.uk

Time and Location: Thu 9.30-11.00am, Building 32L, Room LG.18

Class Work: The work for each week consists of one or more exercises taken from the textbook and some additional problems. Answers to exercises will be posted on the website throughout the term. You are expected to make a reasonable attempt at the classwork in advance of the class. *Two written assignments are required before class in weeks 6 and 11 of MT*. These assignments will be of similar scope and difficulty as exam questions. You are expected to be diligent about meeting the deadlines for submission.

Exams: The final exam will determine your grade on the course entirely. The two parts of the course will be weighted equally. For each part of the course, you will have to answer two short questions and one long problem. The exam will focus entirely on topics covered in lectures and classes.

Weekly Course Program

1. Review: Order Statistics & Game Theory [KA, KC, KF]
2. Independent Private Value Auctions [K1, K2]
3. Revenue Comparisons and Reserve Prices [K2]
4. Revenue Equivalence Principle [K3]
5. Extensions: Risk Aversion [K4]
6. Extensions: Budget Constraints [K4]
7. Mechanism Design [K5, 3]
8. Optimal Mechanisms [K5, 5]
9. Efficient and Balanced Budget Mechanisms [K5, 6]
10. Interdependent Values [K6-7, 4]

Readings

Most materials covered can be found in the main textbook. The slides used in class will be posted. Some of the papers below may be suggested to deepen some topics.

Main Textbook: [K] Auction Theory, Vijay Krishna, Academic Press 2010

Related Papers [Not Required]**:**

[0] Athey & Levine, "Comparing Open and Sealed Bid Auctions: Theory and Evidence", Mimeo 2004

[1]Ausubel, "A Generalized Vickery Auction", Econometric Society Conference, 2000

[2] Dasgupta & Maskin, "Efficient Auctions", Quarterly Journal of Economics, 2000

[3] Krishna & Perry, "Efficient Mechanism Design", Mimeo Penn State University, 1998

[4] Milgrom & Weber, "A Theory of Auctions and Competitive Bidding", Econometrica 50, 1982

[5] Myerson, "Optimal Auction Design", Journal of Operation Research 6, 1982

[6] Myerson & Satterwaite, "Efficient Mechanisms for Bilateral Trading", JET 28, 1983

[7] Reny & Perry, "An Ex-Post Efficient Auction", Econometrica 70, 2002

[8] Vickery, "Counterspeculation, Auctions and Competitive Sealed Traders", Journal of Finance 7, 1961

[9] Wilson, "A Bidding Model of Perfect Competition", Review of Economic Studies 44, 1977

[10] Zheng, "High Bids and Broken Winners", Journal of Economic Theory 100, 2001

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Weekly Course Assignments

Several weekly exercises are taken from Krishna's textbook. Problems assigned are discussed during classes and lag the topics covered in the lectures by approximately a week. Answers to exercises will be posted on the website throughout the term. You are expected to make a reasonable attempt at the classwork in advance of the class. *Two problem sets have to be handed in as written assignments at the end of weeks 5 and 10 of Michealmas Term.* These assignments will be of similar scope and difficulty as exam questions. You are expected to be meet the deadlines for submission.

[2MT] Order Statistics

1. Consider n independent draws from the uniform probability distribution function $f(x) = 1/k$ on $[0, k]$.
   (a) Compute the cumulative distribution function and probability distribution function of the first and second order statistic.
   (b) Compute the probability distribution function of the second order statistic conditional on the first order statistic being equal to y.
   (c) Compute the expected value of the first and the second order statistic.
   (d) Compute the expected value of the second order statistic conditional on the first order statistic being equal to y.
2. Consider 3 independent draws from the exponential probability distribution function $f(x) = e^{-x}$ on $[0, \infty)$.
   (a) Compute the cumulative distribution function and probability distribution function of the first and second order statistic.
   (b) Compute the probability distribution function of the second order statistic conditional on the first order statistic being equal to y.

[3MT] IPV Auctions

1. K2.1
2. Consider a first price auction with 3 buyers with values distributed according to $f(x) = e^{-x}$ on $[0, \infty)$. Find the unique symmetric equilibrium.
3. [Extra] Consider a Dutch auction in which the auctioneer: starts the auction at a price so high no buyer is willing to buy; and then lowers the price gradually until a buyer signals his willingness to buy. When that occurs, the buyer is sold the object at the last price quoted by the auctioneer. Prove that in an IPV setup this Dutch auction has an equilibrium that is equivalent to the symmetric equilibrium of the first price auction.

[4MT] IPV Auctions

1. Consider a first price auction with 3 buyers with values distributed according to $f(x) = e^{-x}$ on $[0, \infty)$. Compute the seller's expected revenues.
2. K2.2
3. K2.4

[5MT] Revenue and Reserve Prices

1. Consider 3 buyers with values distributed according to $f(x) = 1/2$ on $[0, 2]$.
   (a) Compute the distribution of winning prices in the first and the second price auction.
   (b) Compute the mean of the two distributions in part (a). How do these means relate to each other and to revenue in each of the two auctions?

(c) Compute the variance of the two distributions in part (a). How do these variances relate to each other? Can you explain this in light of the discussion in lecture?

2. Consider a first price auction with 3 buyers with values distributed according to $f(x) = e^{-x}$ on $[0, \infty)$.

   (a) Compute the seller's optimal reserve price.
   (b) What are the seller's expected revenues with optimal reserve price in place?

[6MT] **IPV Auctions and Revenue Equivalence**

4. K3.1
5. K2.5
6. Hand In Problem Set I

[7MT] **Revenue Equivalence and IPV Auctions Extensions**

1. K3.2
2. K4.1

[8MT] **IPV Auctions Extensions and Mechanism Design**

1. K4.2
2. Consider three buyers participating in a second price auction. Buyers are financially constrained, and their value-wealth pairs (X_i, W_i) are identically and independently distributed according to a uniform on $[0, 1]^2$.

   (a) Show explicitly that bidding according to $B(x, w) = \min\{x, w\}$ is weakly dominant.
   (b) Compute the probability of winning with value-wealth pair (x, w).
   (c) Find an expression for the expected payment of a buyer with value-wealth pair (x, w).
   (d) Compute expected revenues of the seller.

3. [Extra] K4.5

[9MT] **Revenue Maximizing Mechanisms**

1. K5.1
2. K5.2

[10MT] **Revenue Maximizing and Efficient Mechanisms**

1. Prove explicitly that if the design problem is regular and symmetric then the optimal mechanism reduces to a second price auction with reserve price.
2. K5.3
3. Consider 3 potential buyers with independent private values for an object. The values of bidders a and b are identically distributed according to a uniform distribution on the interval $[-1, 1]$, so that $F_a(x) = F_b(x) = (x + 1)/2$. The value of bidder c is instead uniformly distributed on the interval $[0, 2]$, so that $F_c(x) = x/2$.

   (a) Define the Vickrey-Clarke-Groves (VCG) mechanism and specialize it to the scenario described above.
   (b) Define efficiency, incentive compatibility, and individually rationality. Show that the VCG mechanism satisfies these properties.
   (c) Compute the revenue of the VCG mechanism.

[11MT] **Efficient and Balanced Budget Mechanisms**

1. K5.4
2. [Extra] K5.5
3. Hand In Problem Set II

Hand In Problem Set 1 Francesco Nava
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Please give your answers to your class teacher before class in week 6MT. Thank you!

1. Consider the following variant of a 1^{st} price auction in an IPV setup. Sealed bids are collected. The highest bidder pays his bid, but receives the object only if the outcome of a toss of a fair coin is heads. If the outcome is tails, the seller keeps the object and the high bidders bid. Assume bidder symmetry. [30 Marks]

   (a) Find the unique symmetric equilibrium bidding function. [10 Marks]

   (b) Do buyers bid higher or lower than in a 1^{st} price auction? [5 Marks]

   (c) Find an expression for the seller's expected revenue. [10 Marks]

   (d) Show that the seller's expected revenue is exactly half that of a standard 1^{st} price auction. [5 Marks]

2. [Harder] In a 2^{nd} price auction with entry fee, buyers have to pay a fee E in order to participate in the auction. Buyers first choose whether to participate and pay the fee and then choose how much to bid. Consider an IPV setup with four buyers with values drawn from a uniform distribution on the interval $[0, 1]$. Suppose that only the seller knows how many buyers pay the fee. [40 Marks]

   (a) Find the unique symmetric equilibrium of this auction and write down the expected payment of a buyer with value x. Hint: Pay the fee only if $xG(x) \geq E$. Explain. [15 Marks]

   (b) What entry fee maximizes the equilibrium expected revenues of the seller? [15 Marks]

   (c) Compare the revenues to those of a standard 2^{nd} price auction. [5 Marks]

   (d) Compare the revenues to those of a 2^{nd} price auction with revenue maximizing reserve prices. [5 Marks]

3. Consider three buyers with independent and private values for an object at auction. The value of the object at sale is uniformly distributed on $[0, 1]$ for each buyer. Individuals know their valuation for the object, but ignore that of their competitors. Buyers participate in an auction in which the highest bid wins the object and in which everyone, but for the lowest bidder, has to pay his bid. [30 Marks]

   (a) Compute the probability that a buyer wins the auction if he bids b and everyone follows a strictly increasing strategy $\beta : [0, 1] \to \mathbb{R}$. [5 Marks]

   (b) Use the revenue equivalence theorem to derive the symmetric equilibrium bidding strategy. [15 Marks]

   (c) Compute the revenues of the auctioneer. [10 Marks]

Hand In Problem Set 2 Francesco Nava

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Please give your answers to your class teacher before class in week 11MT. Thank you!

1. Show that the expected value of the virtual valuation is always zero. [10 Marks]

2. Suppose that the buyers have values independently and identically distributed according to a uniform on $[1, 3]$. Let there be 3 buyers.

   (a) Derive the revenue maximizing auction for the seller. [10 Marks]

   (b) Compute the seller's expected revenue. [10 Marks]

3. Consider an environment with three buyers. Buyer 0 has a value that is drawn from a uniform distribution on the interval $[0, 2]$, and every other buyer $\{1, 2\}$ has a value that is independently drawn from a uniform distribution on the interval $[2, 3]$.

   (a) Find the revenue maximizing mechanism amongst those that are IC, IR. [15 Marks]

   (b) Find the revenue maximizing mechanism amongst those that are efficient, IC, IR. Check that it is IC & IR. [15 Marks]

   (c) Is there any mechanism that balances the budget and that is efficient, IC, IR? If it exists, characterize it. [15 Marks]

4. Consider a bilateral trade problem. Assume that the value of the object to be traded to the buyer is uniformly distributed on $[1, 3]$ and that the costs of the producer are uniformly distributed on $[0, 2]$.

   (a) Derive the VCG mechanism & check efficiency IC & IR. [15 Marks]

   (b) Is there any mechanism that balances the budget and that is efficient, IC, IR? Explain. [10 Marks]

Auction Theory

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Motivation

In history auctions have been used extensively to trade goods

Today they are used to:

1. trade commodities [tobacco, flowers, gold]
2. privatize public companies
3. sell long-term securities [US treasuries]
4. sell the rights to use natural resources
5. procure services
6. sell good through the internet [eBay]...

Auctions can be used for different purposes:

1. Revenue Maximization
2. Efficiency
3. ...

Notes

Objectives of the Course are:

1. Understand how buyers ought to bid in different auction formats
2. Quantify the revenues of the seller in different auction formats
3. Understand how the interdependence in valuations and the correlation in information affects behavior
4. Design revenue maximizing selling mechanisms
5. Design efficient selling mechanisms
6. Understand how informational frictions might limit trade

Notes

General Information

Lecturer:	Francesco Nava
Email:	f.nava@lse.ac.uk
Course Website:	moodle.lse.ac.uk
Time and Location:	Thu 9.30am-11.00am, 32L.LG.18
Office Hours:	Wed 1.30pm-3.00pm, 32L.3.20

Notes

Grades & Assignments

Class Work:

- Each week you will be assigned a few exercises (textbook or extra)
- Answers to exercises will be posted sometime after classes
- You are expected to attempt the classwork before your class
- In weeks *5 and 10 of MT written assignments are required*
- You are expected to be diligent about meeting the deadlines

Exams:

- The final exam will determine your grade on the course entirely
- Section 1 requires you to answer 4 short questions on MT & LT
- Section 2 requires you to answer 1 long questions on MT
- Section 3 requires you to answer 1 long questions on LT

Notes

Weekly Course Program

1. Review: Order Statistics & Game Theory [KA, KC, KF]
2. Independent Private Value Auctions [K1, K2]
3. Revenue Equivalence Principle [K3]
4. Extensions: Resale & Budget Constraints [K4]
5. Extensions: Risk Aversion [K4]
6. Optimal Auction Design [K5, 3]
7. Optimal Auction Design [K5, 5]
8. Efficient Mechanisms VCG [K5, 6]
9. Interdependent Values [K6, 4]
10. Linkage Principle [K7, 4]

Notes

Materials

Main Textbook

Auction Theory, Vijay Krishna, Academic Press 2010 [suggested] [K]

Slides

Include all materials required for the evaluations unless otherwise specified

Slides will be posted before each class

Papers

Some of the papers on the syllabus may be suggested to the interested reader

Alternative Textbook

Putting Auction Theory to Work, Paul Milgrom, Cambridge University Press 2004

Notes

Notes

Order Statistics

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Notes

Order Statistics

We wish to order a list of random draws. Let:

- $X_1, ..., X_n$ be independent draws from a distribution F density f.
- Rearrange these draws so that $Y_1 \geq Y_2 \geq ... \geq Y_n$.
- Hence for instance $Y_1 = \max\{X_1, ..., X_n\}$.

The random variable Y_i is referred to as the i^{rh} **order statistic.**

Denote F_i the distribution and f_i the density of Y_i.

Notes

First and Second Order Statistics

To compute the distribution of the first two order statistics notice that:

The event $Y_1 \leq y$ requires $X_i \leq y$ for all i. Thus,

$$F_1(y) = F(y)^n,$$
$$f_1(y) = nf(y)F(y)^{n-1}.$$

The event $Y_2 \leq y$ requires one of two mutually exclusive events to hold:

1. $X_i \leq y$ for all i;
2. $X_i \leq y$ for all $i \neq j$ and $X_j > y$ some j.

Therefore,

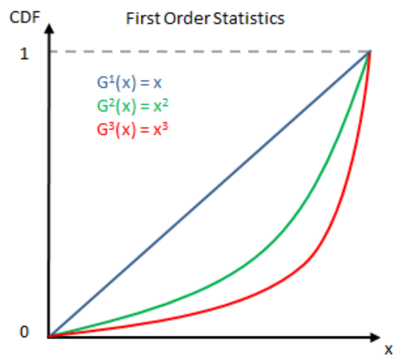
$$\begin{aligned} F_2(y) &= F(y)^n + nF(y)^{n-1}(1 - F(y)) = \\ &= nF(y)^{n-1} - (n-1)F(y)^n, \\ f_2(y) &= n(n-1)(1 - F(y))f(y)F(y)^{n-2}. \end{aligned}$$

Notes

Example: First Order Statistic

Denote by $G^n(y)$ the CDF of the first order statistic in a sample of size n.

Assume that F is uniform on $[0, 1]$. If so:



Notes

Relations among Different Sample Size

For convenience, label:

- the i^{th} order statistic in a sample of size n by $Y_i^{(n)}$;
- its distribution and density by $F_i^{(n)}$ and $f_i^{(n)}$.

Recall $F_1^{(n)}(y) = F(y)^n$ and $F_2^{(n)}(y) = nF(y)^{n-1} - (n-1)F(y)^n$.

Then observe that

$$\begin{aligned} F_2^{(n)}(y) &= nF_1^{(n-1)}(y) - (n-1)F_1^{(n)}(y), \\ f_2^{(n)}(y) &= nf_1^{(n-1)}(y) - (n-1)f_1^{(n)}(y), \\ E(Y_2^{(n)}) &= nE(Y_1^{(n-1)}) - (n-1)E(Y_1^{(n)}). \end{aligned}$$

Additionally, it must be that

$$f_2^{(n)}(y) = n(1 - F(y))f_1^{(n-1)}(y).$$

Notes

Joint Distributions

The joint density of $\mathbf{Y} = (Y_1^{(n)}, ..., Y_n^{(n)})$ is given by

$$f_{\mathbf{Y}}^{(n)}(y_1, ..., y_n) = \begin{cases} n!f(y_1)f(y_2)...f(y_n) & \text{if} \quad y_1 \geq y_2 \geq ... \geq y_n \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

which simply amounts of the probability of $(x_1, ..., x_n)$ adjusted by the number of possible permutations.

From integration for $y_1 \geq y_2$ we also get that

$$f_{1,2}^{(n)}(y_1, y_2) = n(n-1)f(y_1)f(y_2)F(y_2)^{n-2}.$$

For instance when $n = 3$:

$$f_{1,2}^{(3)}(y_1, y_2) = \int_0^{y_2} 3!f(y_1)f(y_2)f(y_3)dy_3 = 6f(y_1)f(y_2)F(y_2).$$

Notes

Conditional Distributions

The density of $Y_2^{(n)}$ conditional on $Y_1^{(n)} = y$ thus satisfies for $y \geq z$

$$f_2^{(n)}(z|Y_1^{(n)} = y) = \frac{f_{1,2}^{(n)}(y,z)}{f_1^{(n)}(y)} = \frac{(n-1)f(z)F(z)^{n-2}}{F(y)^{n-1}}.$$

Similarly the density of $Y_1^{(n-1)}$ conditional on $Y_1^{(n-1)} < y$ satisfies

$$f_1^{(n-1)}(z|Y_1^{(n-1)} < y) = \frac{f_1^{(n-1)}(z)}{F_1^{(n-1)}(y)} = \frac{(n-1)f(z)F(z)^{n-2}}{F(y)^{n-1}}.$$

These coincide not by coincidences. Think about it.

Notes

Notes

Game Theory Review

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Notes

Complete Information Games

Notes

Complete Information

A complete information game $G = \{N, \{A_i, u_i\}_{i \in N}\}$ consists of:

- N the set of players in the game;
- A_i player i's action set;
  - An action profile a is an element the set $A = \times_{j \in N} A_j$;
  - For convenience define $A_{-i} = \times_{j \in N \setminus i} A_j$;
- $u_i : A \to \mathbb{R}$ player i's utility function;
  - Mapping action profiles to payoffs.

Notes

Nash Equilibria

Definition (Nash Equilibrium NE)

A (pure strategy) Nash equilibrium of a game G consists of a strategy profile $a = (a_i, a_{-i}) \in A$ such that

$$u_i(a) \geq u_i(a_i', a_{-i}) \text{ for any } a_i' \in A_i \text{ \& } i \in N.$$

- Properties:
  - strategy profiles are independent;
  - strategy profiles common knowledge;
  - pure strategy Nash equilibria may not exist.
- Example one buyer with value 2$ and one seller:

$B \setminus S$	sale	no sale
2$	0,2	0,0
1$	1,1	0,0

Notes

Incomplete Information Games

Notes

Incomplete Information

An incomplete information game $\Gamma = \{N, \{A_i, \mathbb{X}_i, u_i\}_N, f\}$ consists of:

- N the set of players in the game;
- A_i player i's action set;
- $\mathbb{X}_i$ player i's set of possible signals;
  - A profile of signals is an element $x \in \mathbb{X} = \times_{j \in N} \mathbb{X}_j$;
  - For convenience define $\mathbb{X}_{-i} = \times_{j \in N \setminus i} \mathbb{X}_j$;
- $f \in \Delta(\mathbb{X})$ a distribution over the possible signals;
- $u_i : A \times \mathbb{X} \to \mathbb{R}$ player i's utility function, $u_i(a|x)$.

Notes

Beliefs and Strategies

Information structure:

- X_i denotes the signal as a random variable;
- Player i observes only X_i;
- Conditional observing $X_i = x_i$ his beliefs are

$$f(X_{-i}|x_i) = \frac{f(X_{-i}, x_i)}{\sum_{x_{-i}\in\mathbb{X}_{-i}} f(x_{-i}, x_i)} \in \Delta(\mathbb{X}_{-i}).$$

Strategy Profiles:

- A strategy consists of a map from information to actions $\alpha_i : \mathbb{X}_i \to A_i$;
- A strategy profile is a map $\alpha(X) = (\alpha_1(X_1), ..., \alpha_N(X_N))$.

Notes

Payoffs & Dominance

The **ex-post**, **ex-ante** and **interim expected utility** of strategy profile α are respectively defined by:

$$u_i(\alpha(x)|x) : \mathbb{X} \to \mathbb{R},$$
$$E[u_i(\alpha(X)|X)] = \sum_{\mathbb{X}} u_i(\alpha(x)|x)f(x) \in \mathbb{R},$$
$$E[u_i(\alpha(X)|X)|X_i = x_i] = \sum_{\mathbb{X}_{-i}} u_i(\alpha(x)|x)f(x_{-i}|x_i) : \mathbb{X}_i \to \mathbb{R}.$$

Weak Dominance:

- Strategy α_i weakly dominates α_i' if for any a_{-i} and x

$$u_i(\alpha_i(x_i), a_{-i}|x) \geq u_i(\alpha_i'(x_i), a_{-i}|x) \quad \text{[strict somewhere]}.$$

- Strategy α_i is dominant if it dominates any other strategy α_i'.
- Strategy α_i is undominated if no strategy dominates it.

Notes

Dominant Strategy and Bayes Nash Equilibria

Definitions (Dominant Strategy Equilibrium DSE)

A **Dominant Strategy equilibrium** of an incomplete information game Γ is a strategy profile α that for any $i \in N$, $x \in \mathbb{X}$ and $a_{-i} \in A_{-i}$ satisfies

$$u_i(\alpha_i(x_i), a_{-i}|x) \geq u_i(\alpha_i'(x_i), a_{-i}|x) \quad \text{for any } \alpha_i' : \mathbb{X}_i \to A_i.$$

Definitions (Bayes Nash Equilibrium BNE)

A **Bayes Nash equilibrium** of an incomplete information game Γ is a strategy profile α that for any $i \in N$ and $x_i \in \mathbb{X}_i$ satisfies

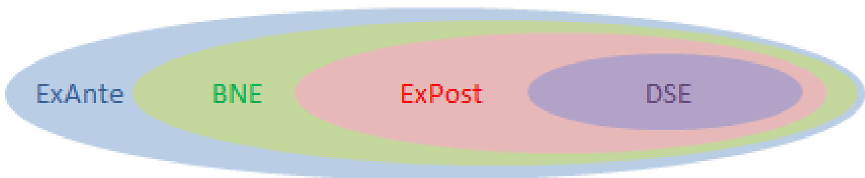
$$E[u_i(\alpha(X)|X)|X_i = x_i] \geq E[u_i(a_i, \alpha_{-i}(X_{-i})|X)|X_i = x_i] \quad \text{for any } a_i \in A_i.$$

- If α is a DSE then it is a BNE

Notes

Equilibrium Properties

- BNE requires for **interim optimality**.
- **Ex-ante** equilibrium is a strategy profile s.t. for any $\alpha_i' : \mathbb{X}_i \longrightarrow \Delta_i$
$$E[u_i(\alpha(X)|X)] \geq E[u_i(\alpha_i'(X_i), \alpha_{-i}(X_{-i})|X)].$$
- **Ex-post** equilibrium is BNE s.t. for any $a_i \in A_i$ and $x \in \mathbb{X}$
$$u_i(\alpha(x)|x) \geq u_i(a_i, \alpha_{-i}(x_{-i})|x).$$
- DSE $\Rightarrow$ Ex-post $\Rightarrow$ BNE $=$ Interim $\Rightarrow$ Ex-ante [by integrating].
- Ex-ante optimality $\Rightarrow$ Interim optimality almost surely.



Notes

BNE Example I: Exchange

A buyer and a seller want to trade an object:

- Buyer's value for the object is 3\$;
- Seller's value is either 0\$ or 2\$ based on the signal, $\mathbb{X}_S = \{L, H\}$;
- Buyer can offer either 1\$ or 3\$ to purchase the object;
- Seller choose whether or not to sell.

$B\backslash S.L$	sale	no sale
3\$	0,3	0,0
1\$	2,1	0,0

$B\backslash S.H$	sale	no sale
3\$	0,3	0,2
1\$	2,1	0,2

- This game for any prior f has a BNE in which

$$\alpha_S(L) = \text{sale},\ \alpha_S(H) = \text{no sale},\ \alpha_B = 1\$.$$

- Selling is weakly dominant for $S.L$.
- Offering 1\$ is weakly dominant for the buyer.

Notes

BNE Example II: Discrete Second Price Auction

Two buyers want to buy object at second price auction:

- Buyer i's value is either 1\$ or 2\$ based on his signal, $\mathbb{X}_i = \{L, H\}$;
- Buyers choose whether to bid 1\$ or 2\$;
- The highest bid wins the object (ties are broken with fair lottery);
- The second highest bid is paid by the winner.

$1.L\backslash 2.L$	2\$	1\$
2\$	-0.5,-0.5	0,0
1\$	0,0	0,0

$1.L\backslash 2.H$	2\$	1\$
2\$	-0.5,0	0,0
1\$	0,1	0,0.5

$1.H\backslash 2.L$	2\$	1\$
2\$	0,-0.5	1,0
1\$	0,0	0.5,0

$1.H\backslash 2.H$	2\$	1\$
2\$	0,0	1,0
1\$	0,1	0.5,0.5

The game has a DSE in which $\alpha_i(L)$ =1\$ and $\alpha_i(H)$ =2\$ for $\forall i \in N$.

Notes

BNE Example III: Discrete First Price Auction

Two buyers want to buy object at second price auction:

- Buyer i's value is either 1\$ or 2\$ based on his signal, $\mathbb{X}_i = \{L, H\}$;
- Buyers choose whether to bid 1\$ or 2\$;
- The highest bid wins the object (ties are broken with fair lottery);
- The highest bid is paid by the winner.

$1.L \backslash 2.L$	2\$	1\$
2\$	-0.5,-0.5	-1,0
1\$	0,-1	0,0

$1.L \backslash 2.H$	2\$	1\$
2\$	-0.5,0	-1,0
1\$	0,0	0,0.5

$1.H \backslash 2.L$	2\$	1\$
2\$	0,-0.5	0,0
1\$	0,-1	0.5,0

$1.H \backslash 2.H$	2\$	1\$
2\$	0,0	0,0
1\$	0,0	0.5,0.5

The game has a DSE in which $\alpha_i(L) = \alpha_i(H) =$1\$ for $\forall i \in N$.

Notes

Extra: Continuous Games Existence

Notes

Extra: Continuous Games & BNE Existence

- A reference for proofs is Athey Econometrica 2001.
- Players' action sets $A_i \subseteq \mathbb{R}$ and are convex.
- Players' types sets $\mathbb{X}_i \subseteq \mathbb{R}$ and are convex.
- Let $U_i(a_i, \alpha_{-i}(S)|x_i) = \int_{x_{-i} \in S} u_i(a_i, \alpha_{-i}(x_{-i})|x) f_{-i}(x_{-i}|x_i) dx_{-i}$.
- Let $U_i(a_i, \alpha_{-i}|x_i) = U_i(a_i, \alpha_{-i}(\mathbb{X}_{-i})|x_i)$.

Definition (Regularity Conditions RC)

Types have a joint density f which is bounded and atomless. $U_i(a_i, \alpha_{-i}(S)|x)$ exists and is finite for any S convex and any non-decreasing function $\alpha_j : \mathbb{X}_j \to A_j$ for $j \neq i$.

Notes

Single Crossing & Existence BNE

Definition (Single Crossing of Incremental Returns SCPIR)

The function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies single crossing property of incremental returns if for any $\overline{a} > \underline{a}$ and $\overline{x} > \underline{x}$

$$h(\underline{a}, \overline{x}) > h(\underline{a}, \underline{x}) \quad \Rightarrow \quad h(\overline{a}, \overline{x}) \geq h(\overline{a}, \underline{x}).$$

Definition (Single Crossing Condition SCC)

The single crossing condition for games with incomplete information is met if for any $i \in N$, whenever $\alpha_j : \mathbb{X}_j \to A_j$ is non-decreasing for any $j \neq i$, player i's objective function $U_i(a_i, \alpha_{-i}|x_i)$ satisfies SCPIR in (a_i, x_i).

Theorem (Existence of BNE)

Assume that SCC and RC hold. Suppose that $u_i(a|x)$ is continuous in a for any $i \in N$. Then there exists a BNE in non-decreasing strategies.

Notes

Independent Private Value Auction

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Notes

The Environment

Notes

Economy

Consider the following environment:

- 1 auctioneer;
- N potential buyers;
- 1 indivisible object owned by the auctioneer;
- Buyer i's valuation X_i for the object;
- X_i **independent** and identically distributed;
- $X_i \overset{\text{cdf}}{\sim} F$ continuous and increasing on $[0, \omega]$;
- F admits a continuous density f and has full support.

Notes

Information & Budgets

Private Values

How much player values the good is independent of others.

Information

Buyer i knows:

- x_i the realization of X_i;
- $X_j \overset{\text{cdf}}{\sim} F$ independently across $j \neq i$.

Budgets

- No buyer faces any budget/liquidity constraint.

Notes

First Price Auction

Notes

First Price Sealed-Bid Auction

Mechanism

- All players submit sealed bids b_i.
- Highest bid wins the object and pays the amount bid.
- For any profile of bids $b = \{b_j\}_{j \in N}$ payoffs are

$$\Pi_i = \begin{cases} x_i - b_i & \text{if} \quad b_i > \max_{j \neq i} b_j \\ 0 & \text{if} \quad b_i < \max_{j \neq i} b_j \end{cases}$$

Tie-breaking Rule

- Equal probability of getting object & paying amongst highest bidders.

Notes

FPA Equilibria & Shading

A strategy profile $\{\beta_j\}_{j \in N}$, $\beta_j : [0, \omega] \to \mathbb{R}_+$ any j, is **symmetric** if

$$\beta_j = \beta_i = \beta \text{ for all } i, j.$$

What is the optimal bid b by buyer i if:

- $j \neq i$ adopt a symmetric, increasing, differentiable strategy β;
- his private value is $X_i = x$.

Note that:

- A bid $b = x$ secures a payoff 0;
- Any bid $b > \beta(\omega)$ is suboptimal;
- $\beta(0) = 0$.

Notes

FPA Equilibria & Shading

Define $Y_1 = \max_{j \neq i} X_j \implies Y_1 \overset{\text{cdf}}{\sim} G = F^{N-1}$.

Since β is increasing $\max_{j \neq i} \beta(X_j) = \beta(Y_1)$.

Player i wins the object if

$$b > \beta(Y_1) \quad \Leftrightarrow \quad \beta^{-1}(b) > Y_1.$$

Expected payoff of i when choosing b thus amounts to

$$G(\beta^{-1}(b))(x - b).$$

Maximizing with respect to b yields

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x - b) - G(\beta^{-1}(b)) = 0.$$

Notes

FPA Equilibria & Shading

As we saw, FOC amount to

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x-b) - G(\beta^{-1}(b)) = 0.$$

Such condition at a symmetric equilibrium, $b = \beta(x)$, becomes

$$\frac{d}{dx}(G(x)\beta(x)) = xg(x).$$

Since $\beta(0) = 0$, integration yields

$$\beta(x) = \frac{1}{G(x)}\int_0^x yg(y)dy = E[Y_1 | Y_1 < x].$$

Necessary not sufficient for symmetric equilibrium existence.

Notes

FPA Symmetric Equilibria

Theorem

Symmetric equilibrium strategies in a first price auction are:

$$\beta_1(x) = E[Y_1 | Y_1 < x]$$

for Y_1 the highest of $N-1$ independently drawn values.

Notes

Proof.

Suppose that all bidders but one follow the strategy $\beta = \beta_1$.
Let x be his value and b his bid and define $z = \beta^{-1}(b)$ [invertible].
Only bids $b \leq \beta(\omega)$ are optimal. Expected profits for such bidder are

$$\begin{aligned}\Pi(x,b) &= G(z)[x-\beta(z)] = \\ &= G(z)[x - E[Y|Y<z]] = \\ &= G(z)x - \int_0^z yg(y)dy = \\ &= G(z)[x-z] + \int_0^z G(y)dy.\end{aligned}$$

Thus regardless of $x \leq z$ or $x \geq z$ it must be

$$\Pi(x,\beta(x)) - \Pi(x,\beta(z)) = G(z)[z-x] - \int_x^z G(y)dy \geq 0.$$

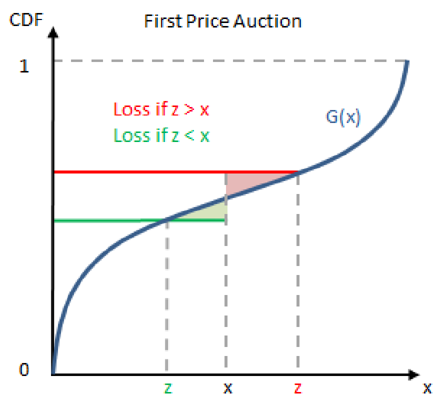
Therefore this is symmetric equilibrium, since $b > \beta(\omega)$ is suboptimal. □

Notes

FPA Symmetric Equilibria Plot

The plot below shows why it must be that

$$\Pi(x,\beta(x)) - \Pi(x,\beta(z)) = G(z)[z-x] - \int_x^z G(y)dy \geq 0.$$



Notes

Degree of Shading and the Uniform Case

Shading

$$\begin{aligned}\beta_1(x) &= \frac{1}{G(x)}\int_0^x y dG(y) = \\ &= \frac{1}{G(x)}\left[[yG(y)]_0^x - \int_0^x G(y)dy\right] = \\ &= x - \int_0^x \frac{G(y)}{G(x)}dy = x - \int_0^x \left[\frac{F(y)}{F(x)}\right]^{N-1} dy.\end{aligned}$$

Example

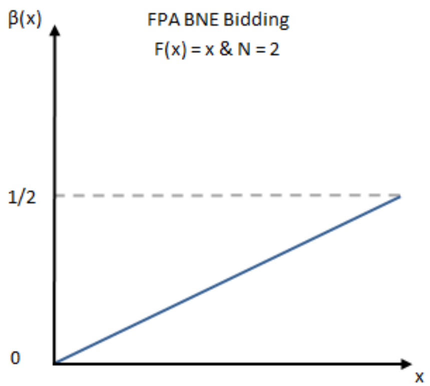
If $F(x) = x$ is uniform on $[0,1]$:

$$\beta_1(x) = x - \frac{1}{x^{N-1}}\left[\frac{y^N}{N}\right]_0^x = \frac{N-1}{N}x.$$

Notes

Uniform Example

For instance with 2 bidders, everyone bids half of the private value.



Notes

Second Price Auction

Notes

Second Price Sealed-Bid Auction

Mechanism

- All players submit sealed bids b_i.
- Highest bid wins the object and pays the second highest.
- For any profile of bids $b = \{b_j\}_{j \in N}$ payoffs are

$$\Pi_i = \begin{cases} x_i - \max_{j \neq i} b_j & \text{if} \quad b_i > \max_{j \neq i} b_j \\ 0 & \text{if} \quad b_i < \max_{j \neq i} b_j \end{cases}$$

Tie-breaking Rule

- Equal probability of getting object & paying amongst highest bidders.

Notes

Equilibrium Behavior

Lemma

Bidding according to $\beta_2(x) = x$ is a weakly dominant strategy in a second price auction.

Proof.

Let $p_i = \max_{j \neq i} b_j$. Player i by bidding x_i gets a payoff

$$\Pi_i = \begin{cases} x_i - p_i & \text{if} \quad x_i > p_i \\ 0 & \text{if} \quad x_i < p_i \end{cases}$$

If he were to bid $b_i < x_i$ he would still earn

$$\Pi_i = \begin{cases} x_i - p_i & \text{if} \quad b_i > p_i \\ 0 & \text{if} \quad b_i < p_i \end{cases}$$

But $\Pr(x_i > p_i) > \Pr(b_i > p_i)$. Similarly $b_i > x_i$. □

Notes

Revenue Comparisons

Notes

Revenue Comparison

In a 1^{st} & 2^{nd} price auction the expected payment of bidder with value x:

$$m_1(x) = \Pr(\text{win}) * 1^{\text{st}}\text{bid} = G(x)E(Y_1|Y_1 < x),$$
$$m_2(x) = \Pr(\text{win}) * E(2^{\text{nd}}\text{bid}|1^{\text{st}}\text{bid} = x) = G(x)E(Y_1|Y_1 < x).$$

The expected ex-ante payment by a particular player is for $k \in \{1, 2\}$

$$E(m_k) = \int_0^\omega \left[\int_0^x yg(y)dy\right] f(x)dx = \int_0^\omega \left[\int_y^\omega f(x)dx\right] yg(y)dy =$$
$$= \int_0^\omega [1 - F(y)]\, yg(y)dy = \int_0^\omega y\,[1 - F(y)]\, f_1^{(N-1)}(y)dy.$$

Therefore the expected ex-ante revenues derived by the auctioneer are

$$E(R_k) = N * E(m_k) = \int_0^\omega y f_2^{(N)}(y)dy = E(Y_2^{(N)}).$$

Notes

Revenue Comparison

Revenue Equivalence first take:

Lemma

In an IPV setup the expected revenues from both a 1^{st} and a 2^{nd} price auction are the same.

Revenues in both formats have the same mean, but different variance.

Mean-preserving Spread (MPS):

Definition

Let $X \overset{\text{cdf}}{\sim} F$ and $Z|X = x \overset{\text{cdf}}{\sim} H(x)$ so that $E(Z|X = x) = 0$. If $Y = [X + Z] \overset{\text{cdf}}{\sim} G$, then we say that G is a mean-preserving spread of F.

Notes

Revenue Comparison

Lemma

In an IPV setup the distribution of equilibrium prices of a 2^{nd} price auction is a MPS of the distribution of equilibrium prices of a 1^{st} price auction.

Proof.

Revenues are as random variables respectively $R_1 = \beta(Y_1^{(N)})$ & $R_2 = Y_2^{(N)}$ thus it must be that

$$E(R_2|R_1 = p) = E(Y_2^{(N)}|Y_1^{(N)} = \beta^{-1}(p)) =$$
$$= E(Y_1^{(N-1)}|Y_1^{(N-1)} < \beta^{-1}(p)) = \beta(\beta^{-1}(p)) = p.$$

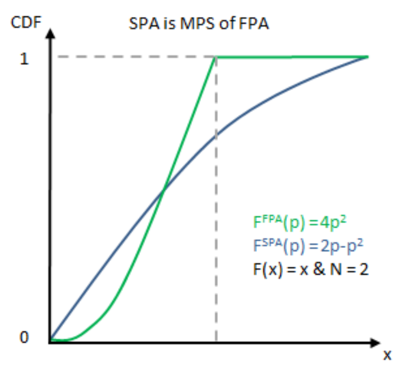
Thus there exists Z such that the distribution of R_2 is the same as that of $R_1 + Z$ and $E(Z|R_1 = p) = 0$. □

Notes

Revenue Comparison Example

A uniform distribution example with $N = 2$ and $\omega = 1$.

FPA is less volatile than SPA:



Notes

Reserve Prices

Notes

Reserve Prices SPA

Reserve Price

- Seller reserves the right not to sell if winning price is below r.
- The reserve price is common knowledge amongst players.

2^{nd} Price Auction

- Winner pays reserve price if second bid is below.
- Makes no difference in behavior still weakly dominant to bid own value.
- The expected payment for bidder with value r is $rG(r)$. If $x > r$ then

$$m_2(x, r) = rG(r) + \int_r^x yg(y)dy.$$

Notes

Reserve Prices FPA

1st Price Auction:

- Note that: (a) $x < r$ never gain and (b) $\beta(r) = r$.
- If $x \geq r$ as before we can get that

$$\beta(x) = E(\max\{Y_1, r\} | Y_1 < x) = r\frac{G(r)}{G(x)} + \frac{1}{G(x)} \int_r^x yg(y)dy.$$

- The expected payment becomes

$$m_1(x, r) = rG(r) + \int_r^x yg(y)dy.$$

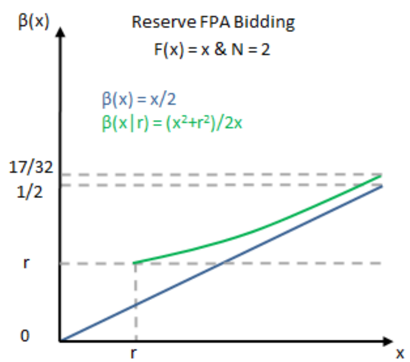
- Again expected revenue in both formats coincide.

Notes

Reserve Price Example FPA

A uniform distribution example with $N = 2$ and $\omega = 1$.

Bidding is more aggressive with reserve prices than without:



Notes

Revenue Effects and Optimal Reserve Prices

The ex-ante expected payment is for $k \in \{1, 2\}$

$$E(m_k(r)) = r(1 - F(r))G(r) + \int_r^{\omega} y(1 - F(y))g(y)dy.$$

If seller attaches value $x_0 \in [0, \omega)$ to the object and sets reserve price r his expected payoff is

$$\Pi_0(r) = N * E(m_k(r)) + F(r)^N x_0.$$

For $\lambda(x) = f(x)/(1 - F(x))$ (hazard rate), optimality requires

$$\begin{aligned}\frac{d\Pi_0}{dr} &= N[1 - F(r) - rf(r)]G(r) + NG(r)f(r)x_0 \\ &= N[1 - (r - x_0)\lambda(r)](1 - F(r))G(r) = 0.\end{aligned}$$

Notes

The Exclusion Principle

If $x_0 > 0$ then $d\Pi_0(x_0)/dr > 0$ thus $r > x_0$.

If $x_0 = 0$ then $d\Pi_0(0)/dr = 0$, but it's local minimum thus $r > x_0$.

Higher payoff by setting $r > x_0$ [Exclude sometimes to raise bids].

First order Necessary condition reduces to

$$(r_* - x_0)\lambda(r_*) = 1.$$

If λ is increasing, the condition also sufficient.

Entry fees can be chosen to mimic the effects of reserve prices.

Notes

Efficiency vs Revenue

Comments on Efficiency

- Though a reserve price maximizes revenues to the seller it may be inefficient.
- If $x_0 = 0$ and $r > 0$ there is a positive probability that the seller retains the object.
- This is inefficient unless all player do not value the object (zero probability).

Commitment

- There needs to be commitment to reserve price and no ex-post resale.
- Secret reserve values affect behavior.

Notes

Notes

Revenue Equivalence Principle

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Notes

Revenue Equivalence

Notes

Main Result

Definition

An auction is **standard** if the person who bids the highest amount is awarded the object.

Comment: A lottery in which the probability of winning depends on the amount bid is not standard.

Theorem (Revenue Equivalence)

If values are iid and bidders risk neutral, then any symmetric increasing equilibrium of any standard auction such that the expected payment of a bidder with 0 *value is* 0*, yields the same expected revenue.*

Notes

Proof.

Fix any standard auction format and a symmetric equilibrium β.
Let $m(x)$ be the expected payment when the value is x.
Let β be so that $m(0) = 0$. Suppose that all, but 1 player follow β.
If he bids $\beta(z)$ he wins if $z \geq Y_1$. His expected payoff is

$$\Pi(z|x) = G(z)x - m(z).$$

Interim optimality requires that

$$d\Pi(z|x)/dz = g(z)x - m'(z) = 0.$$

If β is equilibrium the equality holds at $z = x$. Thus $m'(y) = g(y)y$ and

$$m(x) = m(0) + \int_0^x yg(y)dy = G(x)E(Y_1|Y_1 < x).$$

Thus expected payments do not depend on the auction format. □

Notes

Example: Uniform Revenues

- Uniform distribution defined on $[0, 1]$.
- $F(x) = x$ and $G(x) = x^{N-1}$ and $g(x) = (N-1)x^{N-2}$.
- Expected payments amount to

$$m(x) = \int_0^x (N-1)y^{N-1}dy = \frac{N-1}{N}x^N.$$

- Expected revenues are

$$E(R) = N\int_0^1 m(x)dx = \int_0^1 (N-1)x^N dx = \frac{N-1}{N+1}.$$

Notes

Applications

Notes

Third Price Auction

If $X_i \overset{\text{iid}}{\sim} F$ and $\max X_i = x$ we have that:

$$F_2^{(N-1)}(y|Y_1^{(N-1)} < x) = \frac{F(y)^{N-1} + (N-1)(F(x)-F(y))F(y)^{N-2}}{F_1^{(N-1)}(x)},$$
$$f_2^{(N-1)}(y|Y_1^{(N-1)} < x) = \frac{(N-1)(F(x)-F(y))f_1^{(N-2)}(y)}{F_1^{(N-1)}(x)}.$$

If β_3 is a symmetric BNE by revenue equivalence $m_3(x) = m_2(x)$ so that

$$F_1^{(N-1)}(x)E[\beta_3(Y_2^{(N-1)})|Y_1^{(N-1)} < x] = \int_0^x yg(y)dy,$$
$$\int_0^x \beta_3(y)(N-1)(F(x)-F(y))f_1^{(N-2)}(y)dy = \int_0^x yf_1^{(N-1)}(y)dy,$$
$$\int_0^x \beta_3(y)(F(x)-F(y))f_1^{(N-2)}(y)dy = \int_0^x yf(y)F^{N-2}(y)dy.$$

Notes

Third Price Auction

Thus if β_3 is increasing the following condition holds

$$\int_0^x \beta_3(y)(F(x)-F(y))f_1^{(N-2)}(y)dy = \int_0^x yf(y)F_1^{(N-2)}(y)dy.$$

Taking derivatives with respect to x implies that

$$0 + f(x)\int_0^x \beta_3(y)f_1^{(N-2)}(y)dy = xf(x)F_1^{(N-2)}(x).$$

Simplifying $f(x)$ and differentiating again implies that

$$\beta_3(x)f_1^{(N-2)}(x) = F_1^{(N-2)}(x) + xf_1^{(N-2)}(x).$$

So that the equilibrium bidding strategy must satisfy

$$\beta_3(x) = \frac{F_1^{(N-2)}(x)}{f_1^{(N-2)}(x)} + x.$$

Notes

Uncertain Number of Bidders

$\mathcal{N} = \{1, ..., N\}$ be the set of potential bidders.

Symmetric prior $p_n = \Pr(|\mathcal{A}| = n+1)$ for all bidders.

Consider a standard auction and a symmetric & increasing eq β.

Consider a bidder with value x who bids $\beta(z)$ instead.

If he faces n bidders he wins if $z \geq Y_1^{(n)}$ with prob $G_n(z) = F(z)^n$.

The probability of winning becomes

$$G(z) = \sum_{n=0}^{N-1} p_n G_n(z).$$

The expected payoff thus becomes

$$\Pi(z|x) = G(z)x - m(z).$$

Notes

Uncertain Number of Bidders

2nd Price Auction

- Still weakly dominant to bid one's own value.
- Expected payment is given by

$$m_2(x) = \sum_{n=0}^{N-1} p_n G_n(x) E(Y_1^{(n)} | Y_1^{(n)} < x).$$

1st Price Auction

- Expected payment is given by $m_1(x) = G(x)\beta(x)$.
- By the revenue equivalence $m_1(x) = m_2(x)$ and thus

$$\beta(x) = \sum_{n=0}^{N-1} p_n \frac{G_n(x)}{G(x)} E(Y_1^{(n)} | Y_1^{(n)} < x).$$

- Weighted average of the bids in the known number case.

Notes

All Pay Auction or Lobbying

Highest bid wins object, all bidders pay their bid.

Suppose $\exists \beta_A$ symmetric equilibrium such that $m_A(0) = 0$.

Because bidders always pay and by revenue equivalence

$$\beta_A(x) = m_A(x) = \int_0^x y g(y) dy.$$

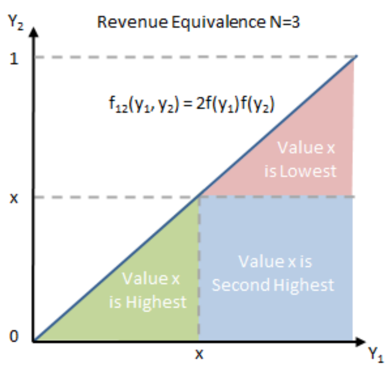
To solve the assigned problem show that:

- Such strategy is a BNE of the auction
- Check that nobody can benefit from unilateral deviations
- ...

Notes

Solving Revenue Equivalence Problems

To solve such problems and correctly write down expected payoffs it is instructive to think of the following plot for $N = 3$:



Notes

IPV Extensions

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Notes

Introduction

Assumptions underlying the revenue equivalence principle:

- Independence
- Risk neutrality
- No Budget Constraints
- Symmetry
- No Resale

Explore consequences of relaxing the last four assumptions.

Notes

Risk Aversion

Notes

Risk Aversion

Risk neutrality $\Rightarrow$ payoff linear in payments $\Rightarrow$ revenue equivalence.

Risk aversion: $u : \mathbb{R}_+ \to \mathbb{R}$ satisfying $u(0) = 0$, $u' > 0$ & $u'' < 0$.

Lemma

In an IPV setup if bidders are risk-averse and symmetric, then the expected revenue in a 1^{st} price auction is grater than in a 2^{nd} price auction.

Proof.

In 2^{nd} price auction bidding one's value remains weakly dominant.
The expected price is thus unchanged.

In 1^{st} price auction if all follow increasing (diff) strategy γ with $\gamma(0) = 0$.
Optimality for a bidder with value x again dictates

$$x \in \arg\max_z G(z)u(x - \gamma(z)).$$

$\square$

Notes

Risk Aversion

Proof.

[Proof continued] The first order condition evaluated at x requires

$$\gamma'(x) = \frac{u(x-\gamma(x))}{u'(x-\gamma(x))}\frac{g(x)}{G(x)}.$$

Were players risk neutral $u(x) = ax$ this would amount to

$$\beta'(x) = (x-\beta(x))[g(x)/G(x)].$$

Since $u(0) = 0$ & $u'' < 0$ we have that $[u(y)/u'(y)] > y$ & thus

$$\gamma'(x) > (x-\gamma(x))[g(x)/G(x)],$$

which implies that if $\beta(x) > \gamma(x)$ then $\gamma'(x) > \beta'(x)$.
But since $\gamma(0) = \beta(0) = 0$ we have that $\gamma(x) > \beta(x)$.
Expected payment in 1st price increases while unchanged in 2nd price! □

Notes

Example: CRRA Utility and 2 Bidders

Suppose that $N = 2$ and that utility satisfies $u(x) = x^{\alpha}$.

If so, relative risk aversion amounts to $xu''(x)/u'(x) = 1-\alpha$.

Observe that $F_{\alpha} = F^{1/\alpha}$ is also a cdf & denote its pdf by $f_{\alpha} = fF^{1/\alpha-1}/\alpha$.

The 1st price equilibrium strategy satisfies $\gamma(0) = 0$ and

$$\begin{aligned}
&\gamma'(x) = (x-\gamma(x))(f(x)/\alpha F(x)) \\
&\Rightarrow \gamma'(x)F(x)^{1/\alpha} + \gamma(x)f(x)F(x)^{1/\alpha-1}/\alpha = xf(x)F(x)^{1/\alpha-1}/\alpha \\
&\Rightarrow \gamma(x)F_{\alpha}(x) = \int_0^x yf_{\alpha}(y)dy.
\end{aligned}$$

Equivalent to risk neutral case but for F_{α} instead of F.

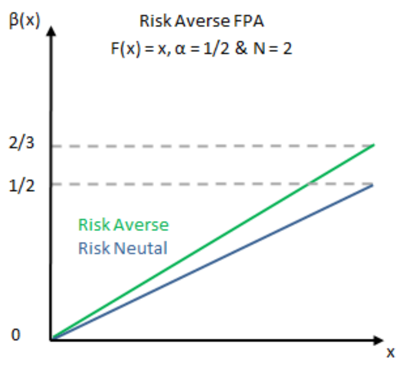
Since $F_{\alpha} \leq F$ the expected revenue is grater with risk aversion.

Notes

Example: CRRA Utility and 2 Bidders

In the previous example for $\alpha = 1/2$ and $F(x) = x$ we get that

$$\gamma(x) = 2x/3 \geq x/2 = \beta(x).$$



Notes

Budget Constraints

Notes

Budget Constraints

Consider the following setting:

- Agent type consists of a value-budget pair (X_i, W_i).
- Bidder with type (x, w) can never bid more than w.
- If he does and defaults, a big penalty is applied.
- Agent types are iid and $(X, W) \overset{\text{pdf}}{\sim} f$ on $[0,1]^2$.
- A bid consists of a map $B : [0,1]^2 \to \mathbb{R}$.

Notes

Second Price Auction

Lemma

In a 2^{nd} price auction it is a weakly dominant strategy to bid $B^{II}(x, w) = \min\{x, w\}$.

Proof.

(a) $B^{II}(x, w) > w$ is dominated. When one wins:

if the second highest bid is above w, default and penalty follow,
if the second bid does not exceed w, bidding w wins as well.

(b) If $x \leq w$ bidder is unconstrained.
The same proof of class notes II shows that $B^{II}(x, w) = x$.

(c) If $x > w$ same argument shows that $B^{II}(x, w) < w$ is dominated. $\square$

Notes

SPA Computing Revenue

Next let us compute expected revenue:

- For $x'' = \min\{x, w\} \;\Rightarrow\; B^{II}(x, w) = B^{II}(x'', 1)$.
- The set of players bidding less than x'' is given by

$$L^{II}(x'') = \left\{(X, W) | B^{II}(X, W) < B^{II}(x'', 1)\right\}.$$

- The probability that type $(x'', 1)$ outbids one bidder is

$$F^{II}(x'') = \int_{L^{II}(x'')} f(X, W)dXdW.$$

- The probability of winning thus is: $G^{II}(x'') = (F^{II}(x''))^{N-1}$.
- Expected utility for type $(x'', 1)$ when bidding $B^{II}(z, 1)$ is

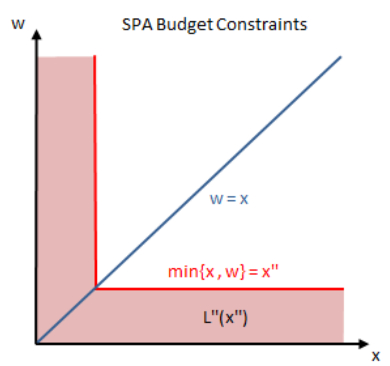
$$G^{II}(z)x'' - m^{II}(z, 1).$$

Notes

SPA Computing Revenue

Expected payments in the second price auction thus amount to

$$m^{II}(x, w) = m^{II}(x'', 1) = \int_0^{x''} ydG^{II}(y).$$



Notes

First Price Auction

Suppose that for an increasing function β the equilibrium strategy is

$$B^I(x, w) = \min\{\beta(x), w\}.$$

It must be that $\beta(x) < x$ or bidder $x < w$ would deviate.

To compute revenue, we proceed just as before:

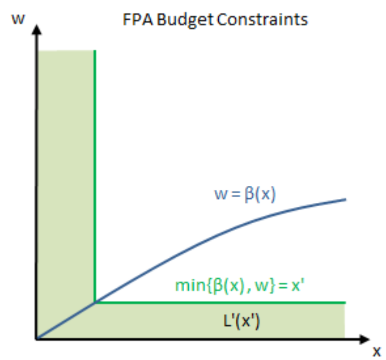
- For x' such that $\beta(x') = \min\{\beta(x), w\} \Rightarrow B^I(x, w) = B^I(x', 1)$.
- The set of players bidding less than x' is given by
$$L^I(x') = \left\{(X, W) | B^I(X, W) < B^I(x', 1)\right\}.$$
- The probability that type $(x', 1)$ outbids one bidder is
$$F^I(x') = \int_{L^I(x')} f(X, W)dXdW.$$
- The probability of winning thus is: $G^I(x') = (F^I(x'))^{N-1}$.

Notes

FPA Computing Revenue

Expected payments in the first price auction thus amount to

$$m^I(x, w) = m^I(x', 1) = \int_0^{x'} ydG^I(y).$$



Notes

BC Revenue Comparison

Lemma

If bidders are financially constrained and the 1st price auction has an equilibrium of the form $B^I(x,w) = \min\{\beta(x), w\}$, then the expected revenue in the 1st price auction exceeds that of the 2nd price auction.

Proof.

Since $\beta(x) < x$ for any x, it must be that $L^I(x) \subset L^{II}(x)$

$F^I(x) \leq F^{II}(x)$ for any $x \in (0,1) \;\Rightarrow\; F^I$ stochastically dominates F^{II}

Expected revenues are:

$$E(R^I) = E(Y_2^{I(N)}) > E(Y_2^{II(N)}) = E(R^{II})$$

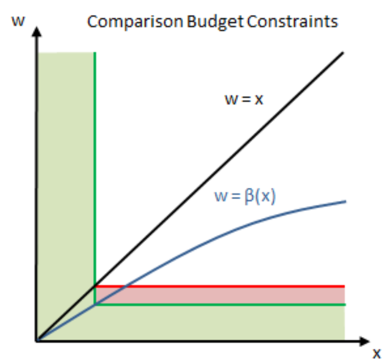
Since $E(Y_2^{a(N)}) = \int_0^1 m^a(x,1) f^a(x)dx$ for any $a \in \{I, II\}$ $\square$

Budget constraints are softer in the 1st price case

Notes

BC Revenue Comparison

The following plot clearly displays the revenue ranking:



Notes

Asymmetric Bidders

Notes

Asymmetric Bidders Two Players Uniform

In 2^{nd} price bidding one's value remains optimal $\Rightarrow$ **efficiency**.

In 1^{st} price strategy changes and outcome may be **inefficient.**

1^{st} Price Auction

- $X_i \overset{\text{cdf}}{\sim} F_i = U[0, \omega_i]$ for a player $i \in \{1, 2\}$.
- Strategies are increasing & differentiable functions β_i $[\phi_i = \beta_i^{-1}]$.
- It must be that $\beta_i(0) = 0$ and $\beta_i(\omega_i) = \overline{b}$ for any $i \in \{1, 2\}$.
- Expected payoff of bidder $i \neq j$ of bidding b is

$$\Pi_i(b, x_i) = (x_i - b)F_j(\phi_j(b)) = (x_i - b)\phi_j(b)/\omega_j.$$

- Optimality and $x_i = \phi_i(b)$ imply

$$\phi_j'(b)(\phi_i(b) - b) = \phi_j(b).$$

Notes

Asymmetric FPA

- By rewriting, adding and integrating for $\phi_i(0) = 0$,

$$(\phi_j'(b) - 1)(\phi_i(b) - b) = \phi_j(b) - \phi_i(b) + b,$$
$$\frac{d}{db}(\phi_1(b) - b)(\phi_2(b) - b) = 2b,$$
$$(\phi_1(b) - b)(\phi_2(b) - b) = b^2.$$

- The last equation implies that

$$\begin{array}{lll} \overline{b} = \omega_1\omega_2/(\omega_1+\omega_2) & \text{by} & \phi_i(\overline{b}) = \omega_i \\ \phi_i'(b) = \phi_i(b)(\phi_i(b) - b)/b^2 & \text{by} & \text{optimality} \end{array}$$

- For $\phi_i(b) = \xi_i(b)b + b$ this becomes

$$\xi_i'(b)b + \xi_i(b) + 1 = \xi_i(b)(\xi_i(b) + 1)$$
$$\Rightarrow \xi_i(b) = \frac{1 - k_i b^2}{1 + k_i b^2} \quad \text{since} \quad \frac{\xi_i'(b)}{\xi_i(b)^2 - 1} = \frac{1}{b}.$$

Notes

Asymmetric FPA

- Therefore $\phi_i(b) = 2b/(1 + k_i b^2)$ and $k_i = (1/\omega_i^2) - (1/\omega_j^2)$.
- Inverting one gets that

$$\beta_i(x) = (1 - \sqrt{1 - k_i x^2})/k_i x.$$

- If $\omega_1 > \omega_2$ note that $\beta_1(x) < \beta_2(x)$.
- The weaker player bids more aggressively.

Lemma (Efficiency)

With asymmetrically distributed values a 2^{nd} price auction always allocates the object efficiently. A 1^{st} price auction with positive probability does not.

- Since $\beta_1(x + \varepsilon) < \beta_2(x)$ for $\varepsilon > 0$ and small enough.

Notes

Asymmetric Revenue Comparison

Consider an example with $\omega_1 = 1/(1-\alpha)$ and $\omega_2 = 1/(1+\alpha)$.

The distribution of selling prices in 2^{nd} price auction is, for $p \in [0, \omega_2]$,

$$\begin{aligned} L^{II}_{\alpha}(p) &= \Pr(\min\{X_1, X_2\} \leq p) = F_1(p) + F_2(p) - F_1(p)F_2(p) \\ &= 2p - (1-\alpha^2)p^2 \end{aligned}$$

The distribution of selling prices in 1^{st} price auction is, for $p \in [0, 1/2]$,

$$\begin{aligned} L^{I}_{\alpha}(p) &= \Pr(\max\{\beta_1(X_1), \beta_2(X_2)\} \leq p) = F_1(\phi_1(p))F_2(\phi_2(p)) \\ &= (1-\alpha^2)(2p)^2 / \left[1 - \alpha^2(2p)^4\right]. \end{aligned}$$

$L^{II}_{\alpha}(p)$ increases with α, $L^{I}_{\alpha}(p)$ decreases with α & $E^{I}_{0}(p) = E^{II}_{0}(p)$.

Thus $E^{I}_{\alpha}(p) > E^{II}_{\alpha}(p)$ for $\forall \alpha > 0$. [Unranked in general].

Notes

Resale

Notes

Resale Two Asymmetric Bidders

Will bidders always allocate the object efficiently if resale is permitted?

- **Information**: the winner & all bids are revealed before resale.
- **Resale**: winner makes a take it or leave it offer.
- Assume that $F_1 \neq F_2$ both on $[0, \omega]$.
- If β_1, β_2 are invertible, $x_1 < x_2$ and $\beta_1(x_1) > \beta_2(x_2)$.
- Then 1 resells the object to 2 at price $p = x_2 \;\Rightarrow\;$ **Efficiency.**
- The next lemma shows that:
  1. β_1, β_2 are not invertible;
  2. the 1^{st} price auction with resale is inefficient.

Notes

Lemma

With asymmetric IPV, the 1^{st} price auction followed by resale does not result in efficiency.

Proof.

[Proof [1/3]] Recall that $\beta_i(0) = 0$ and $\beta_i(\omega) = \overline{b}$ for any $i \in \{1, 2\}$.

Suppose that β_i is invertible and that 2 uses $\beta_2 \Rightarrow$ Efficiency.

Player 1 expected payment if he bids as if his value was z_1 is

$$m_1(z_1) = \beta_1(z_1)F_2(\phi_2\beta_1(z_1)) - \int_{z_1}^{\phi_2\beta_1(z_1)} \max\{z_1, x_2\}\, dF_2(x_2).$$

Second term is positive if $z_1 < \phi_2\beta_1(z_1)$ and negative if $z_1 > \phi_2\beta_1(z_1)$.

Type x_1 expected payoff if he bids as if his value was z_1 is

$$x_1F_2(z_1) - m_1(z_1).$$

$\square$

Notes

Proof.

[Proof [2/3]] If β_1 is invertible optimality at $z_1 = x_1$ requires that

$$m_1'(x_1) = x_1 f_2(x_1).$$

Optimality and $m_1(0) = 0$ imply that

$$m_1(x_1) = \int_0^{x_1} x_2 dF_2(x_2).$$

Efficiency $\Rightarrow$ [1st price + resale] $\cong$ [2nd price]!

If $\beta_1(\omega) = \overline{b} \Rightarrow \phi_2\beta_1(\omega) = \omega \Rightarrow \overline{b} = E[X_2]$ since

$$m_1(\omega) = \overline{b}F_2(\omega) = \int_0^{\omega} x_2 dF_2(x_2).$$

Switching players one gets $\overline{b} = E[X_1] \neq E[X_2]$. A contradiction!

Thus β_1 & β_2 cannot be both strictly increasing! □

Notes

Proof.

[Proof [3/3]] Assume both continuous, non-decreasing & $\beta_2(x_2) = b_2$ for $\forall x_2 \in [x_2', x_2'']$.

Claim: $\exists x_1 \in (x_2', x_2'')$ such that $\beta_1(x_1) \geq b_2$.

Otherwise $x_1'' = [\min_x x \text{ s.t. } \beta_1(x) = b_2] \Rightarrow x_1'' \geq x_2''$.

Also $x_1'' > b_2$ by optimality of type x_2'' which wins at most x_1''.

For $2\varepsilon < x_1'' - b_2$, type $x_1'' - \varepsilon$ would profit by bidding $b_2 + \varepsilon$.

At infinitesimal cost win against any $x_2 \in [x_2', x_2''] \Rightarrow$ A contradiction!

Thus let $x_1 \in (x_2', x_2'')$ be such that $\beta_1(x_1) \geq b_2$.

If $x_2 \in (x_1, x_2'']$ bidder 1 wins auction inefficiently!

Optimality of 1 in resale implies that price $p > x_1$.

Thus with positive probability $p > x_2 > x_1 \Rightarrow$ Inefficient! □

Notes

Mechanism Design IPV

EC319 – Lecture Lecture VI

Nava

LSE

Michaelmas Term

Notes

Introduction

How to design selling mechanisms with desirable properties:

- Revenue Maximization
- Efficiency
- Balanced Budget

For the moment retain the IPV setup:

- N potential buyers, 1 auctioneer and 1 indivisible object.
- Buyer i's valuation $X_i \overset{\text{cdf}}{\sim} F_i$ on $[0, \omega_i] = \mathbb{X}_i$ independently.
- The auctioneer values the object at zero.

Notes

Mechanisms

Revelation Principle and Revenue Equivalence

Notes

Mechanisms

A **mechanism** consists of triple (B, π, μ):

1. Message spaces B_i;
2. An allocation rule $\pi : B \to \Delta^N = \left\{ p \in \mathbb{R}^N_+ \middle| \sum_{i=1}^N p_i \leq 1 \right\}$;
3. A payment rule $\mu : B \to \mathbb{R}^N$.

In a 1st or 2nd price auction, except for ties, satisfy:

- $B_i = \mathbb{X}_i$;
- $\pi_i(b) = \mathbb{I}(b_i > \max_{j \neq i} b_j)$;
- $\mu^I_i(b) = \mathbb{I}(b_i > \max_{j \neq i} b_j) b_i$ for FPA;
- $\mu^{II}_i(b) = \mathbb{I}(b_i > \max_{j \neq i} b_j) \max_{j \neq i} b_j$ for SPA.

A strategy is a map $\beta_i : \mathbb{X}_i \to B_i$.

Equilibrium if $\beta_i(x_i)$ maximizes payoffs given β_{-i} for any x_i and i.

Notes

Revelation Principle

A **direct mechanism** consists of pair (Q, M):

1. An allocation rule $Q : \mathbb{X} \to \Delta^N$;
2. A payment rule $M : \mathbb{X} \to \mathbb{R}^N$.

Theorem (Revelation Principle – Myerson)

For any mechanism and any equilibrium for that mechanism, there exists a direct mechanism in which:

1. *it is an equilibrium for each buyer to report his value truthfully;*
2. *outcomes are the same as in equilibrium of the original mechanism.*

Proof.

Define $Q(x) = \pi(\beta(x))$ and $M(x) = \mu(\beta(x))$ for any $x \in \mathbb{X}$.
Then conditions 1 and 2 are met by definition of equilibrium. $\square$

Notes

Incentive Compatibility Definition

Consider a direct mechanism (Q, M).

The winning probability and the expected payment when reporting z_i, if others are sincere, amount to

$$q_i(z_i) = \int_{\mathbb{X}_{-i}} Q_i(z_i, x_{-i}) dF_{-i}(x_{-i}),$$
$$m_i(z_i) = \int_{\mathbb{X}_{-i}} M_i(z_i, x_{-i}) dF_{-i}(x_{-i}).$$

The expected payoff of type x_i is given by:

$$x_i q_i(z_i) - m_i(z_i).$$

The mechanism is said to be **incentive compatible** (IC) if

$$x_i q_i(x_i) - m_i(x_i) = \max_{z_i \in \mathbb{X}_i} [x_i q_i(z_i) - m_i(z_i)] \equiv U_i(x_i).$$

Notes

Incentive Compatibility Equivalence

Lemma

A direct mechanism is incentive compatible if and only if $U_i'(x_i) = q_i(x_i)$ and $U_i''(x_i) \geq 0$ for all x_i and all i.

Proof.

If: If IC holds we must have that

$$U_i(z_i) \geq z_i q_i(x_i) - m_i(x_i) = U_i(x_i) + (z_i - x_i) q_i(x_i).$$

The same inequality must hold switching x_i with z_i and thus

$$(z_i - x_i) q_i(z_i) \geq U_i(z_i) - U_i(x_i) \geq (z_i - x_i) q_i(x_i). \tag{1}$$

Dividing by $(z_i - x_i)$ and taking limits as $z_i \to x_i$ yields

$$\lim_{z_i \to x_i} \frac{U_i(z_i) - U_i(x_i)}{(z_i - x_i)} = U_i'(x_i) = q_i(x_i).$$

Notes

Incentive Compatibility Equivalence

Proof.

Condition (1) also implies that

$$(z_i - x_i)(q_i(z_i) - q_i(x_i)) \geq 0,$$

which is equivalent to $q_i'(x_i) \geq 0$, and thus $U_i''(x_i) \geq 0$.

Only If: If $U_i'(x_i) = q_i(x_i)$, the expected payoff depends just on Q since:

$$U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t)dt.$$

Thus, for any x_i and any z_i, we get that

$$U_i(z_i) - U_i(x_i) = \int_{x_i}^{z_i} q_i(t)dt \geq (z_i - x_i) q_i(x_i),$$

where the inequality holds as $U_i''(x_i) = q_i'(x_i) \geq 0$. But this is IC. $\square$

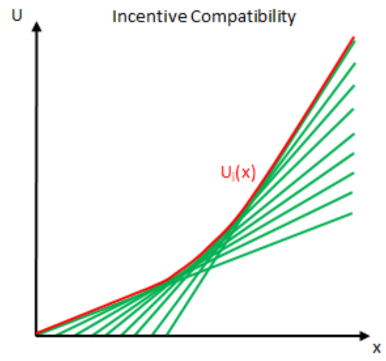
Notes

Incentive Compatibility Comments

The if part of the lemma simply amounted to the Envelope Theorem.

The expected change is payoff $U_i(x_i) - U_i(z_i)$ is independent of M.

Payoff $U_i(x_i)$ is convex as it is the upper-envelope of linear functions:



Notes

Revenue Equivalence

Theorem (Revenue Equivalence – Myerson)

If the direct mechanism (Q, M) is IC, then for any i and x_i the expected payment is

$$m_i(x_i) = m_i(0) + x_i q_i(x_i) - \int_0^{x_i} q_i(t)dt.$$

Proof.

Since $U_i(x_i) = x_i q_i(x_i) - m_i(x_i)$ and $U_i(0) = -m_i(0)$, IC implies that

$$x_i q_i(x_i) - m_i(x_i) = -m_i(0) + \int_0^{x_i} q_i(t)dt.$$

□

Thus expected payments in any two IC mechanisms with the same allocation rule are equivalent up to constant.

Notes

Individual Rationality

A mechanism is said to be **individually rational** (IR) if

$$U_i(x_i) \geq 0 \quad \text{for all } x_i.$$

When IC holds, IR simplifies to requiring

$$U_i(0) \geq 0.$$

In turn this is equivalent to

$$m_i(0) \leq 0.$$

Notes

Optimal Mechanisms

Notes

Expected Revenue

A mechanism is **optimal** if it maximizes revenues subject to IC & IR.

If the seller uses a direct mechanism expected payments are

$$E(m_i(X_i)) = m_i(0) + \int_0^{\omega_i} x_i q_i(x_i) dF_i(x_i) - \int_0^{\omega_i}\int_0^{x_i} q_i(t) dt dF_i(x_i).$$

Notice that

$$\int_0^{\omega_i}\int_0^{x_i} q_i(t) dt dF_i(x_i) = \int_0^{\omega_i}\int_t^{\omega_i} dF_i(x_i) q_i(t) dt =$$
$$= \int_0^{\omega_i} (1 - F_i(x_i)) \frac{q_i(x_i)}{f_i(x_i)} dF_i(x_i).$$

Therefore, the seller's expected revenues are

$$E(R) = \sum_{i\in N}\left[m_i(0) + \int_{\mathbb{X}} \left[x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}\right] Q_i(x) dF(x)\right].$$

Notes

Virtual Valuations

Define the **virtual valuation** as

$$\psi_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} = x_i - \frac{1}{\lambda_i(x_i)}.$$

Virtual valuation can be interpreted as marginal revenue.

Also observe that

$$\begin{aligned} E(\psi_i(X_i)) &= \int_0^{\omega_i} \left[y - \frac{1 - F_i(y)}{f_i(y)}\right] dF_i(y) \\ &= E(X_i) - \int_0^{\omega_i}\int_y^{\omega_i} f_i(x) dx dy \\ &= E(X_i) - \int_0^{\omega_i}\int_0^{x} dy f_i(x) dx \\ &= E(X_i) - \int_0^{\omega_i} x f_i(x) dx = 0. \end{aligned}$$

A design problem is **regular** if $\psi_i' > 0$. $[\lambda_i' > 0 \Rightarrow \psi_i' > 0]$

Notes

Optimal Mechanisms

The **optimal** mechanism solves

$$\max_{(Q,M)} \textstyle\sum_{i\in N} m_i(0) + \int_{\mathbb{X}} \left[\sum_{i\in N}\psi_i(x_i)\,Q_i(x)\right] dF(x)$$
$$\text{s.t.}\quad \text{(IC) } q_i'(x) \geq 0 \;\text{ and }\; \text{(IR) } \; m_i(0) \leq 0.$$

A mechanism is optimal if the following two conditions hold

$$\begin{cases} Q_i(x) > 0 \Leftrightarrow \psi_i(x_i) = \max_j \psi_j(x_j) \geq 0 & (1) \\ M_i(x) = Q_i(x)x_i - \int_0^{x_i} Q_i(t, x_{-i})dt & (2) \end{cases}$$

Observe that:

- (1) + regular $\Rightarrow$ IC,
- (2) $\Rightarrow m_i(0) = 0 \Rightarrow$ IR,
- (1-2) maximize revenue.

Notes

Optimal Mechanisms (for Regular Problems)

Define the smallest value that wins against x_{-i} by:

$$y_i(x_{-i}) = \inf\left\{z_i \middle|\, \psi_i(z_i) \geq \max\{\max_{j\neq i}\psi_j(x_j), 0\}\right\} =$$
$$= \max\{\max_{j\neq i}\psi_i^{-1}(\psi_j(x_j)), \psi_i^{-1}(0)\}.$$

Lemma (Optimal Mechanism – Myerson)

If the design problem is regular there is optimal mechanism satisfying

$$Q_i(x) = \begin{cases} 1 & \textit{if} \;\; x_i > y_i(x_{-i}) \\ 0 & \textit{if} \;\; x_i < y_i(x_{-i}) \end{cases} \qquad M_i(x) = \begin{cases} y_i(x_{-i}) & \textit{if} \;\; Q_i(x) = 1 \\ 0 & \textit{if} \;\; Q_i(x) = 0 \end{cases}$$

Proof.

By simplfying the previous expressions and since

$$\int_0^{x_i} Q_i(t, x_{-i})dt = \begin{cases} x_i - y_i(x_{-i}) & \text{if} \;\; x_i > y_i(x_{-i}) \\ 0 & \text{if} \;\; x_i < y_i(x_{-i}) \end{cases}.$$

$\square$

Notes

Optimal Mechanisms

Symmety implies: $\psi_i = \psi$ and $y_i(x_{-i}) = \max\{\max_{j \neq i} x_j, \psi^{-1}(0)\}$.

Lemma (Symmetric OM)

If the design problem is regular and symmetric then a second price auction with reserve price $r = \psi^{-1}(0)$ is an optimal mechanism.

The **optimal mechanism is inefficient** because:

1. object is unsold when $\max_j \psi_j(x_j) < 0$;
2. $x_i > y_i(x_{-i})$ doesn't imply $x_i \geq \max_{j \neq i} x_{-j}$.

OM favors disadvantaged players to motivate others to bid higher.

If F_1 hazard rate dominates F_2 [F_1 stochastically dominates F_2]

$$\lambda_1(x) \leq \lambda_2(x) \quad \Rightarrow \quad \psi_1(x) \leq \psi_2(x).$$

Notes

Efficient Mechanisms

Notes

Properties: Balanced Budget & Efficiency

Consider values in $\mathbb{X}_i = [\alpha_i, \omega_i] \subset \mathbb{R}$.

A payment rule M is said to be **balaced budget** (BB) if

$$\textstyle\sum_{i \in N} M_i(x) = 0.$$

An allocation rule Q is said to be **efficient** if

$$Q(x) \in \arg\max_{\pi \in \Delta^N} \textstyle\sum_{j \in N} \pi_j x_j.$$

For Q efficient define the maximized social values by

$$\begin{aligned} W(x) &= \textstyle\sum_{j \in N} Q_j(x) x_j, \\ W_{-i}(x) &= \textstyle\sum_{j \in N \setminus i} Q_j(x) x_j. \end{aligned}$$

A 2$^{\text{nd}}$ price auction without reserve price is efficient when $\alpha_i = 0$.

Notes

VCG Efficient Mechanism

Definition (Vickery Clarke Groves)

A VCG mechanism (Q^v, M^v) consists of an efficient allocation rule and of a payment rule that satisfies

$$M_i^v(x) = W(\alpha_i, x_{-i}) - W_{-i}(x).$$

VCG is IC since truth-telling is optimal by efficiency of Q^v,

$$\begin{aligned} & Q_i^v(z_i, x_{-i}) x_i - M_i^v(z_i, x_{-i}) \\ & = Q_i^v(z_i, x_{-i}) x_i + W_{-i}(z_i, x_{-i}) - W(\alpha_i, x_{-i}) \\ & = \textstyle\sum_{j \in N} Q_j^v(z_i, x_{-i}) x_j - W(\alpha_i, x_{-i}) \leq W(x) - W(\alpha_i, x_{-i}). \end{aligned}$$

VCG expected payoff in equilibrium is increasing and is

$$U_i^v(x_i) = E[W(x_i, X_{-i}) - W(\alpha_i, X_{-i})].$$

VCG is IR since U_i^v is increasing and $U_i^v(\alpha_i) = 0$.

Notes

VCG Maximizes Revenues

Lemma (Krishna Perry)

Among all mechanism for allocating an object that are efficient, IC and IR, the VCG mechanism maximizes the expected payment of each agent.

Proof.

Suppose also (Q^*, M^*) satisfies the three properties.

Efficiency requires that $Q^* = Q^v$.

Since both mechanisms are IC revenue equivalence theorem holds.

Thus payoffs differ by no more than a constant $c_i = U_i^*(x_i) - U_i^v(x_i)$.

By IR $c_i \geq 0$ since $U_i^v(\alpha_i) = 0$.

But $U_i^*(x_i) \geq U_i^v(x_i)$ and $Q^* = Q^v$ imply that

$E[M_i^*(x_i, X_{-i})] \leq E[M_i^v(x_i, X_{-i})]$. □

Notes

AGV Balanced Budget Mechanism

Definition (Arrow d'Aspremont Gerard-Varet AGV)

An AGV mechanism (Q^a, M^a) consists of an efficient allocation rule and of a payment rule that satisfies

$$M_i^a(x) = \tfrac{1}{N-1}\textstyle\sum_{j \neq i} E[W_{-j}(x_j, X_{-j})] - E[W_{-i}(x_i, X_{-i})].$$

AGV expected payoff in equilibrium is increasing and is

$$U_i^a(x_i) = E[W(x_i, X_{-i})] - \tfrac{1}{N-1}\textstyle\sum_{j \neq i} E[W_{-j}(X)].$$

AGV balances the budget since

$$\textstyle\sum_{i \in N} \tfrac{1}{N-1} \sum_{j \neq i} E[W_{-j}(x_j, X_{-j})] = \sum_{i \in N} E[W_{-i}(x_i, X_{-i})].$$

AGV is IC for the same reason VCG was

$$\textstyle\sum_{j \in N} Q_j^a(z_i, x_{-i}) x_j \leq \sum_{j \in N} Q_j^a(x) x_j = W(x).$$

Notes

Efficient, BB, IC, IR Mechanism

Lemma (Krishna Perry)

There exists an efficient, IC and IR mechanism that is BB iff the VCG mechanism results in a positive expected surplus.

Proof.

Previous lemma shows that VCG's positive surplus is necessary.

Sufficiency:
By IC & efficiency $\Rightarrow$ revenue equivalence $\Rightarrow$ for $m \in \{v, a\}$:

$$c_i^m = E[W(x_i, X_{-i})] - U_i^m(x_i).$$

Suppose that VCG runs surplus

$$E\left[\sum_{i \in N} M_i^v(X)\right] \geq E\left[\sum_{i \in N} M_i^a(X)\right] = 0.$$

This implies $\sum_{i \in N} c_i^v \geq \sum_{i \in N} c_i^a$. □

Notes

Proof Continued

Proof.

Define $d_i = c_i^a - c_i^v$ for $i > 1$ and $d_1 = -\sum_{i>1} d_i$.

Notice that $d_1 = \sum_{i>1}(c_i^v - c_i^a) \geq (c_1^a - c_1^v)$.

Consider the BB payment rule M^* defined by

$$M_i^*(x) = M_i^a(x) - d_i.$$

If $Q^* = Q^a$, the mechanism (Q^*, M^*) is efficient.

It is also IC since payoffs differ from (Q^a, M^a) by additive constant.

Additionally (Q^*, M^*) is IR because

$$U_i^*(x_i) = U_i^a(x_i) + d_i \geq U_i^a(x_i) + c_i^a - c_i^v = U_i^v(x_i) \geq 0.$$

□

Notes

Bilateral Trade Problem

Notes

Bilateral Trade Problem

- 1 buyer with value $V \sim F_V$ on $[\underline{v}, \overline{v}]$.
- 1 seller with costs $C \sim F_C$ on $[\underline{c}, \overline{c}]$.
- Suppose $\underline{v} < \overline{c}$ and $\underline{c} \leq \overline{v}$ [trade is sometimes beneficial].
- If production & trade occur buyer pays M_V and seller receives M_C.
- In this scenario payoffs become $V - M_V$ and $M_C - C$.
- BB requires $M_V = M_C$.
- Efficiency requires trade to take place whenever $V > C$.

Lemma (Myerson Satterthwaite)

In the bilateral trade problem, there exists no mechanism that is efficient, IC, IR & BB.

Notes

Bilateral Trade Problem

Proof.

Consider the efficient payoffs derived for any profile $x = (v, -c)$:

	$Q_V(x) = 1$	$Q_C(x) = 1$
$u_V(x)$	v	0
$u_C(x)$	$-c$	0

To characterize the VCG mechanism, note that efficiency requires:

$$Q_V^v(x) = \begin{cases} 1 & \text{if} \quad v - c > 0 \\ 0 & \text{if} \quad v - c < 0 \end{cases} = \mathbb{I}(v > c)$$

$$Q_C^v(x) = \begin{cases} 0 & \text{if} \quad v - c > 0 \\ 1 & \text{if} \quad v - c < 0 \end{cases} = \mathbb{I}(v < c)$$

□

Notes

Proof.

For such definitions, social welfare and residual social welfare amount to:

$$W(x) = (v - c)Q_V^v(x) + 0Q_C^v(x) = (v - c)\mathbb{I}(v > c) = \max\{v - c, 0\}$$
$$W_{-V}(x) = -cQ_V^v(x) \quad \& \quad W_{-C}(x) = Q_V^v(x)v$$

Hence, the VCG allocation and payments satisfy:

$$Q_V(v, c) = \mathbb{I}(v > c) = 1 - Q_C(v, c)$$

$$\begin{aligned} M_V(v, c) &= W(\underline{v}, -c) - W_{-V}(x) = (\underline{v} - c)Q_V^v(\underline{v}, -c) + cQ_V^v(x) \\ &= (\underline{v} - c)\mathbb{I}(\underline{v} > c) + c\mathbb{I}(v > c) = \mathbb{I}(v > c)\max\{c, \underline{v}\} \end{aligned}$$

$$\begin{aligned} M_C(v, c) &= W(v, -\overline{c}) - W_{-C}(x) = (v - \overline{c})Q_V^v(1, -\overline{c}) - vQ_V^v(x) \\ &= (v - \overline{c})\mathbb{I}(v > \overline{c}) - v\mathbb{I}(v > c) = -\mathbb{I}(v > c)\min\{v, \overline{c}\} \end{aligned}$$

□

Notes

Proof.

The VCG expected payoff of the buyer of claiming to be z amounts to:

$$
\begin{aligned}
E[(v - \max\{C, 1\})\mathbb{I}(z > C)] &= \int_0^z v dF_C(c) - \int_0^z \max\{c, \underline{v}\}\, dF_C(c) \\
&= \int_0^z v dF_C(c) - \int_0^{\underline{v}} \underline{v} dF_C(c) - \int_{\underline{v}}^z c dF_C(c) \\
&= vF_C(z) - \underline{v}F_C(\underline{v}) - \int_{\underline{v}}^z c dF_C(c).
\end{aligned}
$$

It is efficient. It is IC since truthtelling is optimal,

$$
E[(v - \max\{C, \underline{v}\})\mathbb{I}(z > C)] = vF_C(z) - \underline{v}F_C(\underline{v}) - \int_{\underline{v}}^z c dF_C(c)
$$

$$
[\text{FOC}] \quad (v - z) f_C(z) \leq 0 \quad \Rightarrow \quad v = z.
$$

It is IR since $U_V^v(v) \geq 0$ and $U_C^v(c) \geq 0$ for any v & c.

It is not BB since $\max\{c, \underline{v}\} - \min\{v, \overline{c}\} < 0$ if $v > c$.

Since VCG is revenue maximizing amongst efficient, IC and IR mechanisms, the claim holds (prove as in Krishna Perry first lemma). □

Notes

Notes

Interdependent Value Auctions

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Notes

Interdependent Value Models

Notes

Interdependent Values

- Each bidder has some information about the value of the object.
- The information of any player affects valuations for all others.
- Let $X_i \in [0, \omega_i]$ denote player i's signal and $X = (X_1, ..., X_N)$.
- Player i's value V_i depends on all signals

$$V_i = v_i(X).$$

- Assume v_i is non-decreasing in X_j, increasing in X_i & 2-differentiable.
- In general not all information revealed by signals & values are

$$E[V_i | X = x] = v_i(x).$$

- Assume $v_i(0, ..., 0) = 0$, $E(V_i) < \infty$ and risk neutrality.

Notes

Common Values

- In one extreme case of the model values are private if $v_i(X) = X_i$.
- In the opposite extreme valuations are common (CV) if

$$v_i(X) = v(X) = V.$$

- The ex-post value is common to all and unknown to anyone.
- A special case commonly used has:
  - $V \sim F$,
  - $X_i | V \overset{\text{iid}}{\sim} H(V)$,
  - $E[X_i | V = v] = v$.
- This model is used to describe oil drilling lease auctions. [Mineral Rights]

Notes

The Winner's Curse

- Consider the special case of the CV model.
- Assume bidders play a 1^{st} price auction.
- i's estimate of value before the auction is $E[V|X_i = x_i]$.
- But if he is announced winner his estimate becomes

$$E[V|X_i = x_i, Y_1 < x_i] < E[V|X_i = x_i].$$

- Though each signal is unbiased $E[X_i|V = v] = v$, the max is not

$$E[\max X_i|V = v] > \max E[X_i|V = v] = v.$$

- The **winner's curse** consist of the failure to account for this effect.
- Shading the bids prevents this from happening in equilibrium.

Notes

Affiliated Signals

- Let $x \vee x' = \{\max\{x_i, x_i'\}\}_{i \in N}$ and $x \wedge x' = \{\min\{x_i, x_i'\}\}_{i \in N}$.
- Signals $(X_1, ..., X_N)$ are distributed according to a joint density f.
- $(X_1, ..., X_N)$ are **affiliated** if f satisfies for any $x, x' \in \mathbb{X}$

$$f(x \vee x')f(x \wedge x') \geq f(x)f(x').$$

- Affiliation implies that $\partial^2 \ln f / \partial x_i \partial x_j \geq 0$ if $i \neq j$ and that:
  1. $(X_1, Y_1, ..., Y_{N-1})$ are affiliated for $Y_k = (k\text{-highest variable in } X_{-1})$;
  2. for $Y_1|X_1 \sim G(X_1)$ if $x_1 > x_1'$

$$g(y|x_1)/G(y|x_1) \geq g(y|x_1')/G(y|x_1');$$

  3. if γ is an increasing function and $x_1 > x_1'$

$$E[\gamma(Y_1)|X_1 = x_1] \geq E[\gamma(Y_1)|X_1 = x_1'].$$

Notes

Symmetric Model

Start the analysis of interdependent values by assuming **symmetry**:

- Valuation map is common to all players

  $$v_i(X) = u(X_i, X_{-i}).$$

  and symmetric in X_{-i}. Thus for any permutation πX_{-i} of X_{-i}

  $$u(X_i, X_{-i}) = u(X_i, \pi X_{-i}).$$

- Joint density f is symmetric in X on $[0, \omega]^N$ and signals are affiliated.
- Expected value to bidder i with signal x facing a highest signal of y is

  $$v(x, y) = E[u(X_i, X_{-i}) | X_i = x,\ Y_1 = y].$$

  It is non-decreasing in y, increasing in x and $v(0, 0) = 0 = u(0)$.

Notes

Second Price Auction

Notes

Second Price Auction

Lemma (Milgrom Weber)

Symmetric equilibrium strategies in 2^{nd} price auction are $\beta^{\mathrm{II}}(x) = v(x,x)$.

Proof.

Let any $j \neq 1$ use $\beta = \beta^{\mathrm{II}}$, payoffs of player 1 type x of bidding b are

$$\Pi(b|x) = \int_0^{\beta^{-1}(b)} [v(x,y) - \beta(y)]\,dG(y|x).$$

For $Y_1 = \max_{j\neq 1} X_j$, $Y_1|X_1 \sim G(X_1)$ and $\beta(x) = v(x,x)$.

For $\psi(b) = \beta^{-1}(b)$, FOC for b imply that

$$[v(x,\psi(b)) - v(\psi(b),\psi(b))]g(\psi(b)|x)\psi'(b) = 0.$$

Which in turn implies $x = \psi(b)$ or equivalently $b = \beta(x)$. $\square$

Notes

Remarks on Second Price Symmetric Equilibrium

- β^{II} is such that bidder 1 with value x breaks even by winning against a highest bid $\beta^{\mathrm{II}}(x)$ since

$$E(V_1|X_1 = Y_1 = x) = v(x,x) = \beta^{\mathrm{II}}(x).$$

- β^{II} is not a weakly dominant strategy.
- β^{II} is the unique symmetric equilibrium.
- But there maybe different asymmetric equilibria [even in the symmetric case].
- IPV: 2^{nd} price and English auctions are strategically equivalent.
- Not anymore since dropping out reveals information about valuations.

Notes

English Auction

Notes

English Auction [Open Ascending Price]

The format of English Auction considered here has:

- the auctioneer raising the price gradually starting at 0;
- bidders leaving the room when they stop participating in the auction;
- bidders not allowed to reenter the auction once they quit;
- all bidders in the room observing at which price exits occur;
- auction ending when one bidder remains;
- the price paid being that of the last exit.

The equilibrium discussed next is robust to a change in format that allows players to call out prices [Avery 1998].

Notes

Strategies in an English Auction

- Strategy tell when to drop out depending on several variables:
  1. Individual signal;
  2. Number of players in the room;
  3. Prices at which the remaining players left the room.
- Consider the following strategy for $k < N$:

$$\beta^N(x) = u(x, ..., x),$$
$$\beta^k(x, p_{k+1}, ..., p_N) = u(x, ..., x, x_{k+1}, ..., x_N), \qquad (1)$$

  for x_{k+1} defined by $\beta^{k+1}(x_{k+1}, p_{k+2}, ..., p_N) = p_{k+1}$.
- E.g. $\beta^{N-1}(x, p_N) = u(x, ..., x, x_N)$ for $\beta^N(x_N) = p_N$.
- Which can be done since $\partial\beta^k/\partial x > 0$.

Notes

Symmetric Equilibrium of English Auction

Lemma (Milgrom Weber)

Symmetric ex-post equilibrium strategies in the English auction satisfy (1).

Proof.

Consider bidder 1 with signal $X_1 = x$ facing
$Y_1 = y_1 > ... > Y_{N-1} = y_{N-1}$.

If $x > y_1$ one wins the object and pays the price of the last dropout:

$$p_2 = \beta^2(y_1, p_3, ..., p_N) = u(y_1, y_1, ..., y_{N-1}).$$

One cannot affect price $\Rightarrow$ he prefers winning since $x > y_1$ implies

$$u(x, y_1, ..., y_{N-1}) - u(y_1, y_1, ..., y_{N-1}) > 0.$$

If $x < y_1$ he loses but is better off since the reverse inequality holds.

So, β is independent of f, and is an ex-post equilibrium. $\square$

Notes

First Price Auction

Notes

First Price Auction

Lemma (Milgrom Weber)

Symmetric equilibrium strategies in 1st price auction are

$$\beta^{\mathrm{I}}(x) = \int_0^x v(y,y)dL(y|x) \quad \textit{for} \quad L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)}dt\right).$$

Proof.

[Proof 1/3] (1) $L(x)$ is a distribution on $[0,x]$:

By affiliation $g(t|t)/G(t|t) \geq g(t|0)/G(t|0)$ and

$$-\int_0^x \frac{g(t|t)}{G(t|t)}dt \leq -\int_0^x \frac{g(t|0)}{G(t|0)}dt = -\int_0^x \frac{d\ln G(t|0)}{dt}dt = -\infty.$$

Taking exponentials $\Rightarrow L(0|x) = 0$ and notice $L(x|x) = \exp(0) = 1$.

Finally $\partial L(y|x)/\partial y = L(y|x)g(y|y)/G(y|y) \geq 0$ □

Notes

Proof.

[Proof 2/3] (2) β^{I} is increasing in x:

Notice that $\partial L(y|x)/\partial x = -L(y|x)g(x|x)/G(x|x) \leq 0$.

Additionally $\partial v(y,y)/\partial y > 0$ is increasing by assumption.

Integrating by parts β^{I} get

$$\beta^{\mathrm{I}}(x) = \int_0^x v(y,y)dL(y|x) = v(x,x) - \int_0^x L(y|x)dv(y,y).$$

Therefore:

$$\frac{\partial \beta^{\mathrm{I}}(x)}{\partial x} = -\int_0^x \frac{\partial L(y|x)}{\partial x}\frac{\partial v(y,y)}{\partial y}dy \geq 0.$$

(3) $\beta = \beta^I$ is optimal if followed by others:

Since β is increasing the payoff of a bid $\beta(z)$ after observing x is

$$\Pi(z|x) = \int_0^z (v(x,y) - \beta(z))dG(y|x).$$

Notes

Proof.

[Proof 3/3] First order conditions require

$$\frac{\partial \Pi(z|x)}{\partial z} = (v(x,z) - \beta(z))g(z|x) - \beta'(z)G(z|x) = 0.$$

Notice that

$$\begin{aligned}\beta'(x) &= -\int_0^x \frac{\partial L(y|x)}{\partial x}dv(y,y) = \frac{g(x|x)}{G(x|x)}\int_0^x L(y|x)dv(y,y) \\ &= \frac{g(x|x)}{G(x|x)}[v(x,x) - \beta(x)].\end{aligned}$$

Thus $x > z \;\Rightarrow\; v(x,z) > v(z,z)$ & $g(z|x)/G(z|x) > g(z|z)/G(z|z)$

$$\frac{\partial \Pi(z|x)}{\partial z} > \left[(v(z,z) - \beta(z))\frac{g(z|z)}{G(z|z)} - \beta'(z)\right]G(z|x) = 0.$$

If $x < z$ similar argument shows that $\partial \Pi(z|x)/\partial z < 0$

Therefore optimality requires that $x = z$

Notes

Revenue Comparisons

Notes

Revenues: English vs Second Price Auction

Lemma

Revenues in an English auction are not smaller than those of a 2^{nd} price.

Proof.

First notice that by affiliation if $x > y$

$$\begin{aligned} v(y,y) &= E[u(X)|X_1 = Y_1 = y] = E[u(Y_1, Y_1, ..., Y_{N-1})|X_1 = Y_1 = y] \\ &\leq E[u(Y_1, Y_1, ..., Y_{N-1})|X_1 = x, Y_1 = y]. \end{aligned}$$

Thus revenues in 2nd price auction are weakly smaller

$$\begin{aligned} E[R^{\mathrm{II}}] &= E[\beta^{\mathrm{II}}(Y_1)|X_1 > Y_1] = E[v(Y_1, Y_1)|X_1 > Y_1] \\ &\leq E[E[u(Y_1, Y_1, ..., Y_{N-1})|X_1 = x, Y_1 = y]|X_1 > Y_1] \\ &= E[u(Y_1, Y_1, ..., Y_{N-1})|X_1 > Y_1] \\ &= E[\beta^2(Y_1, ..., Y_{N-1})] = E[R^{\mathrm{En}}]. \end{aligned}$$

Notes

Revenues: First vs Second Price Auction

Lemma

Revenues in a 2^{nd} price auction are not smaller than those of a 1^{st} price.

Proof.

Consider the distribution $K(y|x) = G(y|x)/G(x|x)$.
Note that $y < x$ implies $L(y|x) \geq K(y|x)$ by affiliation since

$$-\int_y^x \frac{g(t|t)}{G(t|t)}dt \geq -\int_y^x \frac{g(t|x)}{G(t|x)}dt = -\int_y^x \frac{d\ln G(t|x)}{dt}dt = \ln\left(\frac{G(y|x)}{G(x|x)}\right).$$

Probability of winning conditional on x is the same in 1^{st} & 2^{nd} price.
But expected payments conditional on x differ [integration by parts]:

$$E[\beta^{\mathrm{II}}(Y_1)|Y_1 < X_1 = x] - \beta^{\mathrm{I}}(x) = \int_0^x v(y,y)d[K(y|x) - L(y|x)] =$$
$$= \int_0^x [L(y|x) - K(y|x)]dv(y,y) \geq 0.$$

Notes

Equilibrium Bidding and the Winner's Curse

Corollary

With symmetric, interdependent values and affiliated signals expected revenues can be ranked as follows: $E[R^{\mathrm{En}}] \geq E[R^{\mathrm{II}}] \geq E[R^{\mathrm{I}}]$.

In 1^{st} price auction players shade their bids to account for winner's curse,

$$\beta^{\mathrm{I}}(x) = \int_0^x v(y,y)dL(y|x) \leq \int_0^x v(y,y)dK(y|x)$$
$$< \int_0^x v(x,y)dK(y|x) = E[V_1|Y_1 < X_1 = x].$$

In a 2^{nd} price auction players shade to account for winner's curse,

$$E[\beta^{\mathrm{II}}(Y_1)|Y_1 < X_1 = x] = \int_0^x v(y,y)dK(y|x) < E[V_1|Y_1 < X_1 = x].$$

Notes

Efficiency

- Auctions studied gave object to highest signal no to highest value.
- But this can be inefficient. Eg: $v_i(x) = x_i + 2x_j$ for $i \neq j \in \{1, 2\}$.
- Valuations satisfy the **single crossing condition** (SCC) if for any $i \neq j$ and any x
$$\frac{\partial v_i(x)}{\partial x_i} \geq \frac{\partial v_j(x)}{\partial x_i}.$$
- In the symmetric case $v_i(x) = u(x_i, x_{-i})$.
- Let u'_j denote the partial wrt j^{th} argument.
- Thus SCC reduces to $u'_1 \geq u'_2$ by symmetry assumption.
- SCC ensures that order of the signals reflects that of the values.

Notes

Lemma (Milgrom Weber)

With symmetric interdependent values and affiliated signals, if SCC holds then all 3 auction formats have an efficient symmetric equilibrium.

Proof.

Consider $x_i > x_j$ and let $\alpha(t) = t(x_i, x_j, x_{-ij}) + (1-t)(x_j, x_i, x_{-ij})$.

By the fundamental theorem of calculus and SCC,

$$u(x_i, x_j, x_{-ij}) = u(x_j, x_i, x_{-ij}) + \int_0^1 \nabla u(\alpha(t))\alpha'(t)dt.$$

By SCC and the definition of gradient,

$$\begin{aligned}\nabla u(\alpha(t))\alpha'(t) &= u'_1(\alpha(t))(x_i - x_j) + u'_2(\alpha(t))(x_j - x_i) \\ &= (u'_1(\alpha(t)) - u'_2(\alpha(t)))(x_i - x_j) \geq 0.\end{aligned}$$

Therefore, $x_i > x_j$ implies $u(x_i, x_j, x_{-ij}) \geq u(x_j, x_i, x_{-ij})$. $\square$

Notes

The Linkage Principle

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Linkage Principle

Notes

The Linkage Principle

- Let values be interdependent and symmetric.
- Let A be a standard auction with a symmetric equilibrium β^A.
- $W^A(z|x)$ = expected payment of winner with signal x & bid $\beta^A(z)$.
- Let $W_j^A(z|x)$ the partial derivative wrt $j \in \{z, x\}$.
- Notice that $\partial W^A(x|x)/\partial x = W_x^A(x|x) + W_z^A(x|x)$.

Theorem (Linkage Principle – Milgrom Weber)

Let A and B be two auctions in which the highest bidder wins and only he pays a positive amount.

Suppose that each has a symmetric and increasing equilibrium such that: (i) $W_x^A(x|x) \geq W_x^B(x|x)$ for any x and (ii) $W^A(0|0) = W^B(0|0) = 0$.

Then the expected revenue in A is not smaller than that of B.

Notes

Proof.

In auction $c \in \{A, B\}$ bidder 1 with signal x maximizes

$$\int_0^z v(x,y)dG(y|x) - W^c(z|x)G(z|x).$$

Optimality at $z = x$ requires that

$$v(x,x)g(x|x) - W^c(x|x)g(x|x) - W_z^c(x|x)G(x|x) = 0$$
$$\Rightarrow W_z^c(x|x) = [v(x,x) - W^c(x|x)]\frac{g(x|x)}{G(x|x)}.$$

Let $\Delta(x) = W^A(x|x) - W^B(x|x)$ and notice that

$$\begin{aligned}\Delta'(x) &= [W_x^A(x|x) - W_x^B(x|x)] + [W_z^A(x|x) - W_z^B(x|x)] \\ &= [W_x^A(x|x) - W_x^B(x|x)] - \Delta(x)[g(x|x)/G(x|x)].\end{aligned}$$

So, $\Delta(0) = 0$ & $[\Delta(x) \leq 0 \Rightarrow \Delta'(x) \geq 0]$ imply $\Delta(x) \geq 0$ any x. $\square$

Notes

Remarks on the Linkage Principle

Example: LP implies $E[R^{\mathrm{II}}] \geq E[R^{\mathrm{I}}]$ since by affiliation

$$W^{\mathrm{I}}(z|x) = \beta^{\mathrm{I}}(z)$$
$$W^{\mathrm{II}}(z|x) = E[\beta^{\mathrm{II}}(Y_1)|X_1 = x, Y_1 < z]$$
$$\Rightarrow W_x^{\mathrm{II}}(z|x) \geq W_x^{\mathrm{I}}(z|x) = 0.$$

Corollary

If LP assumptions hold and signals are independently distributed then:

$$E[R^A] = E[R^B].$$

Proof.

If signals are independent only the bid determines the payment

$$W_x^A(x|x) = W_x^B(x|x) = 0.$$

□

Notes

Applications

Notes

Public Information

- Suppose seller has some information S affecting valuations
$$V_i = v_i(S, X).$$
- Assume $v_i(0) = 0$ and $v_i(S, X) = u(S, X_i, X_{-i})$ symmetric in X_{-i}.
- Also assume $(S, X_1, ..., X_N)$ to be affiliated.
- If information is not revealed again let
$$v(x, y) = E[V_1 | X_1 = x, Y_1 = y].$$
- If information is revealed instead let
$$\overline{v}(s, x, y) = E[V_1 | S = s, X_1 = x, Y_1 = y].$$
- Clearly $\overline{v}$ is increasing in all variables and $\overline{v}(0) = 0$.

Notes

Public Information in a First Price Auction

- If information is not revealed again $W^{\mathrm{I}}(z|x) = \beta^{\mathrm{I}}(z)$.
- If information is revealed, there exists symmetric equilibrium

$$\overline{\beta}^{\mathrm{I}}(s, x) = \int_0^x v(s, y, y) d\overline{L}(y|s, x) \;\; \& \;\; \overline{L}(y|s, x) = \exp\left(-\int_y^x \frac{g(t|s,t)}{G(t|s,t)} dt\right).$$

- The expected payment in this scenario becomes
$$\overline{W}^{\mathrm{I}}(z|x) = E[\overline{\beta}^{\mathrm{I}}(S, z) | X_1 = x].$$
- Thus by affiliation $\overline{W}_x^{\mathrm{I}}(z|x) \geq W_x^{\mathrm{I}}(z|x) = 0$.
- LP implies that expected revenues are higher if info is revealed.
- Similar arguments for English and 2[nd] price auctions work.

Notes

Alternative Linkage Principle (LPA)

$M^A(z|x)$ = expected payment of player with signal x & bid $\beta^A(z)$

Theorem (All Pay Linkage Principle – Krishna Morgan)

Let A and B be two auctions in which the highest bidder wins.

Suppose that each has a symmetric and increasing equilibrium such that: (i) $M_x^A(x|x) \geq M_x^B(x|x)$ for any x and (ii) $M^A(0|0) = M^B(0|0) = 0$.

Then the expected revenue in A is not smaller than that of B.

Proof.

In auction $c \in \{A, B\}$ bidder 1 with signal x maximizes

$\int_0^z v(x,y)dG(y|x) - M^c(z|x)$.

Optimality at $z = x$ requires $M_z^c(x|x) = v(x,x)g(x|x)$.

$\Delta(x) = M^A(x|x) - M^B(x|x) \;\Rightarrow\; \Delta'(x) = [M_x^A(x|x) - M_x^B(x|x)] \geq 0$.

Thus $\Delta(0) = 0$ implies $\Delta(x) \geq 0$ any x. □

Notes

All Pay Auctions or Lobbying

- LPA does not require that losers do not pay.
- In an all pay auction everyone pays his bid
$$M^{\mathrm{AP}}(z|x) = \beta^{\mathrm{AP}}(z).$$
- In a first price auction instead $M^{\mathrm{I}}(z|x) = \beta^{\mathrm{I}}(z)G(z|x)$.
- Thus, since by affiliation $G_x(z|x) \leq 0$,
$$M_x^{\mathrm{I}}(x|x) \leq M_x^{\mathrm{AP}}(x|x) = 0.$$
- By theorem LPA thus: $E[R^{\mathrm{AP}}] \geq E[R^{\mathrm{I}}]$.
- The symmetric equilibrium of the all-pay auction is
$$\beta^{\mathrm{AP}}(x) = \int_0^x v(y,y)g(y|y)dy.$$

Notes

Bidding Rings
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Notes

Introduction

Bidding rings are cartels to manipulate the outcomes of the auction.

Investigating collusion in auctions is substantial antitrust activity.

Basic Setup

Consider the asymmetric IPV model for $X_i \sim F_i$ on $[0, \omega]$.

Let $I \subseteq N$ be the set of buyers in the bidding ring.

and $N \backslash I$ be the set of buyers not in the bidding ring.

Let $Y_1^S = \max_{i \in S} X_i \sim G^S = \prod_{i \in S} F_i$ for any subset of players $S \subseteq N$.

The ring wants the object to go to the member valuing it the most.

Notes

Collusion in a Second Price Auction

Notes

Collusion in a Second Price Auction

- Assume the ring works and members reveal information.
- Ring produces profits to members by reducing price competition.
- Ring submits one bid to the auctioneer $[Y_1^I = X_i$ some $i \in I]$.
- Conditional on ring winning the payment becomes smaller

$$\overline{P}_I = \max\left\{Y_1^{N\setminus I}, r\right\} \leq \max\left\{Y_1^{N\setminus i}, r\right\} = P_i.$$

- Expected payments conditional x are smaller when the ring is in place.
- Surplus made by ring buyer $i \in I$ conditional on value x is

$$\delta_i(x) = rG^{N\setminus i}(r) + \int_r^x ydG^{N\setminus i}(y) - [rG^{N\setminus I}(r) + \int_r^x ydG^{N\setminus I}(y)]G^{I\setminus i}(x).$$

- $\delta_i(x) = m_i(x) - \overline{m}_i(x) \geq 0$ since payments are always lower.

Notes

Gains and Losses from Collusion

- The ex-ante expected profits from the ring become

$$\delta_I = \sum_{i\in I} E[\delta_i(x_i)].$$

- Bidders not in the ring do not benefit or suffer since for $i \in N\backslash I$:

$$\overline{P}_i = \max\left\{Y_1^I, Y_1^{N\backslash I\backslash i}, r\right\} = \max\left\{Y_1^{N\backslash i}, r\right\} = P_i.$$

- So $\delta_i(x) = 0$ any $i \in N\backslash I$ [ring exerts no externality on outsiders].
- Thus gains by the ring correspond to losses by the seller.
- Increasing the ring's size to $J \supset I$ benefits members at expense of seller since

$$\overline{P}_J = \max\left\{Y_1^{N\backslash J}, r\right\} \leq \max\left\{Y_1^{N\backslash I}, r\right\} = \overline{P}_I.$$

Notes

Efficient Collusion

An **efficient collusion** mechanism:

I. allocates the right to represent the ring efficiently and is IC;
II. determines payments made by the ring [possibly negative].

Lemma (Mailath Zemsky)

In a 2^{nd} price auction with reserve price, the expected gains from efficient collusion of any ring member depend on the identity, but not on his value. Moreover the ring does not affect outsiders.

Proof.

Allocation in 2^{nd} price auction satisfies $Q_i(X) = \Pr(X_i > \max\{Y_1^{N\backslash i}, r\})$.

With efficient collusion $(\overline{Q}, \overline{M})$ allocation remains the same $\overline{Q} = Q$.

Since both mechanisms are IC and $\overline{Q} = Q$, revenue equivalence implies:

$\delta_i = m_i(x_i) - \overline{m}_i(x_i) \geq 0.$ □

Notes

Pre-Auction Knockouts (PAKT)

Ring conducts an auction to determine who represents the ring [**ticket**].

Members submit bids, the highest is awarded the ticket.

Conditional on winning the auction he pays to the ring $\tau_i = P_i - \overline{P}_I$.

Ring makes payments of δ_i (defined above) to each bidder.

Such mechanism is efficient, ex-post IC, IR, BB, but **not** ex-post BB.

It isn't ex-post BB since it makes payments δ_i even when it loses.

If VCG is conducted to allocate ticket it runs a surplus.

Thus there exists an efficient, IC, IR, ex-post BB mechanism.

Such mechanism is **not** ex-post IC however [Krishna Perry Lemma].

Notes

Collusion and Reserve Prices

Notes

Reserve Prices when facing Collusion

If seller knows of ring I what reserve price should he set?

- The seller faces $N-I+1$ buyers with values $\{Y_1^I, X_{I+1}, ..., X_N\}$.
- $Z^I \overset{\text{cdf}}{\sim} H^I$ denotes the second highest of such values.
- If $Y_1^N > r$ the seller receives $\overline{P} = \max\{Z^I, r\}$.
- Therefore the expected selling price is

$$r(H^I(r) - G^N(r)) + \int_r^\omega y dH^I(y).$$

- Optimality of the reserve price r^I requires

$$H^I(r^I) - G^N(r^I) - r^I g^N(r^I) = 0.$$

Notes

Lemma (Mailath Zemsky)

The addition of a bidder to a bidding ring causes the optimal reserve price for the seller to weakly increase.

Proof.

Consider a ring with $J = I+1$ members.

The seller faces $N-I$ buyers with values $\{Y_1^J, X_{I+2}, ..., X_N\}$.

If $Z^J \overset{\text{cdf}}{\sim} H^J$ is second highest of such values then $Z^J \neq Z^I$ iff

$$\begin{aligned} &(1)\ \ Z^I = X_{I+1} > Y_1^{N\setminus J} && \Rightarrow \quad Z^I > Y_1^{N\setminus J} = Z^J, \\ &(2)\ \ Z^I = Y_1^I < X_{I+1} && \Rightarrow \quad Z^I > Y_1^{N\setminus J} = Z^J. \end{aligned}$$

But $Z^J \leq Z^I$ implies $H^J(r) \geq H^I(r)$ for any r and thus

$$H^J(r^I) - G^N(r^I) - r^I g^N(r^I) \geq 0,$$

which implies that $r^J \geq r^I$. □

Notes

Uniform Example: Bidding Ring

- Two bidders with values uniformly distributed in $[0, 1]$.
- Without ring the optimal reserve price is

$$\begin{aligned} & NF^{N-1}(r)(1-F(r)) - rNf(r)F^{N-1}(r) = 0 \\ \Rightarrow \quad & 2r(1-r) - 2r^2 = 0 \quad \Rightarrow \quad r = 1/2. \end{aligned}$$

- With an all-inclusive ring revenues of the seller become

$$r(1-F^N(r)) = r(1-r^2).$$

- Optimality then requires

$$1 - 3r^2 = 0 \Rightarrow r = 1/\sqrt{3} > 1/2.$$

- In a 1$^{\text{st}}$ price auction collusion is less tractable and **not self-enforcing,** but results exist for the symmetric case.

Notes

Notes

Mechanism Design with Interdependent Values

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Notes

Introduction

- Consider the setup with interdependent values and affiliated signals.
- Revelation principle applies also in this context.
- Valuations v satisfy the **single crossing condition,** if $\forall i \neq j$

$$\partial v_i(x)/\partial x_i > \partial v_j(x)/\partial x_i$$

at any x such that $v_i(x) = v_j(x) = \max_{k \in N} v_k(x)$.

- A direct mechanism consists of two functions

$$Q : \mathbb{X} \to \Delta^N \quad \& \quad M : \mathbb{X} \to \mathbb{R}^N.$$

- The analysis proceeds as follows:
  1. Efficient direct mechanisms;
  2. Revenue maximizing direct mechanisms.

Notes

Efficient Mechanism

Notes

Without SCC no Mechanism may be Efficient

- Consider a scenario in which $N = 2$ and

$$v_1(x) = x_1 \text{ and } v_2(x) = x_1^2.$$

- Suppose $\mathbb{X}_1 = [0, 2]$ and $\mathbb{X}_2 = c$ [wlog].
- Clearly SCC is violated since

$$\partial v_1(1, c)/\partial x_1 = 1 < 2 = \partial v_2(1, c)/\partial x_1.$$

- Suppose that mechanism (Q, M) is efficient.
- Efficiency and IC for player 1 require that at $y < 1 < z$

$$y - M_1(y, c) \geq 0 - M_1(z, c),$$
$$0 - M_1(z, c) \geq z - M_1(y, c).$$

- Which implies $y \geq z$ a contradiction!

Notes

SCC is necessary for Efficiency

- Suppose there is an efficient mechanism with an ex-post equilibrium [$\Rightarrow$ there is a direct efficient mechanism with an ex-post equilibrium].
- Given x_{-i} if player i wins or loses regardless of x_i then SCC holds.
- Buyer i is **pivotal** at x_{-i} if there exists y and z such that

$$v_i(y, x_{-i}) > \max_{j \neq i} v_j(y, x_{-i}),$$
$$v_i(z, x_{-i}) < \max_{j \neq i} v_j(z, x_{-i}).$$

- Buyer i's ex-post IC at y and z requires that

$$v_i(y, x_{-i}) - M_i(y, x_{-i}) \geq -M_i(z, x_{-i}),$$
$$-M_i(z, x_{-i}) \geq v_i(z, x_{-i}) - M_i(y, x_{-i}).$$

Notes

SCC is necessary for Efficiency

- Rewriting implies

$$v_i(y, x_{-i}) \geq M_i(y, x_{-i}) - M_i(z, x_{-i}) \geq v_i(z, x_{-i}).$$

- Ex-post IC implies mechanism is monotonic in values [if i wins with signal x_i he wins with any signal bigger than x_i].
- Efficiency $\Rightarrow$ i's value remains the highest if signal x_i increases.
- Thus if $v_i(x) = v_j(x)$ a change ∂x_i benefits more i than j,

$$\partial v_i(x)/\partial x_i \geq \partial v_j(x)/\partial x_i.$$

- Therefore SCC is necessary.

Notes

Generalized VCG Mechanism

- Consider a direct mechanism (Q^v, M^v) satisfying

$$Q_i^v(x) = \begin{cases} 1 & \text{if} \quad v_i(x) > \max_{j\neq i} v_j(x) \\ 0 & \text{if} \quad v_i(x) < \max_{j\neq i} v_j(x) \end{cases}$$

$$M_i^v(x) = \begin{cases} v_i(y_i(x_{-i}), x_{-i}) & \text{if} \quad Q_i^v(x) = 1 \\ 0 & \text{if} \quad Q_i^v(x) = 0 \end{cases}$$

for $y_i(x_{-i}) = \inf\{z | v_i(z, x_{-i}) \geq \max_{j\neq i} v_j(z, x_{-i})\}$.

- Such mechanism awards object efficiently & generalizes VCG.
- It induces truthtelling by changing payments $v_i(y_i(x_{-i}), x_{-i})$.

Lemma (Cremer McLean)

If valuations v satisfy SCC, truthtelling is an ex-post equilibrium of the VCG mechanism (Q^v, M^v).

Notes

SCC and the Generalized VCG

Cramer McLean Proof.

Suppose all players but i are truthful.

(I) If $v_i(x) > \max_{j\neq i} v_j(x) \Rightarrow x_i > y_i(x_{-i})$.

If i reports a signal $z_i > y_i(x_{-i})$ he wins and pays $v_i(y_i(x_{-i}), x_{-i}) < v_i(x)$.

since by SCC $v_i(z_i, x_{-i}) > \max_{j\neq i} v_j(z_i, x_{-i})$.

If i reports a $z_i \leq y_i(x_{-i})$ his surplus is zero by SCC.

No deviation $z_i \neq x_i$ is profitable.

(II) If instead $v_i(x) < \max_{j\neq i} v_j(x) \Rightarrow x_i < y_i(x_{-i})$.

For i to win by SCC he would need to bid $z_i > y_i(x_{-i}) > x_i$.

But payments would become $v_i(y_i(x_{-i}), x_{-i}) > v_i(x)$.

Again no deviation is profitable. □

Notes

Comments on Generalized VCG

- With private values this reduces to ordinary 2^{nd} price auction.
- With two buyers to the efficient equilibrium of the English Auction.
- The VCG mechanism is not detail free!
- It presumes that the auctioneer knows the valuation maps.

Notes

Optimal Mechanism

Notes

Optimal Mechanisms

If buyers' info is correlated they are unable to gain any rents from that info!

For simplicity consider a setup with discrete signals

$$\mathbb{X}_i = \{0, \Delta, ..., (t_i - 1)\Delta\}.$$

Π denotes the joint probability distribution on $\mathbb{X}$.

$\Pi_i = [\pi(x_{-i}|x_i)]$ is i's beliefs matrix with t_i rows and $\times_{i \neq j} t_j$ columns.

If signals are independent all rows are equal $rank(\Pi_i) = 1$.

Again assume $v_i(0) = 0$ and $v_i(x_j + \Delta, x_{-j}) \geq v_i(x)$ strict if $i = j$.

SCC in this setup becomes for any $i \neq j$

$$v_i(x) \geq v_j(x) \quad \Rightarrow \quad v_i(x_i + \Delta, x_{-j}) \geq v_j(x_i + \Delta, x_{-j}).$$

Notes

Theorem (Cremer McLean)

If signals are discrete, valuations v satisfy SCC and Π_i is of full rank for any i, then there exist a mechanism in which truthtelling is the efficient ex-post equilibrium in which all buyers get an expected payoff of zero.

Proof.

Consider the truthtelling expected payoff of the VCG mechanism

$$U_i^v(x_i) = \sum_{x_{-i} \in \mathbb{X}_{-i}} [v_i(x) Q_i^v(x) - M_i^v(x)] \pi(x_{-i}|x_i).$$

U_i^v denotes the t_i-vector, since Π_i is of full rank $\exists c_i = [c_i(x_{-i})]_{x_{-i} \in \mathbb{X}_{-i}}$:

$$\Pi_i c_i = U_i^v.$$

Consider mechanism (Q^v, M^c) with payments $M_i^c(x) = M_i^v(x) + c_i(x_{-i})$.

Truthtelling is ex-post IC as in VCG since payoffs differ by constant.

and interim expected payoffs are zero for all buyers by construction. □

Notes

Comments On Cremer McLean Mechanism

Interim IR always holds with equality since

$$U_i^c(x_i) = \sum_{x_{-i} \in \mathbb{X}_{-i}} [v_i(x) Q_i^v(x) - M_i^v(x) - c_i(x_{-i})] \pi(x_{-i}|x_i) = 0.$$

Ex-post IR is not met since $Q_i^v(x) = 0$ & $M_i^c(x) > 0$ can hold at once.

CM interim expected payment is VCG's plus that of a lottery one must participate in $c_i(X_{-i})$.

If values are private CM payment is 2^{nd} price payment plus $c_i(x_{-i})$.

(I) Optimality result hinged on the correlation amongst signals, but not on interdependence in values.

(II) Efficiency result hinged on the interdependence in values, but not on correlation amongst signals.

Notes

Notes

Note 1: Entry Fees Francesco Nava

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This note explains the comparison between entry fees and reserve prices. On page 24 of the textbook, it is claimed that a reserve price r is equivalent to an entry fee:

$$e = rG(r)$$

For sake of simplicity consider a second price auction. In such auction it remains dominant to bid one's valuation even if some players are not participating in the auction, because conditional on winning players cannot affect the trading price. Thus, if only players with value $x \geq r$ participate the expected payment at auction of a type $x \geq r$ is:

$$m(x) = \int_r^x y dG(y)$$

Therefore a player with value $x = r$ is just indifferent about whether to participate in the auction if the entry fee is $e = rG(r)$, since:

$$u(r) = rG(r) - m(r) = rG(r) = e$$

Finally, buyers with value $x' < r$ strictly prefer not to participate in the auction:

$$u(x') = x'G(r) - m(x') < rG(r) = e$$

and buyers with value $x'' > r$ strictly prefer to participate:

$$u'(x'') = G(x'') > 0$$

Therefore an entree fee of $e = rG(r)$ is equivalent to a reserve price of r since in both cases the interim expected payoffs prior to the auction satisfy:

$$U(x) = \begin{cases} xG(x) - rG(r) - \int_r^x y dG(y) & \text{if} \quad x \geq r \\ 0 & \text{if} \quad x < r \end{cases}$$

Note 2: Budget Constraints Francesco Nava
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This is an explicit proof of the argument made in the Extensions slides on page 10.

(1) Recall that since $\beta(x) < x \leq 1$:

$$\begin{aligned}
L^I(x) &= \{(X,W)|\min\{\beta(X),W\} < \min\{\beta(x),1\}\} = \\
&= \{(X,W)|\min\{\beta(X),W\} < \beta(x)\} \\
L^{II}(x) &= \{(X,W)|\min\{X,W\} < \min\{x,1\}\} = \\
&= \{(X,W)|\min\{X,W\} < x\}
\end{aligned}$$

I want to show that $L^I(x) \subset L^{II}(x)$. To show this it suffices to argue that:

$$\min\{\beta(X),W\} < \beta(x) \quad \Rightarrow \quad \min\{X,W\} < x$$

But notice that since $\beta' > 0$:

$$\begin{aligned}
\min\{X,W\} < x \quad &\Leftrightarrow \quad \beta(\min\{X,W\}) < \beta(x) \\
\min\{X,W\} < x \quad &\Leftrightarrow \quad \min\{\beta(X),\beta(W)\} < \beta(x)
\end{aligned}$$

But since $\beta(W) < W$:

$$\min\{\beta(X),W\} < \beta(x) \quad \Rightarrow \quad \min\{\beta(X),\beta(W)\} < \beta(x)$$

Which concludes the argument.

(2) The additional consideration on the revenues in the inequality. Notice that for any $a \in \{I, II\}$:

$$\begin{aligned}
Y_2^{a(N)} \sim F_2^{a(N)}(y) &= F^a(y)^N + N(1-F^a(y))F^a(y)^{N-1} \\
&= NF^a(y)^{N-1} - (N-1)F^a(y)^N
\end{aligned}$$

We want to show that $F^I(x) < F^{II}(x)$ implies $F_2^{I(N)}(x) < F_2^{II(N)}(x)$ for any for any $x \in (0,1)$.

To do so notice that the expression $Nk^{N-1} - (N-1)k^N$ is strictly increasing in k for $k \in (0,1)$ since:

$$\frac{\partial}{\partial k}\left[Nk^{N-1} - (N-1)k^N\right] = N(N-1)[k^{N-2} - k^{N-1}] > 0$$

Therefore since $0 < F^I(x) < F^{II}(x) < 1$:

$$F_2^{I(N)}(x) - F_2^{II(N)}(x) = [NF^I(x)^{N-1} - (N-1)F^I(x)^N] - [NF^{II}(x)^{N-1} - (N-1)F^{II}(x)^N] < 0$$

Which in turn implies from integration by parts that:

$$E(R^I) = E(Y_2^{I(N)}) = \int_0^1 y dF_2^{I(N)}(y) > \int_0^1 y dF_2^{II(N)}(y) = E(Y_2^{II(N)}) = E(R^{II})$$

Note 3: Regularity and IC Francesco Nava
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In the mechanism design slides on page 9, it is claimed that condition (1) and regularity imply incentive compatibility. To prove this it suffices to show that condition (1) and regularity imply that $q_i' \geq 0$, since that was proven to imply IC. Notice that without ties condition (1) is equivalent to:

$$Q_i(x) = 1 \Leftrightarrow \psi_i(x_i) \geq \max\{\max_{j \neq i} \psi_j(x_j), 0\} \tag{i}$$

1. For $\mathbb{I}(\cdot)$ the indicator function, condition (i) implies that:

$$\begin{aligned}
q_i(x_i) &= \int_{\mathbb{X}_{-i}} Q_i(x_i, x_{-i}) dF_{-i}(x_{-i}) \\
&= \int_{\mathbb{X}_{-i}} \mathbb{I}(\psi_i(x_i) \geq \max\{\max_{j \neq i} \psi_j(x_j), 0\}) dF_{-i}(x_{-i}) \\
&= \mathbb{I}(\psi_i(x_i) \geq 0) \int_{\mathbb{X}_{-i}} \prod_{j \neq i} \mathbb{I}(\psi_i(x_i) \geq \psi_j(x_j)) dF_{-i}(x_{-i}) \\
&= \mathbb{I}(\psi_i(x_i) \geq 0) \prod_{j \neq i} \int_{\mathbb{X}_j} \mathbb{I}(\psi_i(x_i) \geq \psi_j(x_j)) dF_j(x_j) \\
&= \mathbb{I}(\psi_i(x_i) \geq 0) \prod_{j \neq i} \int_{\mathbb{X}_j} \mathbb{I}(\psi_j^{-1}\psi_i(x_i) \geq x_j) dF_j(x_j) \\
&= \mathbb{I}(\psi_i(x_i) \geq 0) \prod_{j \neq i} F_j(\psi_j^{-1}\psi_i(x_i))
\end{aligned}$$

   where in the second to last equality we used that by regularity $\psi_j' > 0$ and thus ψ_j is invertible.

2. Notice that if $\psi_i(x_i) < 0$ then $q_i'(x_i) = 0$. If instead $\psi_i(x_i) \geq 0$ we have that:

$$q_i'(x_i) = \psi_i'(x_i) \sum_{k \neq i} \frac{f_k(\psi_k^{-1}\psi_i(x_i))}{\psi_k'(\psi_k^{-1}\psi_i(x_i))} \left[\prod_{j \neq i,k} F_j(\psi_j^{-1}\psi_i(x_i))\right] \geq 0$$

   since any term in the summation is positive given regularity, $\psi_k' > 0$, and since $f_k \geq 0$.

Which proves $q_i'(x_i) \geq 0$ and therefore IC.

Note 4: Optimal Mechanism Francesco Nava

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Some more details on the example developed in class. So we had 2 players $\{a, b\}$. Player a with values uniformly distributed on $[0, 2]$ and player b with uniform values on $[1, 3]$. In class we had shown:

$$\psi_a(x_a) = 2x_a - 2 \quad \text{and} \quad \psi_b(x_b) = 2x_b - 3$$

Therefore, the highest competing virtual valuations are:

$$\begin{aligned}
y_a(x_b) &= \inf\{z_a | \psi_a(z_a) > \max\{\psi_b(x_b), 0\}\} = \\
&= \inf\{z_a | z_a > x_b - 0.5 \quad \cap \quad z_a > 1\} = \\
&= \max\{x_b - 0.5,\ 1\} \\
y_b(x_a) &= \inf\{z_b | \psi_b(z_b) > \max\{\psi_a(x_a), 0\}\} = \\
&= \inf\{z_b | z_b > x_a + 0.5 \quad \cap \quad z_b > 1.5\} = \\
&= \max\{x_a + 0.5,\ 1.5\}
\end{aligned}$$

Thus, for these expressions as desired we get that OM satisfies for $i \in \{a, b\}$:

$$Q_i(x) = \begin{cases} 1 & \text{if} \quad x_i > y_i(x_{-i}) \\ 0 & \text{if} \quad x_i < y_i(x_{-i}) \end{cases} \qquad M_i(x) = \begin{cases} y_i(x_{-i}) & \text{if} \quad x_i > y_i(x_{-i}) \\ 0 & \text{if} \quad x_i < y_i(x_{-i}) \end{cases}$$

Now with the correct expressions it can be easily noted that this mechanism can be inefficient for two reasons:

1. The object is not sold. For instance, if $(x_a, x_b) = (0.5, 1.25)$ since:

$$\begin{aligned}
x_a &= 0.50 < y_a(x_b) = \max\{x_b - 0.5,\ 1\} = 1 \\
x_b &= 1.25 < y_b(x_a) = \max\{x_a + 0.5,\ 1.5\} = 1.5
\end{aligned}$$

2. The object is sold to the low value player. For instance, if $(x_a, x_b) = (1.5, 1.75)$ since:

$$\begin{aligned}
x_a &= 1.50 > y_a(x_b) = \max\{x_b - 0.5,\ 1\} = 1.25 \\
x_b &= 1.75 < y_b(x_a) = \max\{x_a + 0.5,\ 1.5\} = 2
\end{aligned}$$

Note 5: AGV Shifts & BB Francesco Nava

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Due to the algebra mishaps today during lectures let me establish again how to construct the shifts on the AGV mechanism that insure that the mechanism is IR.

1. First notice that by revenue equivalence the expected utility of both the VCG & the AGV mechanisms can differ at most by a constant. For $m \in \{v, a\}$, let:

$$\begin{aligned} c_i^v &= E[W(x_i, X_{-i})] - U_i^v(x_i) \\ c_i^a &= E[W(x_i, X_{-i})] - U_i^a(x_i) \end{aligned}$$

   Suppose that VCG runs surplus. If so:

$$\begin{gathered} E\left[\sum_{i \in N} M_i^v(X)\right] \geq E\left[\sum_{i \in N} M_i^a(X)\right] = 0 \quad \Leftrightarrow \\ \Leftrightarrow \quad \sum_{i \in N} E\left[M_i^v(X)\right] \geq \sum_{i \in N} E\left[M_i^a(X)\right] = 0 \quad \Leftrightarrow \\ \Leftrightarrow \quad \sum_{i \in N} \left[c_i^v - E[W_{-i}(x_i, X_{-i})]\right] \geq \sum_{i \in N} \left[c_i^a - E[W_{-i}(x_i, X_{-i})]\right] = 0 \quad \Leftrightarrow \\ \Leftrightarrow \quad \sum_{i \in N} c_i^v \geq \sum_{i \in N} c_i^a \quad \Leftrightarrow \\ \Leftrightarrow \quad \sum_{i > 1} (c_i^v - c_i^a) \geq (c_1^a - c_1^v) \end{gathered}$$

2. Define $d_i = c_i^a - c_i^v$ for $i > 1$ and $d_1 = -\sum_{i>1} d_i$. Notice that $d_1 \geq (c_1^a - c_1^v)$, by the last observation.

3. Consider the BB payment rule M^* defined by:

$$M_i^*(x) = M_i^a(x) + d_i$$

   Such rule is trivially BB, since:

$$\sum_{i \in N} M_i^*(X) = \sum_{i \in N} M_i^a(X) + \sum_{i \in N} d_i = 0$$

4. If $Q^* = Q^a$ the mechanism (Q^*, M^*) is efficient. It is also IC since payoffs differ from (Q^a, M^a) by additive constant. Additionally (Q^*, M^*) is IR because:

$$U_i^*(x_i) = U_i^a(x_i) + d_i \geq U_i^a(x_i) + c_i^a - c_i^v = E[W(x_i, X_{-i})] - c_i^v = U_i^v(x_i) \geq 0$$

Note 6: Expectation and Maximum Francesco Nava
EC319: Auction Theory Michaelmas Term

In the interdependent value slides on page 4, it is claimed that for the pure common value model:

$$E[\max X_i|V=v] > \max E[X_i|V=v] = v$$

Recall that the special case has:

- $V \sim F$
- $X_i|V \overset{\text{iid}}{\sim} H(V)$
- $E[X_i|V=v] = v$

Therefore we have that:

$$\begin{aligned} E[\max X_i|V=v] &= \int_0^{\omega_i} y dH(y|v)^N \\ \max E[X_i|V=v] &= E[X_i|V=v] = \int_0^{\omega_i} y dH(y|v) = v \end{aligned}$$

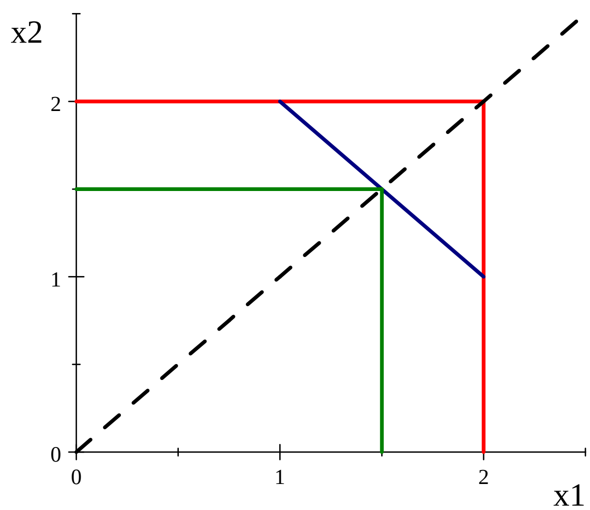
But notice $G(y|v) = H(y|v)^N$ is also a distribution and that it stochastically dominates $H(y|v)$, since $G(y|v) < H(y|v)$ for any $y \in (0,1)$. Thus we have that:

$$E[\max X_i|V=v] = \int_0^{\omega_i} y dG(y|v) \geq \int_0^{\omega_i} y dH(y|v) = \max E[X_i|V=v]$$

In general since the max is a convex operator we have that:

$$E[\max X_i|V=v] \geq \max E[X_i|V=v]$$

by Jensen inequality. That the max is convex can be seen from this plot:



The plot show that a lottery amongst two bundles having the same maximum is preferred to the maximum of the lottery. Indeed consider:

$$(x_1, x_2) = \begin{cases} (2,1) & \text{with probability} \quad 1/2 \\ (1,2) & \text{with probability} \quad 1/2 \end{cases}$$

Then immediately we have that:

$$E[\max X_i] = 2 \geq \max E[X_i] = 1.5$$