EC411 – Microeconomics

Teachers Responsible

Michaelmas Term:	Dr. Francesco Nava	
	Office:	32L 3.20
	Secretary:	32L 1.17
Lent Term:	Prof. Martin Pesendorfer	
	Office:	32L 4.19
	Secretary:	32L 1.17
Schedule:	W1 MT:	Wednesdays 10:00-12:00
	W2-11 MT:	Wednesdays 10:30-12:30
	W1-11 LT:	Wednesdays 10:00-12:00
Location:	Old Theatre	
Examination:	2 Hours Exam in Lent Term	
	2 Hours Exam in Summer Term	

Aims and Objectives

This unit is a graduate level introduction to Microeconomics. The objective is to provide students with a firm grounding in the analytic methods of microeconomic theory used by economists working in research, government, and business. Special emphasis will be placed on the design and the solution of simple economic models and on ensuring that students become familiar with basic optimisation and equilibrium techniques.

Course Summary

The first part of the course focuses on classical theories of consumer and producer behaviour and on the theory of general equilibrium. We begin with a careful analysis of the optimisation problems for price-taking consumers and firms. We then analyse market interaction and the formation of prices in the framework of perfect competition. We conclude with introductions to decision making under uncertainty and game theory.

The second part of the course focuses on models of imperfect competition and information economics. We begin with an analysis of models of monopoly, oligopoly, and bargaining. Then, we study markets with imperfect and incomplete information including search, adverse selection, auctions, signalling, screening, and moral hazard. Special emphasis will be given to economic applications.

Teaching and learning methods

Our approach will emphasise analytical methods of reasoning and thus problem-solving exercises will be a crucial learning tool. Problem sets will be provided and answers to selected problems will be discussed during classes. Two assignments per term will be marked.

Textbooks

The course will draw mainly on the textbook:

- Riley, Essential Microeconomics, Cambridge University Press, 2012.

The following textbooks can also be useful as additional readings:

- Mas-Colell, Whinston, Green, Microeconomic Theory, Oxford University Press, 1995.
- Varian, Microeconomic Analysis, 3rd Edition, Norton, 1992

Course Outline: Term 1

A. Decision Theory, Consumer Theory and Welfare

(1) Riley, Essential Microeconomics, 2012, Chapter 2, 43-78.
(2) Spiegler, Bounded Rationality and Industrial Organization, Appendix, 202-212.
(3) Caplin, Schotter, The Foundations of Positive and Normative Economics, 2008, Chap 8.

Optional Readings:
(a) Gul and Pesendorfer, The Case for Mindless Economics, 2005.
(b) Riley, Essential Microeconomics, 2012, Chapter 1.

B. Producer Theory

(1) Riley, Essential Microeconomics, 2012, Chapter 4, 106-130.

C. Competitive Markets and Equilibrium

(1) Riley, Essential Microeconomics, 2012, Chapter 3, 85-105.
(2) Riley, Essential Microeconomics, 2012, Chapter 5, 139-167.

D. Choice under Uncertainty

(1) Riley, Essential Microeconomics, 2012, Chapter 7, 218-250.

E. Game Theory:

(1) Riley, Essential Microeconomics, 2012, Chapter 9, 303-338.
(2) Shy, Industrial Organization, MIT Press, 1995, Chapter 2, 11-42.

Optional Readings:
(a) Kreps, A Course in Microeconomic Theory, Chapters 11-14.

Course Outline: Term 2

A. Monopoly: Price Discrimination

1) J. Riley, Essential Microeconomics, 2012, Chapter 4.5, 130-138.

2) H. Gravelle and R. Rees, Microeconomics, Longman, 2004, Chapter 9.

3) R. Schmalensee, 'Output and Welfare Implications of Monopolistic Third Degree Price Discrimination', American Economic Review, 1981.

4) E. P. Lazaer, ‘Retail Pricing and Clearance Sales’ American Economic Review, 1986.

B. Oligopoly Theory, Repeated Games and Bargaining

5) J. Riley, Essential Microeconomics, 2012, Chapter 9, 303-346.

6) R. Porter, ‘The Role of Information in U.S. Offshore Oil and Gas Lease Auctions,’ Econometrica, 1995.

7) D. Fudenberg and J. Tirole, Game Theory, MIT Press, 1991, Chapter 4, Section 4.4.

8) A. Mas-Colell, M. Whinston, and J. Green, Microeconomic Theory, Oxford University Press, 1995, Chapter 9, Appendix A, 296-299, and Chapter 12, Sections C and D, 387-405.

9) C. d'Aspremont, J. Jaskold Gabszewicz and J.-F. Thisse, 'On Hotelling's "Stability in Competition"' Econometrica, 1979.

C. Search and Adverse Selection

10) “Search” (unpublished note for lecture).

11) G. Akerlof, 'The Market for Lemons”, Quality Uncertainty and the Market Mechanism', Quarterly Journal of Economics, 1970.

D. Games with Incomplete Information and Auctions

12) J. Riley, Essential Microeconomics, 2012, Chapter 10.1, 347-350.

13) J. Riley, Essential Microeconomics, 2012, Chapter 12.3, 456-462.

14) M. H. Bazerman and W. F. Samuelson, I Won the Auction But Don’t Want the Prize, Journal of Conflict Resolution, 1983.

15) R. Porter, ‘The Role of Information in U.S. Offshore Oil and Gas Lease Auctions,’ Econometrica, 1995.

E. Mechanism Design with Incomplete Information

16) R. Myerson, 'Optimal Auction Design', Mathematics of Operations Research, 1981, 58-73.

17) J. Riley, Essential Microeconomics, 2012, Chapters 12.5 and 12.6, 474-487.

18) A. Mas-Colell, M. Whinston, and J. Green, Microeconomic Theory, Oxford University Press, 1995, Chapter 23D-23F, 891-906.

F. Signalling, Moral Hazard and Screening

19) M. Spence, 'Job Market Signalling', Quarterly Journal of Economics, 1973.

20) B. Salanie, The Economics of Contracts. A Primer, MIT Press, 1997, Chapter 5.

21) M. Rothschild and J. Stiglitz, 'Equilibrum in Competitive Insurance Markets: An Essay on the economics of Imperfect Information', Quarterly Journal of Economics, 1976.

Microeconomic Theory EC411

EC411 Slides 0

Nava

LSE

Michaelmas Term

Notes

Introduction & Objectives

The Michelmas Term part of the course covers:

- Decision theoretic models of choice: with and without uncertainty.
- Classical theories of consumer and producer behaviour.
- Classical theories of markets and the competitive equilibrium.
- Game theory: static and dynamic games and solution concepts.

The objective of the course is to provide students with a firm grounding in the analytic methods of microeconomic theory.

Although the course is centered around classical theories of economic behavior, new developements in these fields will also be discussed.

Notes

General Information

Lecturer:	Francesco Nava
Email:	f.nava@lse.ac.uk
Course Website:	moodle.lse.ac.uk
Lectures:	Wednesday, 10.00 – 12.00, Week 1 Wednesday, 10.30 – 12.30, Week 2-11 Old Bulding, Old Theatre
Office Hours:	Wednesday, 13.30 – 15.00 Building 32L, Room 3.20

Notes

Course Program

1. Decision Theory & Consumer Theory:
   Riley Chapters 1 and 2, Spiegler Appendix.
2. Producer Theory:
   Riley Chapter 4.
3. Markets and Equilibrium:
   Riley Chapters 3 and 5.
4. Choice under Uncertainty:
   Riley Chapter 7.
5. Game Theory:
   Riley Chapter 9 and Shy Chapter 2.
   Alternative: Kreps Chapters 11-14.

Notes

Course Requirements

Classes

- Classes start in week 2 and follow lectures with some lag.
- The work for each week consists of some exercises.
  You are expected to attempt the classwork before the class.
- Two assignments are required for submission: Mon W7, Fri W10.
- Solutions will be posted throughout the term.

LT Exam

- The exam will take place in week 0 LT.
- The exam format has not changed from previous years.

Notes

Materials

Main Textbook

Riley, Essential Microeconomics, Cambridge University Press, 2012. [R]

Lecture Notes

Include materials required for the evaluations unless otherwise specified.
Lecture notes can be found on moodle.

Papers

Some background papers may be suggested to the interested reader.

Supplentary and Alternative Textbook

Mas-Colell, Whinston, and Green, Microeconomic Theory, 1995. [M]
Varian, Microeconomic Analysis, 3rd Edition, 1992. [V]
Kreps, A Course in Microeconomic Theory, Princeton, 1990. [K]

Notes

Preferences and Consumption

EC411 Slides 1

Nava

LSE

Michaelmas Term

Notes

Roadmap: Consumption

- Basic decision theory:
  - definition preferences,
  - utility representation.
- Consumers' optimization problem:
  - utility maximization,
  - expenditure minimization,
  - duality.
- Demand, substitution and income effects.
- Indirect utility function and expenditure function.
- Applications:
  - the Slutsky equation,
  - the Labour market.
- Criticism and alternative theories.

Notes

Simple Decision Theory

Notes

Simple Decision Theory

Our primitive is a preference relation over a set of alternatives.

- $X \subseteq \mathbb{R}^n_+$ is the set of alternatives (e.g. consumption set).
  The elements of X are choices (e.g. consumption bundles).
  A typical element is a vector $x = (x_1, ..., x_n)$.
- $\succsim$ is a preference relation over X.
  When $x \succsim y$, the consumer **weakly prefers** x to y.

Example:

- $X = \{(0,0), (1,0), (0,1), (1,1)\}$;
- $(1,1) \succsim (1,0)$, $(1,1) \succsim (0,0)$, $(1,0) \succsim (0,1)$, $(0,1) \succsim (0,0)$.

Notes

Strict Preferences & Indifference

From $\succsim$, we derive two additional relations on X:

- **Strict Preference** ($\succ$): $x \succ y$ if and only if

$$x \succsim y \text{ and not } y \succsim x.$$

- **Indifference** ($\sim$): $x \sim y$ if and only if

$$x \succsim y \text{ and } y \succsim x.$$

In the previous example, $(1,1) \succ (1,0)$.

Notes

Axioms on Consumer Behavior

Two assumptions on consumer behavior are maintained throughout:

1. **Completeness**: If $x, y \in X$, either $x \succsim y$, or $y \succsim x$, or both.
   In the previous example, preferences are not complete.
2. **Transitivity**: For $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

We say that a preference relation is **rational** if it is transitive and complete.

Exercise:
Show that transitivity extends to strict preference and indifference.

Notes

Other Dimensions

Preferences can be defined over alternatives that do not belong to $\mathbb{R}^n_+$.

For instance, consider:

- preferences over locations:
  Venice $\succsim$ Rome, Rome $\succsim$ Paris, Paris $\succsim$ Venice

- preferences over candidates in an election:

$$\begin{array}{c} X \succsim Y \succsim Z \\ Y \succsim Z \succsim X \\ Z \succsim X \succsim Y \end{array}$$

Aside: Transitivity no longer holds though when we try to aggregate these preferences $X \succsim^A Y \succsim^A Z \succsim^A X$.

Notes

Utility Function

Definition

A utility function is a function $U : X \to \mathbb{R}$ such that, for any $x, y \in X$,

$$U(x) \geq U(y) \text{ if and only if } x \succsim y.$$

If a preference relation can be represented by a utility function, then the preference relation is rational.

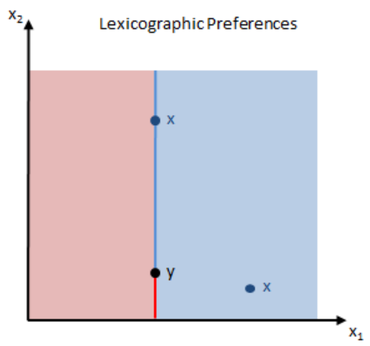
However, not any rational preference relation can be represented by a utility function.

Notes

Example: Lexicographic Preferences

Suppose that $X = \mathbb{R}^2_+$ and that $x \succsim y$ if and only if

$$\text{either } \text{“}x_1 > y_1\text{” or “}x_1 = y_1 \text{ and } x_2 \geq y_2\text{”}.$$



No utility function can represent these rational preferences.

Notes

Representation Theorem

Two additional regularity axioms imply a desirable representation theorem:

Continuity: For any $y \in X$, the sets $\{x \in X : x \succsim y\}$ and $\{x \in X : y \succsim x\}$ are closed.

Monotonicity: For any $x, y \in X$, if $x_i > y_i$, $i = 1, ..., n$, then $x \succ y$.

Theorem

Suppose that $X = \mathbb{R}^n_+$ and that preferences are complete, transitive, continuous, and monotonic. Then there exists a continuous function $U : X \to \mathbb{R}$ such that, for any $x, y \in X$,

$$U(x) \geq U(y) \text{ if and only if } x \succsim y.$$

Notes

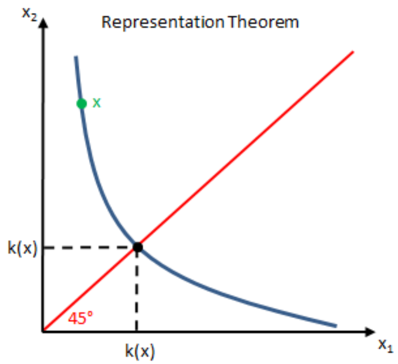
Example: How to Derive a Utility Function

Consider any point x in the consumption set.

Find the scalar $k(x)$, for which $y = (k(x), ..., k(x)) \in X$ satisfies

$$y \sim x.$$

Then, set $U(x) = k(x)$ and note that $x \succ y \implies k(x) > k(y)$.



Notes

Monotonic Transformations & Convexity

Consider a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$.

- If $U(\cdot)$ is a utility function that represents a preference relation $\succsim$, the monotonic transformation $g(U(\cdot))$ also represents $\succsim$.
- Utility is an ordinal concept. A utility function is a numerical representation of a preference relation.

Occasionally a final axiom is imposed on preferences to ensure that the "weakly preferred sets" are convex.

Convexity: (i) X is convex. (ii) For $x, y, z \in X$, if $x \succsim z$ and $y \succsim z$, then $\lambda x + (1 - \lambda) y \succsim z$ for any $\lambda \in [0, 1]$.

Notes

Consumer Theory

Utility Maximization

Notes

The Utility Maximization Problem

Notation:

- p_i is the price of good i;
- $p = (p_1, ..., p_n)$ is the price vector;
- m is the income of the consumer.

A consumer chooses $(x_1, ..., x_n)$ to maximize

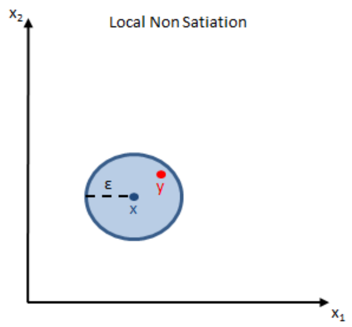
$$U(x_1, ..., x_n)$$
$$\text{subject to } p_1 x_1 + ... + p_n x_n \leq m.$$

Notes

Local Non Satiation

Given $x, y \in \mathbb{R}^n$, let $d(x, y)$ denote the distance between x and y.

Local Non Satiation: For any $x \in X$ and $\varepsilon > 0$, there exists $y \in X$ such that $d(x, y) < \varepsilon$ and $U(x) < U(y)$.



Exercise: Show that LNS implies $px^* = m$ if x^* solves consumer problem.

Notes

Solving the Consumer's Problem

By LNS we replace $px \leq m$ with the budget line $px = m$.

The Lagrangian for the consumer's problem satisfies

$$L = U(x) + \lambda(m - px).$$

FOC for an interior, local maximum satisfy

$$m - px = 0,$$
$$\frac{\partial U}{\partial x_i} - \lambda p_i = 0 \text{ for any } i = 1, ..., n.$$

With two goods, FOC imply that

$$MRS = -\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = -\frac{p_1}{p_2}.$$

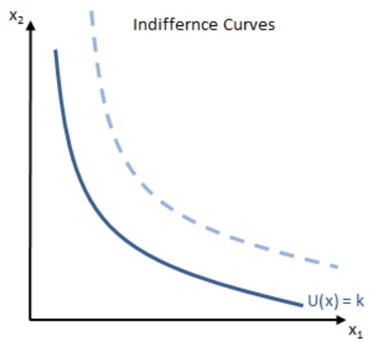
Notes

Indifference Curves

Define an **indifference curve** as the set of bundles for which utility is equal to some constant.

If there are only two commodities, a (level k) indifference curve satisfies

$$U(x_1, x_2) = k.$$



Notes

Marginal Rates of Substitution

Consider a two commodities environment and an indifference curve

$$U(x_1, x_2) = k.$$

Totally differentiating this curve, yields

$$\frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0$$

which in turn implies that

$$\frac{dx_2}{dx_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2} \equiv MRS.$$

We often refer dx_2/dx_1 as the **Marginal Rate of Substitution** (MRS) between good 1 and good 2.

Notes

Examples

Cobb-Douglas:

$$U(x_1, x_2) = x_1^{\alpha} x_2^{\beta} \implies \text{MRS} = -\frac{\alpha x_2}{\beta x_1}.$$

Perfect Substitutes:

$$U(x_1, x_2) = ax_1 + bx_2 \implies \text{MRS} = -\frac{a}{b}.$$

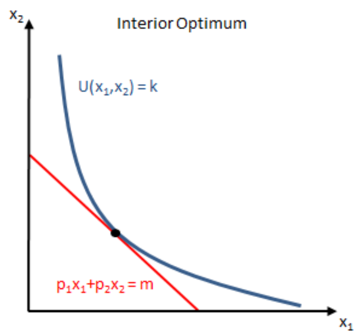
Perfect Complements

$$U(x_1, x_2) = \min\{ax_1, bx_2\} \quad \text{Not Differentiable.}$$

Notes

First Order Conditions

Optimality in the consumer's problem requires the slope of the budget line to be equal to the slope of the indifference curve.



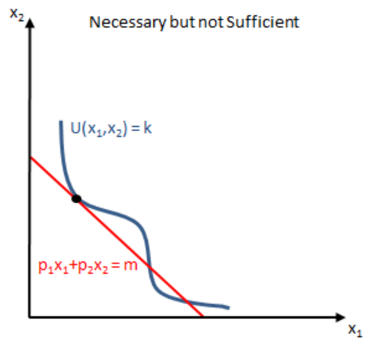
Recall the FOC of the consumer's problem,

$$m - px = 0 \;\; \& \;\; \frac{\partial U}{\partial x_i} - \lambda p_i = 0 \text{ for any } i = 1, ..., n.$$

Notes

Second Order Conditions

SOC guarantee that a solution to FOC is local max:



If the utility function is quasiconcave, KKT FOC also imply global constrained maximum.

Notes

Marshallian Demands

Solving the utility maximization problem, obtain the Marshallian demands,

$$x_i(p, m), \quad \text{for any } i = 1, ..., n.$$

Denote the vector of Marshallian demands by $x(p, m)$.

Marshallian demands are homogeneous of degree zero in (p, m), that is,

$$x(p, m) = x(tp, tm), \quad \text{for any } t > 0.$$

Definition

A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is homogenous of degree k if and only if, for any $t > 0$ and any $x \in \mathbb{R}^n_+$,

$$f(tx) = t^k f(x).$$

Notes

Example: Cobb-Douglas

Consider a utility function, $U(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$.

Optimality requires that:

$$MRS = \frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2} \implies \frac{\beta p_1 x_1}{\alpha} = p_2 x_2.$$

Substituting in the budget line, one obtains that

$$p_1 x_1 \left(1 + \frac{\beta}{\alpha}\right) = m.$$

Marshallian demands hence, satisfy

$$\begin{aligned} x_1(p, m) &= \left(\frac{\alpha}{\alpha + \beta}\right) \frac{m}{p_1} \\ x_2(p, m) &= \left(\frac{\beta}{\alpha + \beta}\right) \frac{m}{p_2} \end{aligned}.$$

The fraction of income spent on x_1 is $\frac{\alpha}{\alpha + \beta}$ and on x_2 is $\frac{\beta}{\alpha + \beta}$.

Notes

Examples: Perfect Complements & Perfect Substitutes

Perfect Complements: $U(x_1, x_2) = \min\{x_1, x_2\}$.

At an optimal choice, $x_1 = x_2$.

Thus, $p_1 x_2 + p_2 x_2 = m$ implies

$$x_2(p, m) = x_1(p, m) = \frac{m}{p_1 + p_2}.$$

Perfect Substitutes: $U(x_1, x_2) = x_1 + x_2$.

If $p_1 < p_2$, then $x_1(p, m) = \frac{m}{p_1}$ and $x_2(p, m) = 0$.

If $p_1 > p_2$, then $x_1(p, m) = 0$ and $x_2(p, m) = \frac{m}{p_2}$.

If $p_1 = p_2$, then any $x(p, m)$ on the budget line.

Notes

Example: Quasi-Linear Utility

Consider a quasi-linear utility function:

$$U(x_1, x_2) = 2\sqrt{x_1} + x_2$$

Optimality requires that:

$$MRS = (x_1)^{-1/2} = \frac{p_1}{p_2}.$$

Thus at an interior optimum Marshallian demands satisfy

$$x_1(p,m) = \left(\frac{p_2}{p_1}\right)^2 \quad \& \quad x_2(p,m) = \frac{m}{p_2} - \frac{p_2}{p_1}.$$

If $\frac{m}{p_2} - \frac{p_2}{p_1} < 0$, x_2 would become negative which is impossible.

If so, the following boundary solution applies

$$x_1(p,m) = \frac{m}{p_1} \quad \& \quad x_2(p,m) = 0.$$

Notes

Price Changes: Ordinary Goods

A good is said to be:

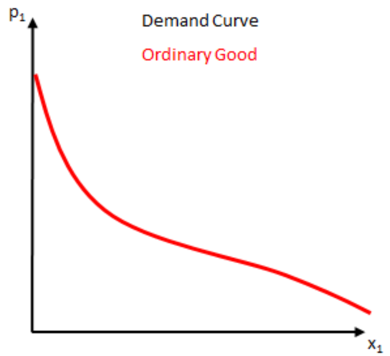
Ordinary if its Marshallian demand increases as its price decreases.

Notes

Price Changes: Demand Curve

Demand curves map equilibrium demands to prices.

If a good is ordinary the demand curve is decreasing.

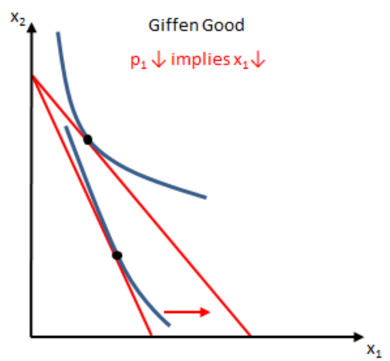


Notes

Price Changes: Giffen Goods

A good is said to be:

Giffen if its Marshallian demand decreases as its price decreases.

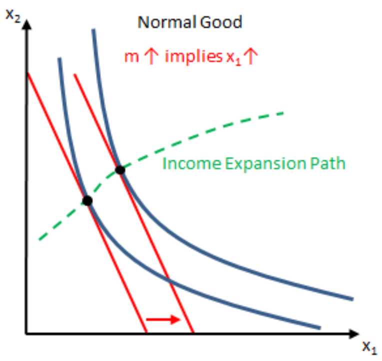


Notes

Income Changes: Normal Goods

A good is said to be:

Normal if its Marshallian demand increases as income increases.

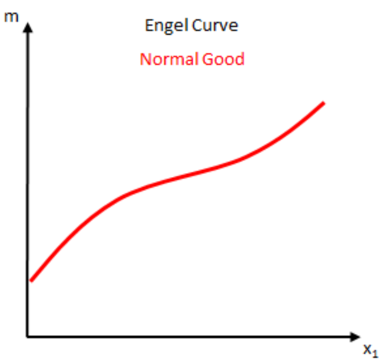


Notes

Income Changes: The Engle Curve

Engle curves map equilibrium demands to income levels.

If a good is normal the Engle curve is increasing.

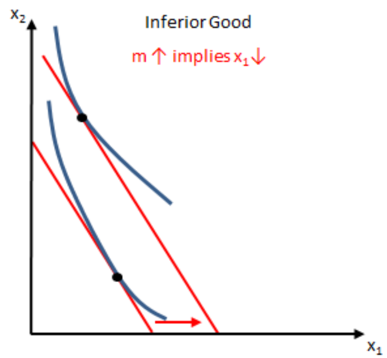


Notes

Income Changes: Inferior Goods

A good is said to be:

Inferior if its Marshallian demand decreases as income increases.



Notes

The Indirect Utility Function

Are we happy about changes in prices and income?

Can preferences be defined directly on choice or budget sets?

The **indirect utility** function is obtained plugging the Marshallian demands in the utility function

$$v(p, m) = U(x(p, m))$$

The indirect utility function depends on the particular representation.

We can think of it as utility function defined over prices and income pairs.

Notes

Properties of the Indirect Utility Function

Lemma

The indirect utility function $v(p, m)$ is:

(i) non-increasing in p and non-decreasing in m;

(ii) $v(p, m)$ is homogeneous of degree zero in (p, m);

(iii) $v(p, m)$ is quasiconvex in p;

(iv) $v(p, m)$ is continuous if $U(x)$ is continuous.

Proof (i) When m increases or p_i decreases, the set of bundles that satisfy the budget constraint becomes larger.

Observation: Under LNS $v(p, m)$ is strictly increasing in m, but not necessarily strictly decreasing in every p_i (eg perfect substitutes).

(ii) Marshallian demands are homogeneous of degree zero in (p, m).

Notes

Properties of the Indirect Utility Function

(iii) Consider:

- prices p, $\bar{p}$, and $p^* = \lambda p + (1-\lambda)\bar{p}$;
- any bundle x^* such that $p^* x^* \leq m$.

It is impossible that $px^* > m$ and $\bar{p}x^* > m$ since:

$$\lambda p x^* > \lambda m \quad \text{and} \quad (1-\lambda)\bar{p}x^* > (1-\lambda) m \quad \Rightarrow \quad p^* x^* > m.$$

Thus, either $px^* \leq m$ or $\bar{p}x^* \leq m$, which in turn implies that

$$v(p^*, m) \leq \max\{v(p, m), v(\bar{p}, m)\}$$

and that lower contour sets are convex.

Exercises:

Draw v's indifference curves to show the last part of the argument.

Assigned for classes: prove (iv).

Notes

Consumer Theory

Expenditure Minimization

Notes

The Expenditure Minimization Problem

Suppose that the consumer chooses x to minimize

$$px$$
$$\text{subject to } U(x) \geq u.$$

This dual problem reverses the role of objective function and constraint.

Assume that U is continuous with LNS preferences.

If the constraint set is not empty, a solution exists.

This is the dual problem of our original problem, since it has the same solution to a corresponding utility maximization problem, when wealth is positive and utility is above 0 utility.

Notes

Hicksian Demands

The solutions to expenditure minimization problem are known as the Hicksian demands,

$$h_i(p, u), \text{ for } i = 1, ..., n.$$

Denote the vector of Hicksian demands by $h(p, u)$.

Hicksian demands are: homogenous of degree 0 in p; and single valued provided that preferences satisfy the convexity axiom.

These are also known as compensated demands, as income changes to compensate for changes in prices so that utility remains constant at u.

Hicksian demand are **not observable**, only Marshallian demands are.

Notes

The Expenditure Function & Its Properties

The expenditure function is

$$e(p, u) = \sum_{i=1}^{n} p_i h_i(p, u).$$

If $U(x)$ is continuous, $U(h(p, u)) = u$.

Lemma

The expenditure function is:

(i) non-decreasing in p_i, for any $i = 1, ..., n$;

(ii) homogeneous of degree 1 in p;

(iii) concave in p;

(iv) continuous in p.

Moreover, $\dfrac{\partial e(p, u)}{\partial p_i} = h_i(p, u)$, *for any $i = 1, ..., n$.*

Notes

Consumer Theory

Duality

Notes

Duality: Expenditure Minimization & Utility Maximization

Both the expenditure function and the indirect utility function capture very important properties of the consumer problem.

Theorem

Suppose that:

- *preferences satisfy LNS and $U(\cdot)$ is continuous;*
- *prices are strictly positive $p \gg 0$.*

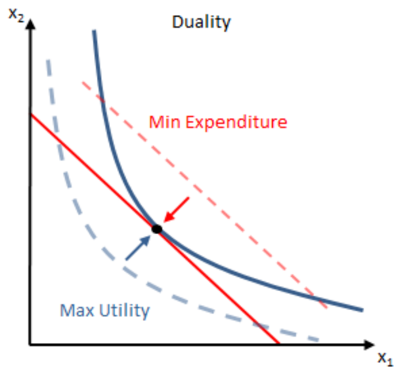
Then:

1. $e(p, v(p, m)) = m$;
2. $v(p, e(p, u)) = u$;
3. $x_i(p, e(p, u)) = h_i(p, u)$, *for* $i = 1, ..., n$;
4. $h_i(p, v(p, m)) = x_i(p, m)$, *for* $i = 1, ..., n$.

The last two properties tie observables to unobservables.

Notes

Graphical Representation of Duality



As prices vary, the Hicksian demand gives us the demand that would arise if expenditure was changed to keep utility constant.

Notes

Proof of Duality Part 3

Suppose that $h(p,u)$ **does not** maximize utility subject to $px \leq e(p,u)$.

Let $\tilde{x}$ be the utility maximizing bundle. Thus,

$$U(\tilde{x}) > U(h(p,u)) = u.$$

Observe that

$$e(p,u) = ph(p,u) = p\tilde{x},$$

where the first equality holds by definition and the second by LNS.

Consider a bundle $\tilde{x} - \varepsilon$, where ε is a small positive vector.

By continuity, we have that $U(\tilde{x} - \varepsilon) > u$ ($\tilde{x}$ is interior wlog).

However, this contradicts the hypothesis that $e(p,u)$ is the minimum expenditure that achieves utility greater than or equal to u, since

$$p(\tilde{x} - \varepsilon) = e(p,u) - p\varepsilon < e(p,u).$$

Notes

Duality Applications

Roy's Identity

Notes

Roy's Identity

Recall that part 2 duality result establishes that $v(p, e(p,u)) = u$.

Differentiating with respect to p_i, obtain that

$$\frac{\partial v\left(p, e(p,u)\right)}{\partial p_i} + \frac{\partial v\left(p, e(p,u)\right)}{\partial m} h_i(p,u) = 0.$$

Exploiting the duality theorem, this simplifies to

$$\frac{\partial v\left(p, m\right)}{\partial p_i} + \frac{\partial v\left(p, m\right)}{\partial m} x_i(p,m) = 0.$$

This is known as **Roy's Identity** and requires that

$$x_i(p,m) = -\frac{\partial v\left(p,m\right)/\partial p_i}{\partial v\left(p,m\right)/\partial m}.$$

As v is not invariant to utility transformations, the derivative of v w.r.t. p is normalized by the marginal utility of wealth to derive demands.

This is a great tool, because we can use the derivatives of the indirect utility function to find demands.

Notes

Example: Cobb-Douglas

The indirect utility of a **specific** Cobb-Douglas utility function is

$$v\left(p,m\right)=\frac{m^2}{p_1p_2}.$$

Thus, **Roy's Identity** in this case requires

$$x_1(p,m)=-\frac{\partial v\left(p,m\right)/\partial p_1}{\partial v\left(p,m\right)/\partial m}=-\frac{-\left(\frac{1}{p_1}\right)^2\frac{m^2}{p_2}}{\frac{2m}{p_1p_2}}=\frac{1}{2}\frac{m}{p_1}.$$

Note that, by identity 2, we also have that

$$\frac{e(p,u)^2}{p_1p_2}=u \quad\Longrightarrow\quad e(p,u)=\sqrt{up_1p_2}$$

We can derive the expenditure function from the indirect utility function and vice versa.

Notes

Duality Applications

Slutsky Equation

Notes

Towards Slutsky Equation

Recall that Marshallian demand curves can be upward sloping.

Does utility maximization impose any restrictions on consumer behavior?

Part 3 duality result establishes that $x_i(p, e(p, u)) = h_i(p, u)$.

Differentiating with respect to p_j we obtain

$$\frac{\partial x_i(p, e(p, u))}{\partial p_j} + \frac{\partial x_i(p, e(p, u))}{\partial m} h_j(p, u) = \frac{\partial h_i(p, u)}{\partial p_j}.$$

By exploiting the duality theorem, this implies that

$$\underbrace{\frac{\partial x_i(p, m)}{\partial p_j} + \frac{\partial x_i(p, m)}{\partial m} x_j(p, m)}_{\theta_{ij}(p,m)} = \underbrace{\frac{\partial h_i(p, v(p, m))}{\partial p_j}}_{\sigma_{ij}(p,m)}.$$

Notes

The Substitution Matrix

Define the **substitution matrix** as the matrix $\theta(m, p)$ satisfying

$$\theta(m, p) = \begin{bmatrix} \theta_{11}(p, m) & \dots & \theta_{1n}(p, m) \\ \dots & \dots & \dots \\ \theta_{n1}(p, m) & \dots & \theta_{nn}(p, m) \end{bmatrix}$$

By construction, the substitution matrix also satisfies

$$\theta(m, p) = \frac{\partial h(p, v(p, m))}{\partial p} = \frac{\partial^2 e(p, v(p, m))}{\partial p^2}.$$

Thus, the substitution matrix must be:

- negative semidefinite (by the concavity of the expenditure function);
- symmetric (since cross-price effects coincide – not intuitive);
- with non-positive diagonal terms.

It depends only on terms that are, at least in principle, observable.

Notes

Slutsky Equation

The **Slutsky equation** decomposes the effect of a change in the price:

$$\frac{\partial x_i(p,m)}{\partial p_j} = \sigma_{ij}(p,m) - \frac{\partial x_i(p,m)}{\partial m} x_j(p,m).$$

Two effects discipline how demand responds to a change in the price:

1. the **substitution effect**, $\sigma_{ij}(p,m)$;
2. the **income effect**, $-\frac{\partial x_i(p,m)}{\partial m} x_j(p,m)$.

Since $\sigma_{ii}(p,m) \leq 0$, the Slutsky equation implies that Giffen goods ($\partial x_i/\partial p_i > 0$) must always be inferior ($\partial x_i/\partial m < 0$).

Notes

Some Intuition: Slutsky Equation

By the envelope theorem $D_p e \equiv de/dp = h$. Therefore, $D_p h = D_p^2 e$.

As e is continuous, differentiable and concave, its Hessian is symmetric and negative semi-definite.

Therefore **compensated own price effects are non-positive**, and **other effects are symmetric**.

Slutsky equation implies that a change in compensated demand h is equal to the change in the uncompensated demand x due to prices only, plus the change in uncompensated demand x due to the change in expenditure multiplied by the change in expenditure.

The RHS of the Slutsky equation gives us computable relationship.

Theory allowed us to derive testable predictions about $D_p h$.

Notes

Graphical Intuition: Slutsky Equation

Consider a normal good i and an increase to p_i:

- Observe that h is flatter than x, as $\partial x_i/\partial p_i \leq \partial h_i/\partial p_i \leq 0$.
- The curves intersect at the initial price as duality implies $x_i^0 = h_i^0$.
- When the price increases, income decreases further.
- Thus, lower income reduces consumption of x_i^1 further.

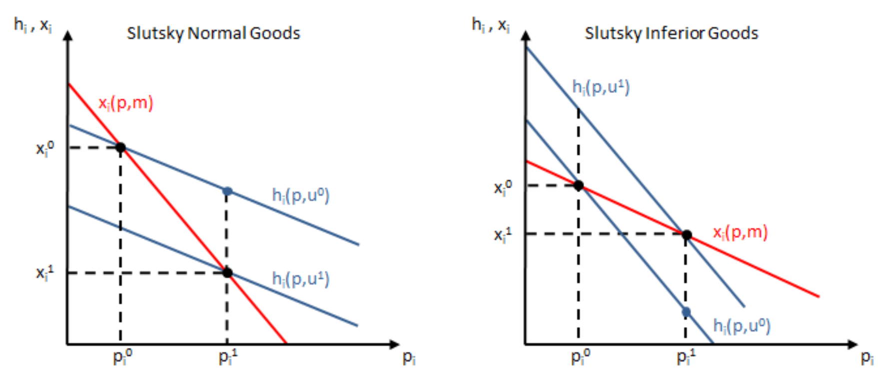
As $\partial x_i/\partial m$ is positive for normal goods, h_i responds less than x_i.

In other words, h_i is compensated by increasing wealth to keep the decision maker at the same utility, so demand does not fall as much.

The converse holds for inferior goods.

Notes

Graphical Intuition: Slutsky Equation



Hicksian is compensated by increasing wealth to keep utility constant.

Notes

Discrete Price Changes: Slutsky Equation

Let the price of good i go from p_i^0 to p_i^1.

Let $u^0 = v(p_i^0, p_{-i}, m)$ and $u^1 = v(p_i^1, p_{-i}, m)$.

The substitution effect can be computed as

$$SE = h_i(p_i^1, p_{-i}, u^0) - h_1(p_i^0, p_{-i}, u^0).$$

The total effect can be computed as

$$\begin{aligned} TE &= x_i(p_i^1, p_{-i}, m) - x_i(p_i^0, p_{-i}, m) \\ &= h_i(p_i^1, p_{-i}, u^1) - h_i(p_i^0, p_{-i}, u^0). \end{aligned}$$

The income effect can be computed as

$$\begin{aligned} IE &= h_i(p_i^1, p_{-i}, u^1) - h_i(p_i^1, p_{-i}, u^0) \\ &= TE - SE. \end{aligned}$$

Notes

Marginal Price Changes: Slutsky Equation

Differentiating the FOC, we obtain that:

$$\begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} d\lambda / dp_1 \\ dx_1 / dp_1 \\ dx_2 / dp_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \lambda \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} d\lambda / dm \\ dx_1 / dm \\ dx_2 / dm \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

These conditions are solved to derive income and substitution effects:

$$\frac{dx_1}{dm} = \frac{p_2 U_{12} - p_1 U_{22}}{|\tilde{H}|}$$

$$\frac{dx_1}{dp_1} = \underbrace{-\frac{\lambda p_2^2}{|\tilde{H}|}}_{\text{substitution effect}} \underbrace{-x_1 \frac{p_2 U_{12} - p_1 U_{22}}{|\tilde{H}|}}_{\text{income effect}}$$

Notes

Endowment in Goods

Let the consumer be endowed with goods $\omega \in \mathbb{R}^n_+$ instead of cash m.

If so, a consumer chooses Marshallian demand $x(p, p\omega)$ by solving

$$\max_{x \in \mathbb{R}^n_+} U(x) \quad \text{subject to} \quad px \leq p\omega.$$

If so, Slutsky equation becomes

$$\frac{dx_i(p, p\omega)}{dp_i} = \frac{\partial h_i(p, u)}{\partial p_i} + (\omega_i - x_i(p, m)) \frac{\partial x_i(p, m)}{\partial m}$$

as prices directly affect the budget $m = p\omega$.

Marshallian demand can be upward sloping even for normal goods provided that $\omega_i > x_i(p, m)$!

Notes

Endowment in Goods: Leisure Demand

Denote by: x_1 leisure; p_1 the hourly wage; x_2 consumption; p_2 its price.

Posit that $\omega = (T, 0)$ where T denotes the number of hours available.

The wage affects demand both as a price and as a determinant of income:

$$\frac{dx_1(p, p\omega)}{dp_1} = \frac{\partial x_1(p, p\omega)}{\partial p_1} + T \frac{\partial x_1(p, p\omega)}{\partial m}.$$

By the classical Slutsky decomposition, we have that

$$\frac{\partial x_1(p, p\omega)}{\partial p_1} = \frac{\partial h_1(p, v(p, p\omega))}{\partial p_1} - x_1(p, p\omega) \frac{\partial x_1(p, p\omega)}{\partial m}.$$

Thus, even if leisure is normal, its demand can increase in wages

$$\frac{dx_1(p, p\omega)}{dp_1} = \frac{\partial h_1(p, v(p, p\omega))}{\partial p_1} + (T - x_1(p, p\omega)) \frac{\partial x_1(p, p\omega)}{\partial m}.$$

and labour supply, $T - x_1(p, p\omega)$, can be downward sloping.

Notes

Welfare

Notes

Welfare

Back to standard utility theory.

We want to use the tools developed (expenditure function, indirect utility, conditional demand) to assess how the welfare of the consumer changes when parameters such as prices change.

Consumers surplus is one way to assess welfare.

But we want to think of it more generally, and still include CS.

A possible approach would be to think of differences in indirect utility.

But that may not help much as well.

Notes

Consumer Surplus

Consider the market for good 1, where $x_1(p, m)$ is demand for good 1.

Suppose that the only price of good 1 decreases from p_1^0 to p_1^1.

Refer to the area under the demand curve as the **consumer surplus** CS.

It measures the "willingness to pay" of a consumer.

CS measures the normalized change in purchasing power of the consumer

$$CS = \int_{p_1^1}^{p_1^0} \frac{\partial e(p, v(p, m))}{\partial p_1} dp_1 = \int_{p_1^1}^{p_1^0} x_1(p, m) dp_1 = \int_{p_1^0}^{p_1^1} \frac{\partial v(p, m)/\partial p_1}{\partial v(p, m)/\partial m} dp_1.$$

Utility is normalized by $\partial v/\partial m$ so to express surplus in dollars and not utiles.

Notes

Alternative Criteria: A Perfect Substitutes Example

Consider a utility map $u(x_1, x_2) = x_1 + x_2$ and income $m = 1$.

A price change from $(2, 1)$ to $(1/2, 1)$ increases utility from 1 to 2.

How much is this price change worth to the consumer?

Criterion 1: At the old prices, how much should income grow for the consumer to achieve utility 2.

At prices $(2, 1)$ the consumer needs an income of 2 to attain the target utility of 2. If so, the change is worth 1.

Criterion 2: At the new prices, how much should income fall for the consumer to achieve utility 1.

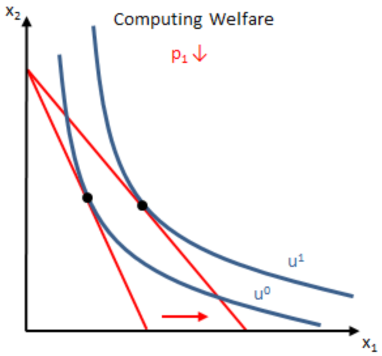
At $(1/2, 1)$ the consumer needs an income of $1/2$ to attain the initial utility of 1. If so, the change is worth $1/2$.

Notes

Alternative Criteria: Monetary Measures of Welfare

Suppose that prices change from a vector p^0 to a vector p^1. Let:

- $u^0 = v(p^0, m)$ denote the initial utility;
- $u^1 = v(p^1, m)$ denote the final utility.



Notes

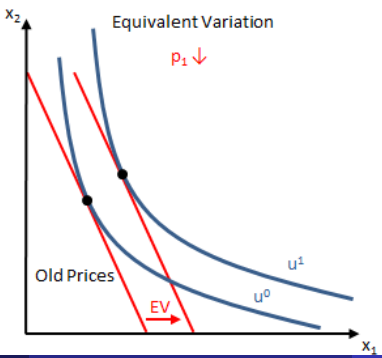
Alternative Criteria: Equivalent Variations

The **equivalent variation** is defined as

$$EV = e(p^0, u^1) - m = e(p^0, u^1) - e(p^0, u^0).$$

It is the change in income that the consumer needs at the old prices p^0 to be as well off as at the new prices p^1,

$$u^1 = v(p^0, m + EV).$$



Notes

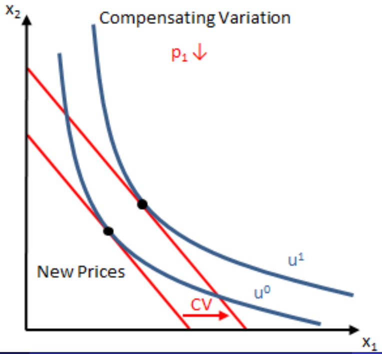
Alternative Criteria: Compensating Variations

The **compensating variation** is defined as

$$CV = m - e(p^1, u^0) = e(p^1, u^1) - e(p^1, u^0).$$

It is the change income that the consumer needs at the new prices p^1 to be as well off as at the old prices p^0,

$$u^0 = v(p^1, m - CV).$$



Notes

Alternative Criteria: CV vs EV

If $u^1 > u^0$, $EV > 0$ and $CV > 0$.

However, in general $EV \neq CV$.

EV and CV use different reference points:

- EV compensates for maintaining the status quo;
- CV compensates for changing the status quo.

Notes

Example: Cobb-Douglas

Recall that for Cobb-Douglas preferences we had:

$$v(p,m) = \frac{m^2}{p_1 p_2} \quad \& \quad e(p,u) = \sqrt{u p_1 p_2}.$$

Again consider an example with: $m = 1$, $p^0 = (1,1)$, $p^1 = (1/2, 1)$.

If so, find that:

- $u^0 = 1$, $u^1 = 2$;
- $EV = e(p^0, u^1) - m = \sqrt{2} - 1 = 0.41$;
- $CV = m - e(p^1, u^0) = 1 - \sqrt{1/2} = 0.29$;
- $CS = [\log(1) - \log(1/2)]/2 = 0.35$.

Notes

Comparing Welfare Criteria

Suppose that $p_1^1 < p_1^0$ and $p_i^1 = p_i^0 = p_i$, $i = 2, ..., n$.

If p^t denotes date $t \in \{0,1\}$ prices, welfare measures can be written as

$$EV = e(p^0, u^1) - e(p^1, u^1) = \int_{p_1^1}^{p_1^0} \frac{\partial e(p, u^1)}{\partial p_1} dp_1 = \int_{p_1^1}^{p_1^0} h_1(p, u^1) dp_1,$$

$$CV = e(p^0, u^0) - e(p^1, u^0) = \int_{p_1^1}^{p_1^0} \frac{\partial e(p, u^0)}{\partial p_1} dp_1 = \int_{p_1^1}^{p_1^0} h_1(p, u^0) dp_1,$$

where the second equalities hold by Fundamental Theorem of Calculus.

Similarly, consumer surplus amounted to

$$CS = \int_{p_1^1}^{p_1^0} \frac{\partial e(p, v(p,m))}{\partial p_1} dp_1 = \int_{p_1^1}^{p_1^0} x_1(p,m) dp_1.$$

Notes

Comparing Welfare Criteria

Differentiating $x_1(p, m) = h_1(p, v(p, m))$ with respect to m implies

$$\frac{\partial x_1}{\partial m} = \frac{\partial h_1}{\partial u}\frac{\partial v}{\partial m}.$$

As LNS implies $\partial v/\partial m > 0$, $\partial x_1/\partial m$ has the same sign as $\partial h_1/\partial u$.

So, for $p_1 \in [p_1^1, p_1^0]$, we obtain that $v(p, m) \in [u^0, u^1]$.

Consequently, for normal goods, we have that

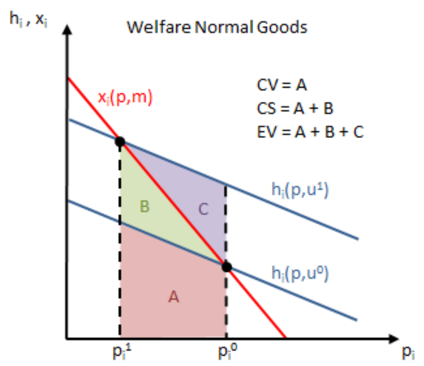
$$x_1(p, m) \in [h_1(p, u^0), h_1(p, u^1)],$$

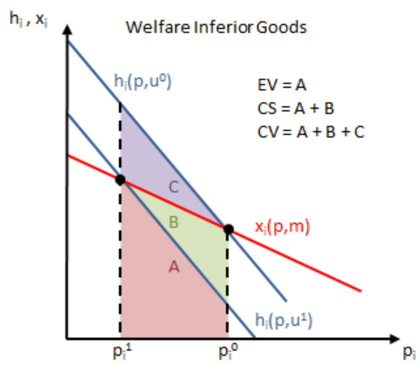
whereas for inferior goods we have that

$$x_1(p, m) \in [h_1(p, u^1), h_1(p, u^0)].$$

Notes

Comparing Welfare Criteria: For Price Declines





- For normal goods: $CV \le CS \le EV$.
- For inferior goods: $EV \le CS \le CV$.

Notes

Example: Comparing Welfare Criteria

Let p_{-i} be the price vector that excludes p_i, and consider

$$v(p, m) = a(p) + b(p_{-i})m.$$

Marshallian demand of good i is independent of m,

$$x_i(p, m) = -\frac{\partial a(p)/\partial p_i}{b(p_{-i})}.$$

The expenditure function satisfies

$$e(p, u) = -\frac{a(p)}{b(p_{-i})} + \frac{u}{b(p_{-i})} \quad \Rightarrow \quad h_i(p, u) = -\frac{\partial a(p)/\partial p_i}{b(p_{-i})}.$$

Hence, since $h_i(p, u) = x_i(p, m)$,

$$EV = CV = CS \;\text{ for changes in } p_i.$$

Intuition: The marginal utility of income is independent of p_i. Hence changes are independent of the base price p_i.

Notes

Example: Quasi-Linear Utility

Consider the utility function: $u(x_1, x_2) = 2\sqrt{x_1} + x_2$.

At an interior solution Marshallian demands satisfy:

$$x_1(p, m) = \left(\frac{p_2}{p_1}\right)^2 \quad \& \quad x_2(p, m) = \frac{m}{p_2} - \frac{p_2}{p_1}.$$

The indirect consequently satisfies:

$$v(p, m) = \frac{m}{p_2} + \frac{p_2}{p_1} \quad \Rightarrow \quad b(p_{-1}) = \frac{1}{p_2}.$$

At an interior solution $EV = CV = CS$ for changes in p_1.

Notes

Extra: Final Considerations on Welfare

Remark: If two alternative final prices p^1 and p^2 are such that

$$v(p^1, m) = v(p^2, m) = u^1.$$

It must be that $EV^1 = EV^2$, since

$$v(p^0, m + EV^1) = v(p^0, m + EV^2) = u^1.$$

Because the final utility and initial prices coincide, so do EV.

It is **not** necessarily true however that $CV^1 = CV^2$.

Since prices differ, so may CV's,

$$v(p^1, m - CV^1) = v(p^2, m - CV^2) = u^0.$$

We need to know the expenditure function which we can recover from x.

Thoughts: Why do we measure welfare through utility?
What about well being or happiness? Can it be patronizing?

Notes

Advances in Decision Theory

Notes

Advances in Decision Theory

The utility representation is neutral with respect to the working of the human mind.

The motives, the reasoning, and the emotions of the consumer are:

1. Implicit;
2. Relevant is so far as they appear in actual choices.

On the contrary, behavioral economics attempts to model **explicitly** psychological aspects of decisions.

In particular, it considers several relevant departures from our standard axiomatic model of behavior.

Notes

Reference Dependent Preferences

The **endowment effect**: consumers value more commodities they own relative to commodities they do not own.

More generally, the preferences of consumer can depend on reference points such as expectations or targets.

For example, a consumer may be ex-ante indifferent between free trip to X and 1,000£. However, if the consumer expects to travel to X, he may be unwilling to give up the trip for 1,050£.

Notes

Experimental Evidence: Mugs?

Random students were given Cornell University mugs which were for sale for 6$ at the campus store.

The remaining students were given nothing.

After this the two groups were asked respectively:

- How much would you willing to sell for?
- How much would you be willing to buy the mug for?

The reported selling price was 5.25$.
The reported buying price only 2.75$.

Implications:

- loss vs gain effect;
- less trade than standard theory.

Notes

Prospect Theory: Kahneman and Tversky (1979)

Preferences depend not just on outcomes, but also on how outcomes differ from some reference point.

Biases in cognition and perception also affect preferences.

> *When we respond to attributes such as brightness, loudness, or temperature, the past and present context of experience defines an adaptation level, or reference point, and stimuli are perceived in relation to this reference point. Thus, an object at a given temperature may be experienced as hot or cold to the touch depending on the temperature to which one has adapted. (Kahneman and Tversky 1979)*

Not only are sensory perceptions reference-dependent but so are judgments and evaluations about outcomes in one's life.

Notes

Prospect Theory: Kahneman and Tversky (1979)

Do I have a good income? Depends on the standards of my society.

Should I be happy with a 5% raise? Depends on what I was expecting.

Kahneman and Tversky (1979) introduce four main features:

- Reference Dependence
- Loss Aversion
- Diminishing Sensitivity
- Nonlinear Probability Weighting

Koszegi & Rabin 2006-2008 expand this model by endogenizing reference points with rational expectations.

Notes

Reference Dependence: Koszegi & Rabin Style

KR assume that the preferences are determined by a map $u(x|r)$ where x denotes a consumption bundle and r **reference point**.

Preferences of a player in particular satisfy

$$u(x|r) = v(x) + g(x|r).$$

where $v(x)$ is the "consumption utility" and $g(x|r)$ the "gain-loss utility".

Consumption utility $v(x)$ satisfies $v' > 0$ and $v'' < 0$, and thus exhibits decreasing marginal utility.

Utility is often also additively separable across dimensions

$$v(x) = \sum_{i=1}^{n} v_i(x_i) \quad \text{and} \quad g(x|r) = \sum_{i=1}^{n} g_i(x_i|r_i).$$

Notes

Reference Dependence: Koszegi & Rabin Style

The gain-loss function is often a "universal gain-loss function" satisfying

$$g_i(x_i|r_i) = \mu(v_i(x_i) - v_i(r_i)).$$

The expression says that gains and losses are a function only of the change in consumption utility relative to the reference point.

The following assumptions are imposed on universal gain-loss functions:

1. $\mu(v)$ is continuous for any v and twice differentiable for $v \neq 0$.
2. $\mu(v)$ is strictly increasing, and $\mu(0) = 0$.
3. $\mu(y) + \mu(-y) < \mu(v) + \mu(-v)$ for $y > v > 0$.
4. $\mu''(v) \leq 0$ for $v > 0$, and $\mu''(v) \geq 0$ for $v < 0$.
5. $\mu'_-(0)/\mu'_+(0) > 1$ for

$$\mu'_+(0) = \lim_{v\to 0} \mu'(|v|) \text{ and } \mu'_-(0) = \lim_{v\to 0} \mu'(-|v|).$$

Notes

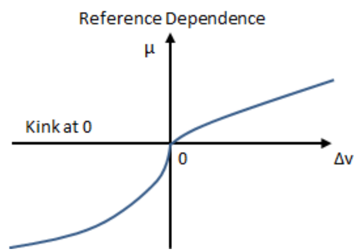
Reference Dependence: Gain-Loss Function

Local concavity at 0 is captured by assumption 5.

Loss aversion on large stakes comes from the 3.

Loss aversion on small stakes comes from the 5.

Diminishing sensitivity comes from 4.



Notes

Reference Dependence: Example

A person has utility over two goods mugs, x_1, and money, x_2.

Let $v_1(x_1) = 6x_1$, let $v_2(x_2) = x_2$, and let μ be piecewise-linear

$$\mu(v) = \begin{cases} v & v \geq 0 \\ 2v & v < 0 \end{cases}$$

First, consider a mugless person not expecting to buy a mug, $r = (0,0)$.

If she buys one mug at a price p, she gets:

$$\begin{aligned} u(1,-p|0,0) &= v_1(1) + v_2(-p) + \mu(v_1(1) - v_1(0)) + \mu(v_2(-p) - v_2(0)) \\ &= 6 - p + \mu(6) + \mu(-p) = 6 - p + 6 - 2p = 12 - 3p. \end{aligned}$$

If she doesn't buy then she gets:

$$u(0,0|0,0) = v_1(0) + v_2(0) + \mu(v_1(0) - v_1(0)) + \mu(v_2(0) - v_2(0)) = 0.$$

So if $12 - 3p > 0$ she will buy. Her willingness to pay for the mug is 4.

Notes

Reference Dependence: Koszegi & Rabin Style

Next, consider a person with a mug not expecting to sell it $r = (1,0)$.

If she sells her mug at a price q, she gets:

$$\begin{aligned} u(0,q|1,0) &= v_1(0) + v_2(q) + \mu(v_1(0) - v_1(1)) + \mu(v_2(q) - v_2(0)) \\ &= q + \mu(-6) + \mu(q) = 2q - 12. \end{aligned}$$

If she doesn't sell then she gets:

$$u(1,0|1,0) = v_1(1) + v_2(0) + \mu(v_1(1) - v_1(1)) + \mu(v_2(0) - v_2(0)) = 6.$$

So if $2q - 12 \geq 6$ she will sell. Her willingness to accept for the mug is 9.

The endowment effect applies here as a consumer who owns the mug values it 9 whereas one who does not own it values it only 4.

Notes

Reference Dependence: Personal Equilibrium

An apparent problem with such preferences is that:

- Actions are chosen based on expectations (a person expecting to buy will pay more than one who does not).
- Expectations are constrained by actions (expectations are rational).

We need a solution concept to get around this circularity. Solution:

- Agents correctly perceive the environment and their responses.
- Reference points generated by these beliefs maximize expected utility.

Definition

A consumption bundle x is a **personal equilibrium** if $u(x|x) \geq u(x'|x)$ for any other consumption bundle x'.

Notes

Reference Dependence: Personal Equilibrium Example

Botond Goes Shoe Shopping:

- Botond likes to go shoe shopping.
- Botond's consumption utility for money x_1 and shoes x_2 is given by

$$v_1(x_1) = x_1 \quad \text{and} \quad v_2(x_2) = 40x_2.$$

- Botond's universal gain loss function is

$$\mu(v) = \eta v \quad \text{for} \quad v \geq 0 \quad \text{and} \quad \mu(v) = \lambda \eta v \quad \text{for} \quad v < 0.$$

- Botond knows the price of the shoes.

Notes

Reference Dependence: Personal Equilibrium Example

When is it a personal equilibrium for Botond to buy shoes?

If so, the reference point is $r = (-\bar{p}, 1)$, and if Botond buys

$$u(-\bar{p}, 1 | -\bar{p}, 1) = -\bar{p} + 40 + \mu(0) + \mu(0) = -\bar{p} + 40$$

If Botond does not buy

$$u(0, 0 | -\bar{p}, 1) = \mu(-40) + \mu(\bar{p}) = -40\lambda\eta + \bar{p}\eta$$

Botond buys if the utility of buying is greater than if he does not.

The indifference point is when they are equal

$$u(-\bar{p}, 1 | -\bar{p}, 1) = u(0, 0 | -\bar{p}, 1) \quad \Leftrightarrow \quad \bar{p} = 40\frac{1+\lambda\eta}{1+\eta}.$$

There is PE in which Botond expects to buy and buys if $p \leq \bar{p}$.

If $\lambda = 3$ (losses are 3 times worse than gains) and $\eta = 1$ (gains are as important as levels) the most Botond will pay is $80.

Notes

Reference Dependence: Personal Equilibrium Example

When can it be a personal equilibrium for Botond not buy the shoes?

If so, the reference point is $r = (0, 0)$, and if Botond buys

$$u(-\hat{p}, 1 | 0, 0) = -\hat{p} + 40 + \mu(-\hat{p}) + \mu(40) = -(1+\lambda\eta)\hat{p} + (1+\eta)40$$

If Botond doesn't buy

$$u(0, 0 | 0, 0) = 0 + 0 + \mu(0) + \mu(0) = 0$$

The indifference point is when they are equal

$$u(-\hat{p}, 1 | 0, 0) = u(0, 0 | 0, 0) \quad \Leftrightarrow \quad \hat{p} = 40\frac{1+\eta}{1+\lambda\eta}.$$

There is PE in which Botond expects to not buy and does not if $p \geq \hat{p}$.

If $\lambda = 3$ and $\eta = 1$, Botond will not be able to resist buying the shoes for any price lower than $20.

Notes

Reference Dependence: Personal Equilibrium Example

When $p \in [\hat{p}, \bar{p}]$, Botond may buy or not depending on his expectations.

If he expects to buy he will, and if he does not that he will not.

Definition

Let $\mathcal{X}$ be the set of personal equilibria. Then x^* is a **preferred personal equilibrium** if it is a PE and if

$$u(x^*|x^*) \geq u(x|x) \quad \text{for all} \quad x \in \mathcal{X}.$$

To find the PPE when $p \in [\hat{p}, \bar{p}]$, calculate payoffs in the two PE as a function of the price p.

In the buy PE Botond's payoff amounts to $u(-p, 1|-p, 1) = -p + 40$, while in the no buy PE to $u(0, 0|0, 0) = 0$.

So when $p > 40$ then not buying is the only PPE; when $p < 40$ buying is the only PPE; and when $p = 40$ both are PPE.

Notes

Comments on Reference Dependence

Reference point r is derived endogenously and is based on "psychological principles".

Alternative theories also explain how consumers develop endogenously reference points based on expectations about future outcomes.

Gul and Pesendorfer show that reference dependence is equivalent to preferences failing transitivity.

- A decision maker with the above utility representation is simply a decision maker whose behavior does not exhibit transitivity for whichever reason.
- The "psychological principles" are void of empirical content.

Notes

Multi-selves Model

Multi-selves models capture preferences that are not stationary.

Now consumers choose subsets of consumption bundles (menus), knowing that in the future they will have to choose an bundle from the menu.

Define preferences over any two menus as follows if $A, B \subseteq X$,

$$A \succsim B \text{ means set } A \text{ is preferred to set } B.$$

Axiom A1: For every $A, B \subseteq X$ either $A \cup B \sim A$ or $A \cup B \sim B$.

Theorem

A preference relation over menus satisfies A1 if and only if there exist two functions $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$ such that for every $A, B \subseteq X$

$$A \succsim B \quad \textit{if and only if} \quad \max_{x \in \arg\max_{z \in A} v(z)} u(x) \geq \max_{x \in \arg\max_{z \in B} v(z)} u(x).$$

Here v represents preference when choosing from a menu, u preferences when choosing between menus.

Notes

Temptation and Self-Control: Gul & Pesendorfer Style

If consumers choose menus, a consumer who is afraid of temptation may prefer a smaller menus to a bigger ones. Eg: Broccoli and Chocolate.

A representation theorem by Gul and Pesendorfer shows why under appropriate assumptions the utility $W(A)$ of a set $A \subset X$ is:

$$W(A) = \max_{x \in A} (u(x) - c(x, A))$$

where $u(x)$ is the consumption utility, and $c(x, A)$ the temptation cost.

Furthermore, temptation costs can be represented by the functional

$$c(x, A) = \max_{y \in A} V(y) - V(x)$$

This is interpreted as the cost of choosing x over the most tempting element of A.

Notes

Temptation and Self-Control: Gul & Pesendorfer Style

Preferences over menus satisfy **set betweenness** if $A \succsim B$ implies $A \succsim A \cup B \succsim B$.

Set betweenness is necessarily satisfied if preferences are represented by W.

In fact, GP show that set betweenness, independence and continuity are necessary and sufficient for preferences to be represented by W.

Limitations:

- The functions $u(x)$ and $c(x, A)$ are only abstract constructions.
- Only the preferences over subsets ($A \succsim B$) have empirical content.
- If $u(x)$ and $c(x, A)$ have no empirical or psychological content, how useful is this representation for dynamic models where choices from the set A are actually made? The model is silent with respect to whether the consumer resists temptation.

Notes

Where to find more on this?

The general debate: *mindless or mindful economics?*

Reading list:

- Spiegler book appendix.
- Koszegi & Rabin 2006, 2007, 2008.
- Gul & Pesendorfer 2001, 2004.

Notes

Production and Firms

EC411 Slides 2

Nava

LSE

Michaelmas Term

Notes

Classical Theory of Production

Our unit of analysis in terms of behavior is a **Firm**.

We won't discuss the ownership of a firm, or its management and organization. We only ask how to operate the firm to maximize profits.

The firm is treated as a black box with no owners, workers or managers. Just a transition box that transforms inputs into ouputs.

The main ingredients of the classical theory of production are:

- **Technology Constraints**
  (Limits to production due to technological or legal constraints);
- **Profit Maximizing Behavior**
  (Firms maximize a particular objective function, profits);
- **Free Entry & Price Taking Behavior.**

This theory imposes little discipline on technology, but lots on payoffs. Firms could maximize sales, survival, welfare of workers and/or managers.

Notes

Roadmap: Production

The aim of the section is to introduce a simple model that allows us to predict the behavior of firms in the market.

The analysis proceeds as follows:

- Basic definitions, vocabulary and technological constraints.
- Firms' optimization problem:
  - profit maximization,
  - cost minimization,
  - duality.
- Supply, profit function and cost function.
- The short run and the long run.
- Partial equilibrium and taxes.
- Criticism and alternative theories.

Notes

Technology

Notes

The Technology

Definition

A **production plan** $z = (z_1, ..., z_m)$ is a vector of net outputs.

For a given production plan z, we say that:

- z_i is an **input** if $z_i < 0$;
- z_i is an **output** if $z_i > 0$.

The **production possibility set**, Y, is the set of technologically feasible production plans.

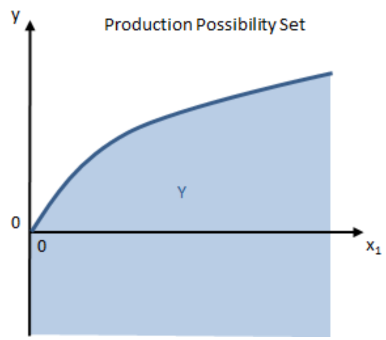
Notes

Examples: Single Output

With only one output, we can write a production plan as $(y, -x)$ where:

- y denotes the amount of output and
- $x = (x_1, ...x_n)$ is a nonnegative vector of n inputs.

For the case, of one input we have:



Notes

Properties of Y and Isoquants

We usually assume Y to be: non-empty, closed, and that can do nothing.

Although sunk costs or a positive level of inputs in place may change this.

Restrict the analysis to the case of **one** output.

The **Input Requirement Set** is defined as

$$V(y) = \{x \in \mathbb{R}^n \mid (y, -x) \in Y\}.$$

An **Isoquant** is defined as

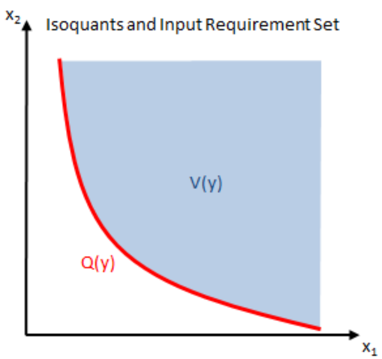
$$Q(y) = \left\{x \in \mathbb{R}^n \mid x \in V(y) \text{ and } x \notin V(y') \text{ if } y' > y\right\}.$$

Notes

Example: Isoquants

If $Y = \left\{(y, -x_1, -x_2) \mid y \le \sqrt{x_1} + \sqrt{x_2}\right\}$, we have that

$$V(\hat{y}) = \left\{(x_1, x_2) \mid \hat{y} \le \sqrt{x_1} + \sqrt{x_2}\right\},$$
$$Q(\hat{y}) = \left\{(x_1, x_2) \mid \hat{y} = \sqrt{x_1} + \sqrt{x_2}\right\}.$$



Notes

The Production Function

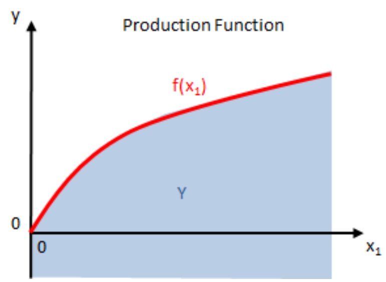
Definition

A **production function** is a map $y = f(x)$ where $f(x)$ is the maximum output obtainable from x in Y.

There are close connections between f and the properties of Y.

We can either make restrictions on Y or on f.

With 1 output and n inputs, Y is convex $\Rightarrow f$ is concave.



Notes

Example: Cobb-Douglas Production Function

Consider the following production function,

$$y = x_1^{\alpha} x_2^{\beta}.$$

The equation describing its isoquant for $y = 2$ is

$$2 = x_1^{\alpha} x_2^{\beta} \implies x_2 = \left(\frac{2}{x_1^{\alpha}}\right)^{\frac{1}{\beta}}.$$

The slope of this isoquant is

$$\frac{dx_2}{dx_1} = -\frac{\alpha}{\beta}\left(\frac{2}{x_1^{\alpha+\beta}}\right)^{\frac{1}{\beta}}.$$

It expresses the trade-off between x_1 and x_2 that keeps output constant.

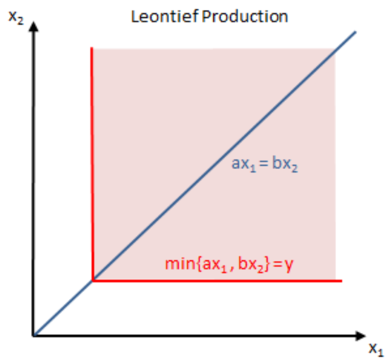
Notes

Example: Leontief Production Function

Consider the following production function,

$$y = \min\{ax_1, bx_2\}.$$

Its isoquants are non-differentiable and plotted below



Notes

Technical Rate of Substitution

The slope of isoquants is found by totally differentiating the production function and evaluating such derivative at $dy = 0$.

Consider a differentiable production function $y = f(x_1, x_2)$.

Totally differentiating the production function we obtain

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2.$$

Setting $dy = 0$, we find the **Technical Rate of Substitution** (TRS),

$$\frac{dx_2}{dx_1} = -\frac{(\partial f/\partial x_1)}{(\partial f/\partial x_2)}.$$

Notes

Technical Rate of Substitution

At a fixed output level y, the TRS expresses how much we need to change x_2, if we had changed x_1, in order to keep output at y.

The TRS is the ratio of the marginal products,

$$MP_i = (\partial f / \partial x_i) .$$

For the Cobb Douglas production function

$$TRS = -\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta-1}} = -\frac{\alpha x_2}{\beta x_1}.$$

Relatedly: at fixed inputs x, the **Marginal Rate of Transformation** (MRT) expresses how much we need to change output y_1, if we had changed output y_2, in order to keep inputs at x.

Notes

Returns to Scale

A production function $f(x)$ can exhibit:

- Constant Returns to Scale (CRS), if

$$f(tx) = tf(x) \quad \text{for any } t > 0.$$

- Decreasing Returns to Scale (DRS), if

$$f(tx) < tf(x) \quad \text{for any } t > 1 \text{ and } x \quad \text{for which } f(x) > 0.$$

- Increasing Returns to Scale (IRS):

$$f(tx) > tf(x) \quad \text{for any } t > 1 \text{ and } x \quad \text{for which } f(x) > 0.$$

For instance, convexity of Y and $0 \in Y$ together imply non-increasing returns to scale.

Notes

Example: Linear Technology

Consider a linear production function,

$$y = ax_1 + bx_2.$$

The map displays CRS, since

$$f(tx) = atx_1 + btx_2 = t(ax_1 + bx_2) = tf(x).$$

Notes

Example: Cobb-Douglas Technology

For a Cobb Douglas production function $y = x_1^\alpha x_2^\beta$ we have,

$$f(tx) = (tx_1)^\alpha (tx_2)^\beta = t^{\alpha+\beta} x_1^\alpha x_2^\beta.$$

Therefore, the production function displays:

- CRS when $\alpha + \beta = 1$, since
$$f(tx) = t^{\alpha+\beta} x_1^\alpha x_2^\beta = t x_1^\alpha x_2^\beta = tf(x).$$
- DRS when $\alpha + \beta < 1$, since
$$f(tx) = t^{\alpha+\beta} x_1^\alpha x_2^\beta < t x_1^\alpha x_2^\beta = tf(x) \quad \text{for } t > 1 \ \ \& \ \ x_1^\alpha x_2^\beta \neq 0.$$
- IRS when $\alpha + \beta > 1$, since
$$f(tx) = t^{\alpha+\beta} x_1^\alpha x_2^\beta > t x_1^\alpha x_2^\beta = tf(x) \quad \text{for } t > 1 \ \ \& \ \ x_1^\alpha x_2^\beta \neq 0.$$

Notes

Homogeneity & Homoteticity

Definition

A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is **homogenous** of degree k if and only if

$$f(tx) = t^k f(x) \quad \text{for any } t > 0 \;\; \& \;\; x \in \mathbb{R}^n_+.$$

Example: $y = x_1^\alpha x_2^\beta$ is homogenous of degree $\alpha + \beta$.

Definition

A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is **homothetic** if a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ and a homogenous of degree 1 function $h : \mathbb{R}^n_+ \to \mathbb{R}$ exist such that

$$f(x) = g(h(x)) \quad \text{for any } x \in \mathbb{R}^n_+.$$

Exercise: If a function is homothetic, TRS at x equals TRS at tx.
Hint: If a map is homogeneous of degree k, its partial derivatives are?

Notes

Profit Maximization

Notes

Intro to Profit Maximization

We have introduced assumptions about firms:

- Firms are profit maximizing, price taking entities.
- Firms have a technology defined by production possibilities set Y.
- We think of Y as non-empty, closed, and that can do nothing.
- The boundary of Y can be defined as the production function.
- The technology is what we make assumptions on.
- We have introduced concepts of returns to scale.

Now we turn to profit maximization:

- The highest profit line on Y will determine the solution.
- But a solution may not necessarily exist.
- For example if the production function exhibits IRS. GRAPH.
- For existence we need Y to be either strictly convex or bounded.

Notes

Profit Maximization

Firms are price-takers by **perfect competition**.

We employ the following conventions:

- y denotes output;
- p denotes the price of output;
- $x = (x_1, ..., x_n)$ denotes a vector of inputs;
- $w = (w_1, ..., w_n)$ denotes the vector of input prices;
- $wx = \sum_{i=1}^{n} w_i x_i$ denotes expenditure on inputs.

A firm chooses $(y, -x)$ to maximize

$$py - wx \text{ subject to } (y, -x) \in Y.$$

The highest profit line intersecting Y identifies the solution if it exists.
But no solution may exist (e.g. if Y exhibits IRS).

Notes

Profit Maximization

By profit maximization, a firm produces the maximal output given inputs.

Thus, the problem can be restated using the production function $f(x)$.

A firm chooses $(y, -x)$ to maximize

$$py - wx \text{ subject to } y = f(x).$$

Hence, a firm chooses x to maximize

$$pf(x) - wx$$

So with single output the firm essentially chooses only inputs.

Notes

Input Demands & Output Supply

Solving the profit maximization problem, we obtain:

- the vector of (unconditional) **factor or input demands**

$$x(p, w) = (x_1(p, w), x_2(p, w), ..., x_n(p, w));$$

- the **supply function** for output

$$y(p, w) = f(x(p, w)).$$

Factor demands and the supply function are homogeneous of degree zero in (p, w),

$$x(p, w) = x(tp, tw) \text{ and } y(tp, tw) = y(p, w).$$

Do we always have a solution? No, for example CRS with some prices, or IRS. We need Y to be bounded from above and strictly convex.

Notes

First Order Necessary Conditions

Recall that $x(p, w)$ maximizes $pf(x) - wx$.

This is an constrained optimization problem with linear non-negativity constraints, and a possibly concave objective function.

If $x(p, w)$ is an interior solution, **First Order Conditions** require

$$p\frac{\partial f(x)}{\partial x_i}\bigg|_{x=x(p,w)} - w_i = 0 \text{ for any } i = 1, 2, ..., n.$$

FOC for the profit maximization problem require:

- the marginal product of an input to equal its price normalized by the output price;
- the marginal rate of transformation between two inputs to equal their price ratio.

Notes

Second Order Sufficient Conditions

Define second order partial derivatives as follows,

$$f_{ij}(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

Also define the Hessian matrix associated to the optimization problem as

$$H(x) = \begin{bmatrix} pf_{11}(x) & ... & pf_{1n}(x) \\ & ... & \\ & ... & \\ pf_{n1}(x) & ... & pf_{nn}(x) \end{bmatrix}.$$

Necessary conditions also require $H(x)$ to be negative semidefinite at the solution $x = x(p, w)$.

If f is concave, or Y is strictly convex, FOC are sufficient for $x(p, w)$ to be a global maximum (not necessarily unique), as $H(x)$ is negative definite.

SOC only discipline f as the rest of the objective function is linear.

Notes

The Profit Function

Define the profit function $\pi(p, w)$ as

$$\pi(p, w) = pf(x(p, w)) - wx(p, w).$$

Recall that this is a maximum value function.

Example: Consider the production function $y = \sqrt{x_1} + \sqrt{x_2}$.

FOC require that:

$$\frac{1}{2}p(x_1)^{-\frac{1}{2}} - w_1 = 0 \quad \Rightarrow \quad x_1(p, w) = \left(\frac{p}{2w_1}\right)^2;$$
$$\frac{1}{2}p(x_2)^{-\frac{1}{2}} - w_2 = 0 \quad \Rightarrow \quad x_2(p, w) = \left(\frac{p}{2w_2}\right)^2.$$

Therefore, it follows that:

$$y(p, w) = \frac{p}{2w_1} + \frac{p}{2w_2} \quad \& \quad \pi(p, w) = \frac{p^2}{4}\left(\frac{1}{w_1} + \frac{1}{w_2}\right).$$

Notes

Profit Maximization Examples

Example: Consider the production function $y = \sqrt{\min\{x_1, x_2\}}$.

First note that to maximize profits $x_1 = x_2$.

Hence, x_1 is chosen to maximize

$$p\sqrt{x_1} - (w_1 + w_2)\,x_1.$$

The unconditional factor demands are

$$x_1(p, w) = x_2(p, w) = \left(\frac{p}{2(w_1 + w_2)}\right)^2.$$

Therefore, it follows that:

$$y(p, w) = \frac{p}{2(w_1 + w_2)} \quad \& \quad \pi(p, w) = \frac{p^2}{4}\left(\frac{1}{w_1 + w_2}\right).$$

Notes

Profit Maximization Examples

Example: $y = \min\{x_1, x_2\}$. If $p > w_1 + w_2$, profits are unbounded. The firm's supply is perfectly elastic at $p = w_1 + w_2$ (CRS).

Example: $y = x_1 + x_2$. If $p > \min\{w_1, w_2\}$, profits are unbounded. The firm's supply is perfectly elastic at $p = \min\{w_1, w_2\}$ (CRS).

Example: $y = (x_1 + x_2)^2$. The maximum profits are unbounded and the firm's supply is not well defined (IRS).

The previous examples establish that:

- we **don't always have a solution**;
- we **cannot always use first order conditions;**
- we sometimes have **multiple solutions.**

Notes

Properties of the Profit Function

Lemma

The profit function $\pi(p, w)$ satisfies the following properties

1. $\pi(p, w)$ *is non-decreasing in* p;
2. $\pi(p, w)$ *is non-increasing in* w_i, *for any* $i = 1, ..., n$;
3. $\pi(p, w)$ *is homogeneous of degree 1 in* (p, w);
4. $\pi(p, w)$ *is convex in* (p, w);
5. $\pi(p, w)$ *is continuous in* (p, w).

These are predictions that can be **tested**.

The intuition for the first two parts is obvious (plot to show).

Notes

Proving Properties of the Profit Function

1. Take $p' \geq p$. Then,

$$\pi(p, w) \leq p' y(p, w) - wx(p, w) \leq \pi(p', w).$$

2. The proof for w_i is analogous and left as an exercise.

3. By definition, for any $(y', -x') \in Y$ we have

$$py(p, w) - wx(p, w) \geq py' - wx'.$$

Hence, for $t > 0$ and for any $(y', -x') \in Y$,

$$tpy(p, w) - twx(p, w) \geq tpy' - twx'.$$

Thus, $(y(p, w), -x(p, w))$ is optimal at (tp, tw) and

$$\pi(tp, tw) = t\pi(p, w).$$

Notes

Proving Properties of the Profit Function

4. Consider two price vectors, (p, w) and (p', w').

For some $\lambda \in (0, 1)$ define:

$$(p'', w'') = \lambda (p, w) + (1 - \lambda) (p', w') .$$

With these definitions, note that:

$$\begin{array}{rl} (1) & \pi(p, w) \geq py(p'', w'') - wx(p'', w''); \\ (2) & \pi(p', w') \geq p'y(p'', w'') - w'x(p'', w''). \end{array}$$

Multiplying (1) by λ and (2) by $(1 - \lambda)$, and summing

$$\lambda\pi(p, w) + (1 - \lambda)\pi(p', w') \geq p''y(p'', w'') - w''x(p'', w''),$$
$$\text{which implies } \lambda\pi(p, w) + (1 - \lambda)\pi(p', w') \geq \pi(p'', w'').$$

5. Exercise (optional).

Notes

Hotelling's Lemma

The profit function obtains by plugging $y(p,w)$ and $x(p,w)$ in profits,

$$\pi(p,w) = py(p,w) - wx(p,w).$$

Can we reverse the operation to recover $y(p,w)$ and $x(p,w)$?

The Envelope Theorem then implies the following result.

Lemma

Hotelling's Lemma states that

$$\frac{\partial \pi(p,w)}{\partial p} = y(p,w) \quad \& \quad \frac{\partial \pi(p,w)}{\partial w_i} = -x_i(p,w), \text{ for } i = 1, ..., n.$$

If Y is well behaved, price taking behavior and fixed prices imply a dual description of technology.

Such a description has a great virtue in applications, as we can compute supply directly from the envelope theorem.

Notes

Implications of Hotelling's Lemma

By Hotelling's Lemma and the convexity of the profit function we have

$$\frac{\partial y(p,w)}{\partial p} = \frac{\partial^2 \pi(p,w)}{\partial p^2} \geq 0,$$

$$\frac{\partial x_i(p,w)}{\partial w_i} = -\frac{\partial^2 \pi(p,w)}{\partial w_i^2} \leq 0.$$

This is a simple **Slutzky equation** with **testable predictions**.

Example: Consider the production function $y = \sqrt{x_1} + \sqrt{x_2}$.

Immediately obtain that

$$\pi(p,w) = \frac{p^2}{4}\left(\frac{1}{w_1} + \frac{1}{w_2}\right), \quad y(p,w) = \frac{\partial \pi(p,w)}{\partial p} = \frac{p}{2w_1} + \frac{p}{2w_2}, \ldots$$

Notes

Implications of Hotelling's Lemma

Thus, by convexity of the profit function it follows that:

- **own-price effects are positive for output**;
- **own-price effects are negative for inputs**.

As diagonal terms of a positive semidefinite matrix, are positive.

This is known as the **law of supplies**: quantities respond in the same direction as price changes. Generally it can be written as

$$(p - p')(y - y') \geq 0.$$

There are no wealth effects, just substitution effects since prices have no effect on constraints now.

Another testable implication is that

$$\frac{\partial x_j(p, w)}{\partial w_i} = \frac{\partial x_i(p, w)}{\partial w_j} = -\frac{\partial^2 \pi(p, w)}{\partial w_i \partial w_j}.$$

Cross-price effects are symmetric (not very intuitive, but testable).

Notes

Extra: Alternative Proof Strategy

We have shown by exploiting Hotelling that

$$\frac{\partial x_i(p, w)}{\partial w_i} < 0 \;\; \& \;\; \frac{\partial y(p, w)}{\partial p} > 0.$$

To prove this directly, note that FOC require

$$pDf(x(p, w)) = w.$$

By totally differentiating FOC wrt w_1, we obtain

$$pD^2 f(x(p, w)) \frac{\partial x(p, w)}{\partial w_1} = \begin{bmatrix} pf_{11} \dots pf_{1n} \\ \dots \\ \dots \\ pf_{n1} \dots pf_{nn} \end{bmatrix} \begin{bmatrix} \partial x_1 / \partial w_1 \\ \dots \\ \dots \\ \partial x_n / \partial w_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

Notes

Extra: Alternative Proof Strategy

$$\frac{\partial x_1}{\partial w_1} = \frac{\begin{vmatrix} 1 & pf_{12} \dots pf_{1n} \\ 0 & pf_{22} \dots pf_{2n} \\ & \dots \\ 0 & pf_{n2} \dots pf_{nn} \end{vmatrix}}{|H|} = \frac{\begin{vmatrix} pf_{22} \dots pf_{2n} \\ \dots \\ pf_{n2} \dots pf_{nn} \end{vmatrix}}{|H|}$$

Since the Hessian of f is negative semidefinite:

- the denominator is either zero or has the same sign as $(-1)^n$.
- the numerator is either zero or has the same sign as $(-1)^{n-1}$.

Hence, $\partial x_1/\partial w_1 \leq 0$, if H well defined so that $|H| \neq 0$.

Profit maximization $\Rightarrow$ input demands are non-increasing in own price.

We will derive this explicitly later. Remember that with consumers it depended on income and substitution effects.

Notes

Extra: Alternative Proof Strategy

To determine the sign of $\partial y/\partial p$, differentiate FOC wrt p

$$Df(x(p,w)) + p\frac{\partial x(p,w)}{\partial p}^T D^2 f(x(p,w)) = 0$$

$$\Rightarrow \quad Df(x(p,w)) = -p\frac{\partial x(p,w)}{\partial p}^T D^2 f(x(p,w)).$$

If so, with some manipulation, we find that

$$\frac{\partial y(p,w)}{\partial p} = \frac{\partial f(x(p,w))}{\partial p} = Df(x(p,w))\frac{\partial x(p,w)}{\partial p}$$

$$= -p\frac{\partial x(p,w)}{\partial p}^T D^2 f(x(p,w))\frac{\partial x(p,w)}{\partial p} \geq 0.$$

The inequality holds since f must be negative semidefinite whenever FOC identify a maximum.

Notes

Cost Minimization

Notes

The Cost Minimization Problem

Suppose that a firm must produce y units of output.

Now, the firm chooses inputs x to minimize costs,

$$wx$$
$$\text{subject to } y = f(x).$$

The solutions are the **conditional factor or input demands**

$$x(w,y) = \begin{bmatrix} x_1(w,y) \\ x_2(w,y) \\ \dots \\ x_n(w,y) \end{bmatrix}.$$

The **cost function** is the value function of this problem

$$c(w,y) = wx(w,y) = w_1x_1(w,y) + \dots + w_nx_n(w,y).$$

Notes

Cost Minimization: Solution

The Lagrangian of this program satisfies

$$L = wx + \lambda\,(y - f(x))\,.$$

If $x(w, y)$ is an interior solution and $f_i = \partial f(x)/\partial x_i$, the FOC require that

$$\frac{\partial L}{\partial \lambda} = y - f(x(w, y)) = 0,$$
$$\frac{\partial L}{\partial x_i} = w_i - \lambda f_i(x(w, y)) = 0, \text{ for any } \ i = 1, ..., n.$$

Such conditions simplify to

$$\frac{w_i}{w_j} = \frac{f_i}{f_j}, \text{ for any } \ i, j = 1, ..., n.$$

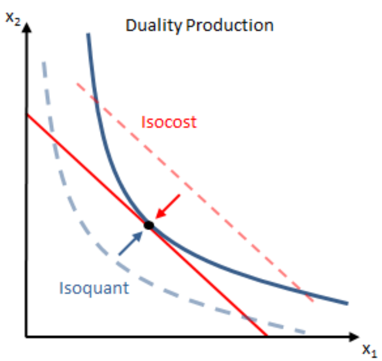
At a solution, the multiplier characterizes the marginal cost of production (i.e. marginal value of relaxing tech constraint).

Notes

Cost Minimization: Plot

For the case of two factors, $-w_1/w_2$ is the slope of the **isocost** curve

$$w_1 x_1 + w_2 x_2 = k.$$



Notes

Cost Minimization: Intuition

Why should we use cost minimization?

- it always holds locally even if the profit function is not well behaved;
- when Y is convex, there is a one-to-one mapping with profit max;
- in other words, profit max implies cost min so they do not give us more information;
- if firms are only price takers in the input market then works whereas profit max does not.

Notes

Cost Minimization: Sufficient Conditions

Consider a critical point of the Lagrangean, (x^*, λ^*),

$$f(x^*) = y \quad \text{and } \lambda^* f_i(x^*) = w_i \quad \text{for any} \quad i = 1, ..., n.$$

If $f(x)$ is concave, x^* is a global minimum.

For local SOC, instead, study Bordered Hessian of the problem,

$$\tilde{H} = \begin{bmatrix} 0 & -f_1 & \dots & -f_n \\ -f_1 & -\lambda f_{11} & \dots & -\lambda f_{1n} \\ \dots & \dots & \dots & \dots \\ -f_n & -\lambda f_{n1} & \dots & -\lambda f_{nn} \end{bmatrix}.$$

If at (x^*, λ^*) the largest $n-1$ leading principal minors are negative, x^* is a strict local minimum.

[SOC for a minimization problem require the largest $n-e$ leading minors to have the sign of $(-1)^e$, where e is the number of binding constraints].

Notes

Cost Minimization: Example I

Again, consider the production function $f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$.

FOC of the cost minimization problem require:

$$w_i - \frac{\lambda x_i^{-1/2}}{2} = 0 \text{ for } i \in \{1, 2\}.$$

Therefore, we obtain that:

$$\frac{w_1}{w_2} = \sqrt{\frac{x_2}{x_1}} \implies \sqrt{x_2} = \sqrt{x_1}\frac{w_1}{w_2} \implies y = \sqrt{x_1}\left(1 + \frac{w_1}{w_2}\right).$$

Consequently, conditional input demands, and costs satisfy:

$$x_i(w, y) = \left(\frac{w_j}{w_2 + w_1}\right)^2 y^2 \text{ for } j \neq i \in \{1, 2\};$$

$$c(w, y) = \frac{w_1 w_2}{w_2 + w_1} y^2.$$

Notes

Cost Minimization: Example II

Next, consider the production function $f(x_1, x_2) = \min\{ax_1, x_2\}$.

Optimality implies that $ax_1 = x_2 = y$.

Consequently, conditional input demands, and costs satisfy:

$$x_1(w, y) = \frac{y}{a};$$

$$x_2(w, y) = y;$$

$$c(w, y) = \left(\frac{w_1}{a} + w_2\right) y.$$

Notes

Cost Minimization: Example III

Finally, consider the production function $f(x_1, x_2) = ax_1 + x_2$.

By substituting the constraint with equality, we may choose x_1 to minimize

$$w_1 x_1 + w_2 (y - ax_1) = (w_1 - w_2 a) x_1 + w_2 y.$$

Consequently, conditional input demands satisfy:

$$\begin{array}{ll} \text{if } w_1 < aw_2 & x_1(w,y) = y/a \quad \& \quad x_2(w,y) = 0 \\ \text{if } w_1 > aw_2 & x_1(w,y) = 0 \quad \& \quad x_2(w,y) = y \\ \text{if } w_1 = aw_2 & ax_1(w,y) + x_2(w,y) = y \end{array}$$

while costs of production satisfy:

$$c(w,y) = \min\{\frac{w_1}{a}, w_2\} y.$$

These examples illustrate that **FOC approach is not ideal**. Although a solution exists by continuity, sometimes we cannot use FOC to find it.

Notes

Properties of the Cost Function

Lemma

The cost function $c(y, w)$ satisfies the following properties:

1. $c(w, y)$ *is non-decreasing in* w_i, *for* $i = 1, ..., n$;
2. $c(w, y)$ *is homogeneous of degree* 1 *in* w;
3. $c(w, y)$ *is concave in* w;
4. $c(w, y)$ *is continuous in* w.

Proof: Exercise.

The Envelope Theorem then implies the following result.

Lemma

Shephard's Lemma states that

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y) \quad \text{for any } i = 1, ..., n.$$

Notes

Properties of the Cost Function

The concavity of the cost function again implies that

$$\frac{\partial x_i(w,y)}{\partial w_i} \leq 0 \text{ for any } i = 1, ..., n.$$

By revisiting again the properties of the implied demands we can find usual conclusions about homogeneity and price effects:

- **own-price effects are negative for inputs**;
- **cross-price effects are symmetric.**

To conclude: Sometimes cost functions characterize completely technology (when technology is convex and monotone). Cost functions are an important tool, and are generally observable. The properties derived have testable implications and were derived from very few assumptions.

Notes

Short vs Long Run Production

Notes

Short Run vs Long Run Production

Given any cost function by $c(y)$, it is possible to restate the profit maximization problem as the firm choosing y to maximize

$$py - c(y)$$

FOC for this problem require

$$p - \frac{dc(y)}{dy} = 0 \implies \text{Price} = \text{Marginal Cost.}$$

So far we have only focused on **long run** (as all inputs were variable).

Now decompose total cost into **variable and fixed costs**:

$$c(y) = \underset{\text{dependent on } y}{v(y)} + \underset{\text{independent of } y}{F}$$

Notes

Fixed Costs and the Short Run

Some costs are fixed:

- because of the physical nature of inputs;
- because some inputs cannot be adjusted in the short run.

Fixed costs:

- are **sunk** and incurred even when output is zero;
- are **ignored in the short run** maximization problem.

Average Cost is defined as

$$AC(y) \equiv AVC(y) + AFC(y) \equiv \frac{v(y)}{y} + \frac{F}{y}.$$

If $AC(y)$ is increasing in y we have **diseconomies of scale**.
If $AC(y)$ is decreasing in y we have **economies of scale**.

Notes

Minimum Average Cost

First consider minimizing the average cost of production

$$\min_y AC(y) = \min_y \frac{v(y)}{y} + \frac{F}{y}.$$

FOC for AC minimization simply require

$$\frac{dAC(y)}{dy} = \frac{1}{y}\left(\frac{dc(y)}{dy} - AC(y)\right) = 0.$$

The average cost is minimized when it coincides with the marginal cost

$$\frac{dc(y)}{dy} = AC(y).$$

Notes

The Short Run Problem

Let F be sunk in the short run.

Profits can be written as

$$py - c(y) = y(p - AVC(y)) - F$$

A firm sells only if short term profits are positive

$$y(p - AVC(y)) \geq 0.$$

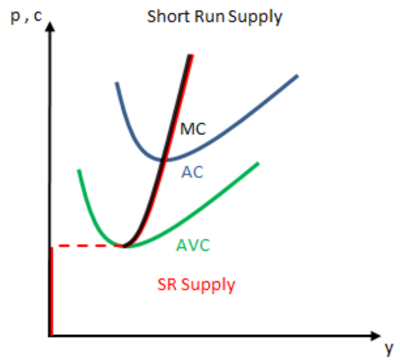
Profit maximization requires the firm:

- to set $y = 0$, if $p < \min_y v(y)/y$;
- to choose y so that $p = c'(y)$ otherwise.

Notes

The Short Run Supply Curve

The short run supply curve of a firm is plotted here



AC decreases before its minimum (the intersection with MC) and then increases.

AC decreases at first because both the AFC and the AVC decrease.

But AC increases once capacity is reached as AVC increases.

Notes

The Short Run Aggregate Supply

Consider a market with m firms.

Let $y_i(p)$ denote the short run supply of firm i.

The **short run industry (or aggregate) supply** is determined as follows:

$$S(p) = \sum_{i=1}^{m} y_i(p).$$

Aggregation is straightforward here.

It is, as if one big firm had made an optimal supply decision.

Production is efficiently allocated among firms as it minimizes costs!

All individual properties are satisfied at the aggregate level.

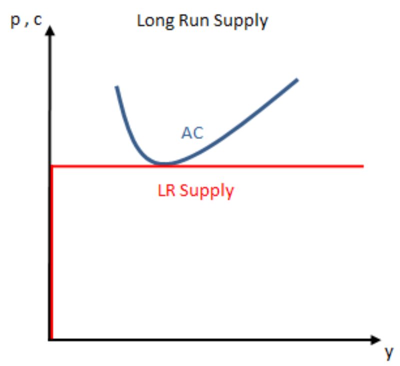
These powerful results are due to price taking behavior.

Notes

Free Entry & Long Run Aggregate Supply

In the **long run**, firms can enter and exit the industry.

With no barriers to entry, the **long run industry supply is perfectly elastic** at a price equal to the **minimum average cost**.



Notes

Markets

Notes

Producer Surplus

Consider the market for one commodity, where $y^i(p)$ is firm i's supply.

Suppose that the price of output increases from p^0 to p^1.

By Hotelling's Lemma, the change in profits for firm i is

$$\Delta^i = \int_{p^0}^{p^1} \frac{d\pi^i(p)}{dp} dp = \int_{p^0}^{p^1} y^i(p) dp.$$

We refer to Δ^i as the **producer surplus**. It identifies the area under the supply curve and measures the "willingness to pay" of a producer.

If $S(p)$ is the aggregate supply, it follows that

$$\sum_{i=1}^{m} \Delta^i = \int_{p^0}^{p^1} S(p) dp.$$

Notes

Partial Equilibrium & Elasticities

Let $D(p)$ be the **aggregate demand** for this commodity.

A **market clearing price** is a price that solves

$$S(p) = D(p).$$

A **partial equilibrium** in this market consist of a market clearing price and of the corresponding optimal demand and supply decisions.

The responsiveness of demand and supply to price changes is measured by:

- the **elasticity of demand**,

$$\varepsilon^D(p) = \frac{dD(p)}{dp} \frac{p}{D(p)} = \frac{\text{\% change in demand}}{\text{\% change in price}};$$

- the **elasticity of supply**,

$$\varepsilon^S(p) = \frac{dS(p)}{dp} \frac{p}{S(p)} = \frac{\text{\% change in supply}}{\text{\% change in price}}.$$

Notes

Taxes

Consider a market in which:

- p_S denotes the price received by producers;
- p_D denotes price paid by consumers;
- t denotes the tax rate.

This setup captures both:

- **quantity taxes**, $p_D = p_S + t$;
- **value taxes** $p_D = p_S(1 + t)$.

Equilibrium still requires market clearing (all units supplied are sold).

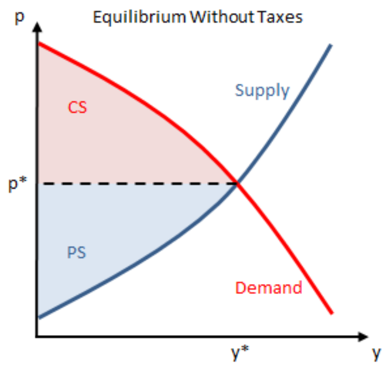
Therefore we must have that

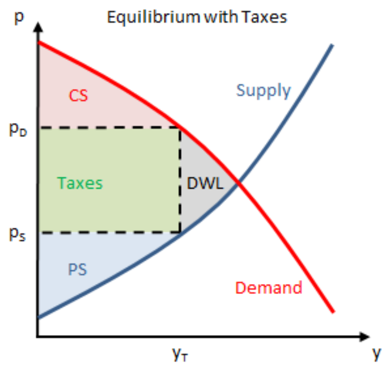
$$D(p_D) = S(p_S).$$

Notes

Quantity Taxes

The plot depicts the equilibrium in a market with quantity taxes.





The amount that consumers and producers would want to pay for removing the tax is exceeds the revenue raised because of the **deadweight loss**.

This is the classic argument for the inefficiency of taxes.

Notes

Towards General Equilibrium

Our brief introduction to equilibrium analysis was constrained to a single market, as input prices were kept constant throughout.

However, when a price changes, prices in other markets may vary too as consumers and firms readjust consumption and production decisions.

More generally, we carried out a **partial equilibrium** analysis.

This can be misleading, as comparative statics neglect changes in other markets which may have a feedback effect on the market considered.

Next we solve for a **general equilibrium** with all the markets at once.

Notes

New Theories of the Firm

Notes

Departures from the Classical Theory of the Firm

Several limitations of the classical theory have been addressed:

- Profit maximization may not be the true **objective of a manager**.
- **Property rights and firm ownership** may affect the equilibrium in a market with frictions. For instance, a supplier may prefer a **merger** with a client to bargaining in the market over prices and quantities, as the merger may solve incentive problems. A key prediction of Williamson's theory (validated empirically) is that firms are more likely to merge if what they produce or their assets are specific to each other and not to the rest of the market.
- **Non-profit** organizations may be successful without targeting profits. Like consumers their objectives are pinned down by preferences, and techniques extend to cover such scenarios.

Notes

Managers Objectives

The objectives of manager may differ from profits for several reasons.

Example 1: Managers may dislike effort and care only about their wage. If so, incentives (bonuses) may be required to induce the optimal effort decision. But when incentives are costly to provide, the firm might prefer induce a suboptimal effort decision. You will discuss this in LT.

Example 2: Managers may have career concerns. If managers are paid more in the market when they are perceived to be talented. They may care more about appearing talented and establishing a reputation, rather than about firm profits.

Reputational concerns can be beneficial and can induce managers to exert effort. However, they are occasionally harmful when managers take decisions in their best interest rather than the firm's.

Notes

Extra: Career Concerns I

Managers gain higher salaries when their reputation is high.

The state of the world w is either good or bad, $w \in \{G, B\}$.

The state is B with probability $p > 0.5$.

The correct action for the manager to take is:

- to invest in state G;
- not to invest in state B.

The manager receives a signal $s \in \{G, B\}$ about w, $\Pr(s = w|w) = q$.

The signal is private information (it is why the manager is hired).

For a good manager $q = 1$, for every other manager $q = 0.5$.

The manager knows his talent, while the market believes that he is good with probability α.

Notes

Extra: Career Concerns II

Payoffs are restricted so that:

- investment is profit maximizing only if the manager is good and signal is $s = G$;
- the manager's preferences are fully determined by the updated belief α' about his talent.

If managers maximize profits the market perceives the manager:

- as good if an investment is observed, $\alpha' = 1$;
- as less likely to be good if no investment is observed, $\alpha' < \alpha$.

If manager maximize reputations:

- bad managers know that those who invest are believed to be good;
- it is not an equilibrium for them to maximize firm profits;
- bad managers to invest more than they should to show off.

Notes

General Equilibrium

EC411 Slides 3

Nava

LSE

Michaelmas Term

Notes

Roadmap: General Equilibrium & Markets

The aim is to introduce a general model of behavior of consumers and firms that captures the consequences of spillovers across markets.

Since markets are interdependent, extending the theory of partial equilibrium to general equilibrium is a natural step.

The analysis proceeds as follows:

- Exchange Economies.
- Definition of a Walrasian Equilibrium.
- Properties of Walrasian Equilibria.
- Efficiency & Pareto Efficiency.
- Welfare Theorems.
- General Equilibrium with Production.

Notes

Exchange Economies

Notes

Exchange Economies

First we abstract from production and study how goods are traded by consumers endowed of a fixed amounts.

Consider an **exchange economy**:

- with n commodities and k consumers;
- in which each consumer j is described by:
  - a utility function $U^j(x^j) = U^j\left(x_1^j, ..., x_n^j\right)$;
  - an initial endowment $\omega^j = \left(\omega_1^j, ..., \omega_n^j\right)$.

A **consumption bundle** for consumer j is $x^j = \left(x_1^j, ..., x_n^j\right)$.

An **allocation** $x = (x^1, ..., x^k)$ assigns a bundle to each consumer.

An allocation x is **feasible** if $\sum_{j=1}^k x^j \leq \sum_{j=1}^k \omega^j$.

A feasible allocation obtains by redistributing or destroying endowments.

Notes

Walrasian Equilibrium

Definition

A **Walrasian equilibrium** $(\bar{x}, \bar{p})$ consists of an allocation $\bar{x} = (\bar{x}^1, ..., \bar{x}^k)$ and of a price vector $\bar{p} = (\bar{p}_1, ..., \bar{p}_n)$ such that;

1. $\bar{x}$ is feasible;
2. $\bar{x}^j$ maximizes $U^j(x^j)$ subject to $\bar{p}x^j \leq \bar{p}\omega^j$ for all $j = 1, ..., k$.

Walrasian equilibria are also referred to as competitive or general equilibria.

A competitive equilibrium consists of an allocation and of a price vector st:

- consumers choices are optimal when taking as given such prices;
- the implied demands are feasible (i.e. all markets clear).

Notes

Introductory Example: Market Clearing

Consider a simple economy with:

- $k = 2$ and $n = 2$;
- $\omega^1 = (3, 1)$ and $U^1(x^1) = \min\{x_1^1, x_2^1\}$;
- $\omega^2 = (1, 3)$ and $U^2(x^2) = x_1^2 x_2^2$.

Solving the consumers' problems we obtain that

$$x_1^1(p, p\omega^1) = \frac{3p_1 + p_2}{p_1 + p_2} \qquad x_2^1(p, p\omega^1) = \frac{3p_1 + p_2}{p_1 + p_2}$$

$$x_1^2(p, p\omega^2) = \frac{p_1 + 3p_2}{2p_1} \qquad x_2^2(p, p\omega^2) = \frac{p_1 + 3p_2}{2p_2}$$

Equilibrium conditions require the two markets to clear:

$$x_1^1 + x_1^2 = \frac{3p_1 + p_2}{p_1 + p_2} + \frac{p_1 + 3p_2}{2p_1} = 4;$$
$$x_2^1 + x_2^2 = \frac{3p_1 + p_2}{p_1 + p_2} + \frac{p_1 + 3p_2}{2p_2} = 4.$$

Notes

Introductory Example: Equilibrium

From the equilibrium condition for good 1 obtain

$$\frac{(p_1 + 3p_2)(p_2 - p_1)}{2p_1(p_1 + p_2)} = 0 \quad \Rightarrow \quad p_1 = p_2.$$

Substituting in the condition for the market for good 2

$$2 + 2 = 4.$$

Key observations:

- Equilibrium in the market for good 1 implies
  Equilibrium in the market for good 2.
- The equilibrium price vector is not unique.
  Only the price ratio is pinned down, $\bar{p}_1/\bar{p}_2 = 1$.
- The equilibrium allocation satisfies:
  $x_1^1 = 2$, $x_2^1 = 2$, $x_1^2 = 2$, and $x_2^2 = 2$.

Notes

Introductory Example: Edgeworth Box

A useful exercise is to plot the example in and Edgeworth Box.

Proceed by:

- plotting the Edgeworth box and the feasible allocations;
- finding the endowment, and drawing budget constraints;
- plotting the equilibrium itself;
- showing that $p_1/p_2 > 1$ cannot be an equilibrium, since consumer 1 demands of good 2 more than 2 wants to give.

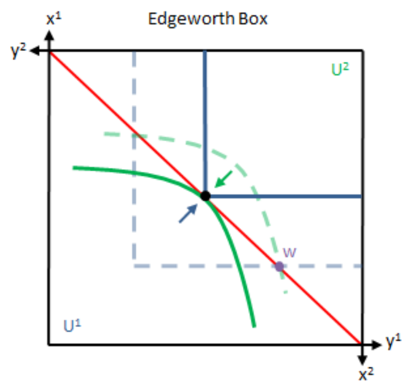
Show more generally that an equilibrium is a price ratio for which consumers's demands are optimal and markets clear.

At an interior solution consumers equalize MRS. But boundary solutions, multiplicity of equilibria and non-existence are other possible phenomena.

Notes

Introductory Example: Edgeworth Box

The plot of Edgeworth Box in this example thus amounts to:



The solution is interior and equalizes the MRS of the two consumers.

Notes

Normalizing Prices

Recall that:

- $x_i^j(p, p\omega^j)$ denotes j's demand for good i;
- $x^j(p, p\omega^j)$ denotes vector of j's demands;
- $x_i^j(p, p\omega^j)$ is homogeneous of degree 0 in p.

Fact

Because demands are homogeneous of degree 0 if $\bar{p}$ is an equilibrium price vector, $t\bar{p}$, $t > 0$, is also an equilibrium price vector.

Hence, we can normalize one price.

Without loss, set $\bar{p}_i = 1$ for one arbitrary good i.

Call good i the **numeraire good.**

Notes

Excess Demand & Walras Law

Define the vector of j's **excess demands** as

$$z^j(p, p\omega^j) = x^j(p, p\omega^j) - \omega^j.$$

Define the vector of **aggregate excess demands** as

$$z(p) = \sum_{j=1}^k z^j(p, p\omega^j).$$

Define the **aggregate excess demand for good** i as

$$z_i(p) = \sum_{j=1}^k \left(x_i^j(p, p\omega^j) - \omega_i^j\right).$$

Lemma (Walras Law)

If LNS holds, the value of excess demand is equal to zero, $p_i z_i(p) = 0$.

Notes

Walras Law: Proof

Proof: As LNS (Local Non Satiation) holds,

$$pz^j(p, p\omega^j) = p\left(x^j(p, p\omega^j) - \omega^j\right) = 0, \text{ for any } j = 1, ..., k.$$

Summing over all players then implies that

$$pz(p) = p\sum_{j=1}^k z^j(p, p\omega^j) = \sum_{j=1}^k pz^j(p, p\omega^j) = 0.$$

The latter can be restated in terms of goods as follows

$$pz(p) = \sum_{i=1}^n p_i \sum_{j=1}^k \left(x_i^j(p, p\omega^j) - \omega_i^j\right) = \sum_{i=1}^n p_i z_i(p) = 0.$$

Observe that $p_i z_i(p) \leq 0$ for all i since:

- prices are non-negative, $p_i \geq 0$;
- excess demands are non-positive by feasibility $z_i(p) \leq 0$.

But the latter than implies that $p_i z_i(p) = 0$ for all i!

Notes

Some Remarks: Assumptions

Competitive equilibria are reduced descriptions of an economy in which institutions of exchange remain unmodelled (a market can be conceived as an auctioneer, as a central market place, or as a market maker).

We do not explain how consumers trade. Agents act as price-taking utility-maximizing machines, markets are conceived as frictionless institutions in which exchange takes place at once (**invisible hand**).

Some strong assumption were invoked to derive our results.

These required markets to be frictionless, and required consumers:

- to be able to buy and sell any amount of goods at current prices;
- to be aware of the current prices;
- to take prices as given;
- to face the same prices;
- to maximize utility.

Notes

Some Remarks: Limitations & Evidence

Limitations arise because the model cannot capture environments in which:

- goods trade at different prices at different locations;
- consumers ignore, but dynamically forecast of future prices;
- consumers cannot buy or sell quantities without affecting your prices.

State contingent commodities can bypass some of these limitations.

Evidence: In controlled experiments, subjects were induced to have different preferences, and put into an exchange economy, buying and selling commodities. A middle man who ignored the aggregate demand and supply, was chosen to clear markets buying from sellers, while selling to buyers. This operation was repeated numerous times to allow learning about the environment. Subjects learnt quickly the prices at which markets would clear, and these prices were close to equilibrium prices. The evidence supported the Walrasian Equilibrium as an appropriate solution concept for large centralized markets (Plott 1986).

Notes

Efficiency and Welfare

Notes

Pareto Efficiency

A common justification for markets is that markets attain efficiency.

The next definition is crucial for the welfare properties of equilibria.

Definition

An allocation $\bar{x} = (\bar{x}^1, ..., \bar{x}^k)$ is **Pareto efficient** if and only if:

1. it is feasible;
2. there is no other feasible allocation $\hat{x} = (\hat{x}^1, ..., \hat{x}^k)$ such that:
   (a) $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$ for any $j \in \{1, ..., k\}$;
   (b) $U^j(\hat{x}^j) > U^j(\bar{x}^j)$ for some $j \in \{1, ..., k\}$.

Comments:

- there are no prices involved in the definition of Pareto efficiency;
- in an EB: find the inefficient, efficient, boundary efficient allocations.

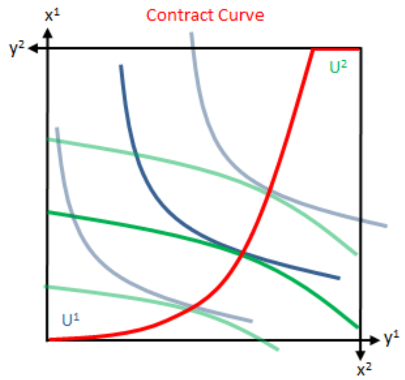
Notes

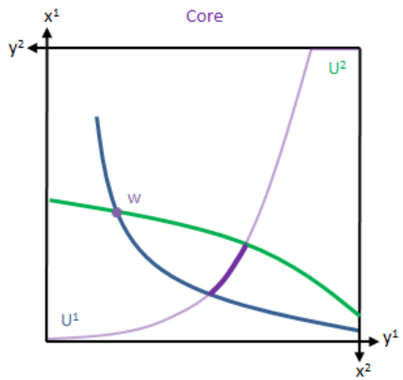
The Pareto Problem & The Core

A Pareto efficient (PE) allocation $(\bar{x}^1, \bar{x}^2)$ maximizes $U^1(x^1)$ subject to:

$$U^2(x^2) = U^2(\bar{x}^2) \quad \text{and} \quad x^1 + x^2 = \omega^1 + \omega^2.$$

The **contract curve** identifies all PE allocations, while the **core** identifies those PE allocations in which players are better off than at the endowment.





Notes

Optimality in the Pareto Problem

If $(\bar{x}^1, \bar{x}^2)$ is interior, FOC for the Pareto problem imply

$$MRS^1 = MRS^2 \;\; \& \;\; \bar{x}^1 + \bar{x}^2 = \omega^1 + \omega^2.$$

Example: Consider and economy with:

- $k = 2$ and $n = 2$;
- $\omega^1 = (3, 1)$ and $U^1(x^1) = x_1^1 x_2^1$;
- $\omega^2 = (1, 3)$ and $U^2(x^2) = x_1^2 x_2^2$.

FOC for the Pareto problem require:

$$x_2^1 / x_1^1 = x_2^2 / x_1^2, \;\; x_1^1 + x_1^2 = 4, \;\; \text{and} \;\; x_2^1 + x_2^2 = 4.$$

It follows immediately that

$$x_2^1 / x_1^1 = (4 - x_2^1) / (4 - x_1^1) \quad \Rightarrow \quad x_2^1 = x_1^1.$$

The contract curve coincides with the 45° line.

Notes

Example: Pareto Problem

Consider and economy with:

- $k = 2$ and $n = 2$;
- $\omega^1 = (0, 0)$ and $U^1(x^1) = x_1^1 x_2^1$;
- $\omega^2 = (4, 2)$ and $U^2(x^2) = x_1^2 + x_2^2$.

The interior KKT FOC for the Pareto problem require:

$$x_2^1 / x_1^1 = 1, \quad x_1^1 + x_1^2 = 4, \quad \text{and} \quad x_2^1 + x_2^2 = 2.$$

Such conditions cannot be satisfied when $U^1(x^1) > 4$.

Looking at corners in this scenario, we find that $x_2^1 = 2$.

Thus, the contract curve can be expressed as

$$x_2^1 = \min\{x_1^1, 2\}.$$

Notes

The First Welfare Theorem

Theorem (The First Welfare Theorem)

If LNS holds, any competitive equilibrium allocation is Pareto efficient.

Proof: Let $(\bar{x}, \bar{p})$ be a competitive equilibrium.

By definition of competitive equilibrium, $\bar{x}$ is feasible.

Suppose however that $\bar{x}$ is not Pareto efficient.

If so, there exists another feasible allocation $\hat{x}$ such that

(a) $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$ for any $j \in \{1, ..., k\}$;

(b) $U^j(\hat{x}^j) > U^j(\bar{x}^j)$ for some $j \in \{1, ..., k\}$.

Notes

The First Welfare Theorem

Proof Continued: Recall that $\hat{x}$ satisfies
(a) $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$ for any $j \in \{1, ..., k\}$;
(b) $U^j(\hat{x}^j) > U^j(\bar{x}^j)$ for some $j \in \{1, ..., k\}$.

First, observe that (a) implies that $\bar{p}\hat{x}^j \geq \bar{p}\omega^j$ for any j.

If $\bar{p}\hat{x}^j < \bar{p}\omega^j$ for some j, by LNS there exists $\tilde{x}^j$ such that

$$\bar{p}\tilde{x}^j < \bar{p}\omega^j \quad \text{and} \quad U^j(\tilde{x}^j) > U^j(\hat{x}^j) \geq U^j(\bar{x}^j).$$

But this is impossible as $\bar{x}^j$ maximizes utility.

Next observe that (b) implies that $\bar{p}\hat{x}^j > \bar{p}\omega^j$ for some j.

This follows, as $\bar{x}^j$ maximizes $U^j(x^j)$ subject to $\bar{p}x^j \leq \bar{p}\omega^j$.

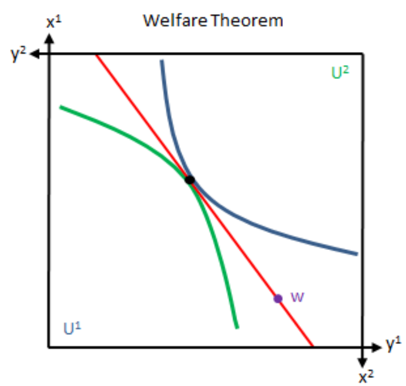
Hence, $\sum_{j=1}^k \bar{p}\hat{x}^j > \sum_{j=1}^k \bar{p}\omega^j$, contradicting the feasibility of $\hat{x}$. QED

Notes

First Welfare Theorem: Plot

Key assumptions:

- Price taking behavior [agents take prices as a given]
- No externalities [utility only depends on consumption]
- Complete markets [any commodity can be traded]



Notes

Comments on the First Welfare Theorem

Interior competitive equilibria are always Pareto efficient as consumers set their MRS equal to the price ratio. Therefore, MRS coincide across players and efficiency obtains.

General Comments on Pareto Efficiency:

- Although Pareto Efficiency is a meaningful concept to evaluate welfare it completely abstracts from fairness. A dictator consuming all goods would yield Pareto efficiency. Other welfare criteria have been contemplated such as max-min utility, or utility aggregation.
- The Jungle Economics: instead of prices, it is possible to consider power relations. An equilibrium allocation is such that no one can be better off by taking something available to him. This model also has an equilibrium which is Pareto efficient.

Notes

Example Externalities

Consider an economy with:

- $k = 2$ and $n = 2$;
- $\omega^1 = (3, 1)$ and $U^1(x^1) = \min\{x_1^1, x_2^1\} - 2x_1^2$;
- $\omega^2 = (1, 3)$ and $U^2(x^2) = x_1^2 x_2^2 - 2x_2^1$.

The equilibrium allocation again satisfies:

- $x_1^1 = 2,\ x_2^1 = 2,\ x_1^2 = 2,\ x_2^2 = 2.$

Both consumers are worse off after trade.

The equilibrium is Pareto dominated.

The initial utilities are $U^1 = -1$, $U^2 = 1$.

Equilibrium utilities are $U^1 = -2$, $U^2 = 0$.

Notes

The Second Welfare Theorem (Easy)

Theorem (The Second Theorem of Welfare – Easy)

Consider a Pareto efficient allocation $\bar{x}$.

Suppose that a competitive equilibrium $(\hat{p}, \hat{x})$ exists when $\omega = \bar{x}$.

If so, $(\hat{p}, \bar{x})$ is a competitive equilibrium.

Proof: Observe that for any consumer $j = 1, ..., k$:

- since $\bar{x}^j$ is in the budget constraint of j, $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$;
- since $\bar{x}$ is Pareto efficient, $U^j(\hat{x}^j) = U^j(\bar{x}^j)$.

Hence, $\bar{x}^j$ is optimal given $\hat{p}$.

Notes

The Second Welfare Theorem (Hard)

Theorem (The Second Theorem of Welfare – Hard)

Consider a Pareto efficient allocation $\bar{x}$ and suppose that:

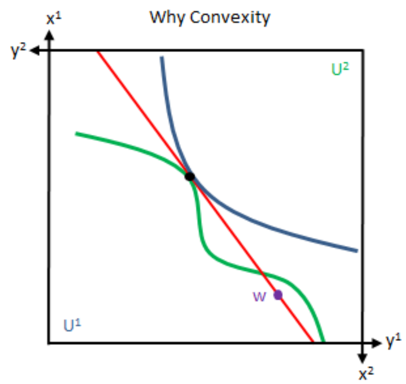
1. *The allocation $\bar{x}$ is interior.*
2. *The preferences of every consumer are*
   - *convex,*
   - *continuous,*
   - *strictly monotonic.*

Then, $\bar{x}$ is a competitive equilibrium allocation when $\omega = \bar{x}$.

Notes

Convexity

The importance of convex preferences is detailed by the plot below:



Consumer 2 does not maximize utility at ω.

If preferences are not convex, existence cannot guaranteed.

Notes

Comments on the Second Welfare Theorem

The second welfare states that any efficient allocation can be decentralized into a competitive equilibrium if transfers are feasible.

To decentralize an efficient allocation planner would need:

- information about preferences and endowments;
- power to enact transfers;
- to make sure that all prices are known.

This exercise will fail however if there are: externalities, market power, public goods, or incomplete information.

Existence of a competitive equilibrium is proven via fixed point theorems, if preferences are continuous, convex and locally non-satiated.

GE is a closed interrelated system, as opposed to partial equilibrium.
It is suitable to address problems which relate to the whole economy.
Its beauty lies in the ambitious results obtained with few free parameters.

Notes

Production Economies

Notes

General Equilibrium with Production

Consider an economy with:

- n commodities;
- k consumers;
- m firms.

Let $f^i : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ denote the production function of firm i.

Let each firm maximizes profits and denote the profit functions by

$$\pi^i(p) = \max_{x_i} pf^i(x^i) - px^i.$$

Let α^j_i be consumer j's share of the profits of firm i.

Consumer j's budget constraint becomes

$$px^j \leq p\omega^j + \sum_{i=1}^m \alpha^j_i \pi_i(p).$$

Notes

Walrasian Equilibrium with Production

Definition

A **Walrasian equilibrium** $(\bar{x}, \tilde{x}, \bar{p})$ consists of an allocation $(\bar{x}, \tilde{x}) \in \mathbb{R}_+^{(k+m)n}$ and of a price vector $\bar{p} \in \mathbb{R}_+^n$ such that:

1. $\tilde{x}^i$ solves for all $i = 1, ..., m$

$$\max_{x_i} pf^i(x^i) - px^i;$$

2. $\bar{x}^j$ solves for all $j = 1, ..., k$

$$\max_{x^j} U^j(x^j) \text{ st } px^j \leq p\omega^j + \sum_{i=1}^m \alpha_i^j \pi_i(p);$$

3. $(\tilde{x}, \hat{x})$ is feasible:

$$\sum_{j=1}^k \bar{x}^j + \sum_{i=1}^m \tilde{x}^i \leq \sum_{j=1}^k \omega^j + \sum_{i=1}^m f^i(\tilde{x}^i).$$

All the main results derived for exchange economies extend to production economies with minor modifications.

Notes

Example Production: Walrasian Equilibrium

Consider the following economy:

1. Two firms, 1 and 2, produce goods y_1 and y_2 using the same input x. Production functions satisfy

$$y_1 = \sqrt{x_1} \text{ and } y_2 = \sqrt{x_2},$$

where x_i is the amount of x used for producing y_i.

2. Two consumers, A and B, have utility functions

$$U^A = y_1^A y_2^A \text{ and } U^B = y_1^B y_2^B.$$

Consumers derive no utility from input x.

3. The total endowment of y_1 and y_2 is zero.
Consumer A owns firm 1 and 5 units of x.
Consumer B owns firm 2 and 3 units of x.

Notes

Example Production: Walrasian Equilibrium

Let p_i denote the price of output y_i, let r denote the price of input x.
Normalize $p_2 = 1$.

Firm profit maximization FOC require that

$$\frac{p_1}{2\sqrt{x_1}} - r = 0 \quad \text{and} \quad \frac{1}{2\sqrt{x_2}} - r = 0.$$

Such conditions pin down factor demands,

$$x_1^s = \left(\frac{p_1}{2r}\right)^2 \quad \text{and} \quad x_2^s = \left(\frac{1}{2r}\right)^2,$$

output supplies,

$$y_1^s = \frac{p_1}{2r} \quad \text{and} \quad y_2^s = \frac{1}{2r},$$

and consequently profit functions,

$$\pi_1 = \frac{(p_1)^2}{4r} \quad \text{and} \quad \pi_2 = \frac{1}{4r}.$$

Notes

Example Production: Walrasian Equilibrium

The firms' problems imply that:

- the income of consumer A is $m^A = 5r + (p_1)^2/4r$;
- the income of consumer B is $m^B = 3r + 1/4r$.

From FOC of the consumers' problems find output demands. For output 1,

$$y_1^A = \frac{1}{2p_1}\left(\frac{(p_1)^2}{4r} + 5r\right) \quad \text{and} \quad y_1^B = \frac{1}{2p_1}\left(\frac{1}{4r} + 3r\right).$$

Then by Walras law, the equilibrium market clearing conditions are

$$y_1 \text{ market:} \quad \frac{1}{2p_1}\left(\frac{1+(p_1)^2}{4r} + 8r\right) = \frac{p_1}{2r}$$

$$x \text{ market:} \quad \left(\frac{p_1}{2r}\right)^2 + \left(\frac{1}{2r}\right)^2 = 8$$

The competitive equilibrium satisfies

$$p_1 = 1, \quad r = 1/4, \quad y_1^A = y_2^A = 9/8, \quad \text{and} \quad y_1^B = y_2^B = 7/8.$$

Notes

Pareto Efficiency with Production

The definition of Pareto efficient allocation only changes to the extent that feasibility does.

Definition

An allocation $(\bar{x}, \tilde{x}) \in \mathbb{R}_{+}^{(k+m)n}$ is **Pareto efficient** if and only if:

1. $(\bar{x}, \tilde{x})$ is feasible:

$$\sum_{j=1}^{k} \bar{x}^{j} + \sum_{i=1}^{m} \tilde{x}^{i} \leq \sum_{j=1}^{k} \omega^{j} + \sum_{i=1}^{m} f^{i}(\tilde{x}^{i});$$

2. there is no other feasible allocation $(\hat{x}, \check{x})$ such that:
   (a) $U^{j}(\hat{x}^{j}) \geq U^{j}(\bar{x}^{j})$ for any $j \in \{1, ..., k\}$;
   (b) $U^{j}(\hat{x}^{j}) > U^{j}(\bar{x}^{j})$ for some $j \in \{1, ..., k\}$.

Notes

Example Production: Pareto Efficiency

Extend the above example to general production functions

$$y_1 = f_1(x_1) \quad \text{and} \quad y_2 = f_2(x_2).$$

Suppose that total input endowment is $x_1 + x_2 = \gamma$.

By allocating the input x between the two production processes, obtain the **production possibility frontier**

$$y_2 = T(y_1).$$

Formally, if $f_1^{-1}(y_1)$ denotes the inverse of $f_1(x_1)$, we have that

$$T(y_1) = f_2\left(\gamma - f_1^{-1}(y_1)\right).$$

Notes

Example Production: Pareto Efficiency

Recall that the marginal rate of transformation, MRT, is

$$\frac{dT}{dy_1} = -\frac{\partial f_2/\partial x}{\partial f_1/\partial x}.$$

Pareto efficient allocations can now be found by choosing $(y_1^A, y_2^A, y_1^B, y_2^B)$ so to maximize

$$U^A(y_1^A, y_2^A) \text{ subject to}$$
$$U^B(y_1^B, y_2^B) = \bar{u} \text{ and } y_2^A + y_2^B = T\left(y_1^A + y_1^B\right).$$

FOC immediately yield that

$$MRS^A = MRS^B = MRT.$$

Notes

Concluding Comments

GE is a parsimonious and flexible theory, which relies on price taking, individual maximization, and market clearing to derive implications about market outcomes.

It can accommodate time, location and state contingent pricing once commodity spaces are enlarged (an empirically relevant extension).

Conclusions on welfare further require the absence of externalities and private information, and the completeness of markets.

Memorandum:

Homework 1 is online and has to be handed by the end of week 7 MT.

Notes

Uncertainty

EC411 Slides 4

Nava

LSE

Michaelmas Term

Notes

Roadmap: Choice under Uncertainty

We introduce the classical model of decision making under uncertainty.

This model is an essential benchmark to analyze behavior in numerous markets (finance, insurance, betting) and in any type of game.

The analysis proceeds as follows:

- Lotteries & Expected Utility.
- Axioms on Preferences.
- Representation Theorem.
- Applications to Insurance.
- Attitudes towards Risk.
- Experimental Evidence.
- New Theories.

Notes

Decision Theory: Uncertainty

Notes

Uncertainty

There are N possible **outcomes** (or states).

Denote the **set of possible** outcomes by

$$X = (x_1, ..., x_N).$$

A **simple lottery** L is a probability distribution $(p_1, ..., p_N)$ over outcomes in which:

- p_i denotes the probability of any outcome $i \in \{1, ..., N\}$.

A **compound lottery** $(\alpha^1 L^1, ..., \alpha^k L^k)$ is a probability distribution over simple lotteries in which:

- L^j denotes a simple lottery for any $j \in \{1, ..., k\}$;
- α^j denotes the probability of lottery L^j.

Notes

From Compound to Simple Lotteries

Consider any compound lottery $(\alpha^1 L^1, ..., \alpha^k L^k)$.

Let p_i^j is the probability of outcome i in lottery L^j.

If so, there exists a corresponding **reduced simple lottery** $(p_1, ..., p_N)$ st:

$$p_i = \sum_{j=1}^{k} \alpha^j p_i^j, \quad \text{for any } i = 1, ..., N.$$

Example: Consider the following scenario:

- $X = (\$1, \$2)$,
- $L_1 = (1/2, 1/2)$,
- $L_2 = (1, 0)$.

For a compound lottery $(0.5L_1, 0.5L_2)$,

The corresponding reduced lottery is $(0.75, 0.25)$.

Notes

Axioms: Completeness, Transitivity, Continuity

Assume that decision makers regard any compound lottery as equivalent to its reduced simple lottery.

The preferences of the decision maker are defined over the set of simple lotteries $\mathcal{L}$, and denoted by $\succeq$.

Preferences satisfy **continuity** if for any $L, L', L'' \in \mathcal{L}$, the following two sets are closed:

$$\{\alpha \in [0,1] : (\alpha L', (1-\alpha)L'') \succeq L\} \text{ and } \{\alpha \in [0,1] : L \succeq (\alpha L', (1-\alpha)L'')\}.$$

The following two axioms on preferences are invoked throughout:

[A1] $\succeq$ is complete and transitive.

[A2] $\succeq$ satisfies continuity.

A1 and A2 imply that preferences can be represented by a utility function.

Notes

Axioms: Independence

A third and final axiom known as the **independence axiom** is needed to derive a classical and fundamental representation theorem:

[A3] $\succeq$ satisfies independence.

Preferences satisfy **independence** if for any $L, L', L'' \in \mathcal{L}$ and $\alpha \in [0,1]$:

$$L \succeq L' \ \text{ if and only if } \ (\alpha L, (1-\alpha)L'') \succeq (\alpha L', (1-\alpha)L'').$$

Example: $L = 100\$$, $L' = 10\$$, $L'' =$ Trip to Paris.

A3 implies that $L \succeq L'$ if and only if

$$(\alpha L, (1-\alpha)[\text{Trip}]) \succeq (\alpha L', (1-\alpha)[\text{Trip}]).$$

The utility of from the trip cancels out. A decision maker should thus keep preferring more money to less regardless of whether she wins the trip.

Notes

Representation Theorem

Theorem (Representation Theorem)

If A1-A3 hold, there exists a vector of utilities $(u_1, ..., u_N)$ such that, for any two lotteries $L = (p_1, ..., p_N)$ and $L' = (p'_1, ..., p'_N)$,

$$L \succeq L' \ \text{ if and only if } \ \textstyle\sum_{i=1}^N p_i u_i \geq \sum_{i=1}^N p'_i u_i.$$

Furthermore, any other vector of utilities $(\bar{u}_1, ..., \bar{u}_N)$ represents the same preferences if and only if for some constants $b > 0$ and a:

$$\bar{u}_i = a + b u_i.$$

The key features of the representation imply that:

- preferences are linear in probabilities;
- the utility of a lottery is the **expected utility** of the outcomes;
- only affine transformations of utility represent the same preferences.

Notes

Intuition: Independence Axiom

Consider the following scenario:

- $X = (x_1, x_2, x_3)$,
- $L = (1, 0, 0)$,
- $L' = (0, 1, 0)$,
- $L'' = (0, 0, 1)$.

Note that the representation theorem implies

$(\alpha L, (1-\alpha)L'') \succeq (\alpha L', (1-\alpha)L'')$ **iff** $\alpha u_1 + (1-\alpha)u_3 \geq \alpha u_2 + (1-\alpha)u_3$.

This in turn immediately implies the independence axiom, as

$$\alpha u_1 + (1-\alpha)u_3 \geq \alpha u_2 + (1-\alpha)u_3 \ \textbf{ iff } \ u_1 \geq u_2 \ \textbf{ iff } \ L \succeq L'.$$

Notes

Risk Attitudes

Notes

Attitudes towards Risk

Now assume that outcomes in X are quantifiable so that $X = \mathbb{R}_+$.

This is a good assumption for money, consumption, wealth...

A lottery now consists of a cumulative distribution

$$F : \mathbb{R}_+ \to [0,1],$$

where $F(x)$ is the probability that the outcome is smaller or equal to x.

The preferences have an expected utility representation

$$U(F) = \int_X u(x) dF(x),$$

where $u(x)$ is the utility of x with certainty.

If the density $f(\cdot)$ is well defined, then

$$U(F) = \int_X u(x) f(x) dx.$$

Assume that $u(\cdot)$ is strictly **increasing and continuous**.

Notes

Attitudes towards Risk

A decision maker is **risk averse** if and only if, for any lottery F,

$$\int_X u(x) dF(x) \leq u\left(\int_X x dF(x)\right).$$

A decision maker is **risk neutral** if and only if, for any lottery F,

$$\int_X u(x) dF(x) = u\left(\int_X x dF(x)\right).$$

A decision maker is **risk loving** if and only if, for any lottery F,

$$\int_X u(x) dF(x) \geq u\left(\int_X x dF(x)\right).$$

Fact

A decision maker is risk averse if and only if $u(\cdot)$ is concave.
A decision maker is risk loving if and only if $u(\cdot)$ is convex.

Notes

Attitudes towards Risk

A decision maker is **strictly risk averse** if and only if, for any lottery F that does not assign probability equal to one to the outcome $\int_X x dF(x)$,

$$\int_X u(x)dF(x) < u\left(\int_X x dF(x)\right)$$

A decision maker is **strictly risk loving** if and only if, for any lottery F that does not assign probability equal to one to the outcome $\int_X x dF(x)$,

$$\int_X u(x)dF(x) > u\left(\int_X x dF(x)\right)$$

Examples:

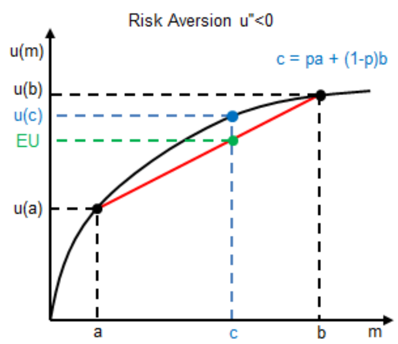
- strictly risk averse, $u(x) = \sqrt{x}$;
- risk neutral, $u(x) = x$;
- strictly risk loving, $u(x) = x^2$.

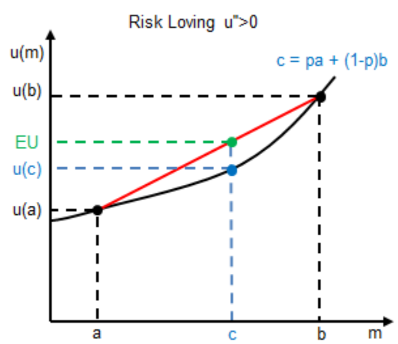
Notes

Attitudes towards Risk

Consider a lottery $X = \{a, b\}$ and $L = \{p, 1-p\}$.

If so, graphically observe that:





Notes

Example: Insurance

Consider the following scenario:

- M denotes the initial wealth;
- L denotes a loss that may be incurred;
- p denotes the probability of the loss;
- S denotes the amount of coverage;
- r denotes unit price of coverage (unit premium).

Two outcomes are possible: $\begin{cases} M - rS & \text{with probability } 1-p \\ M - L + S - rS & \text{with probability } p \end{cases}$.

A risk-averse consumer chooses $S \leq L$ to maximize expected utility,

$$(1-p)\, u\,(M - rS) + pu\,(M - L + (1-r)\,S)\,.$$

The FOC immediately yields

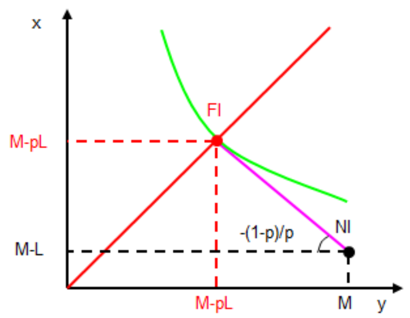
$$r\,(1-p)\,u'\,(M - rS) = (1-r)\,pu'\,(M - L + (1-r)\,S)\,.$$

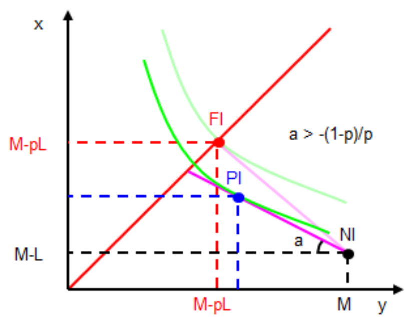
Exercise: Show that, if $r = p$, a risk averse consumer chooses $S = L$.

Notes

Example: Insurance

Demand for insurance can be derived exactly as demand for other goods:





The consumer partially insures whenever the price exceeds the risk $r > p$.

Notes

Measures of Risk Aversion

Intuitively, the more concave is a utility function the more risk averse is the decision maker.

Since $u(\cdot)$ and $bu(\cdot)$, $b > 0$, represent the same preferences, $u''(\cdot)$ is not a satisfactory measure of risk aversion.

Define the **coefficient of absolute risk aversion** as:

$$r_A(x) = -\frac{u''(x)}{u'(x)}.$$

Example:

Consider preferences $u(x) = -e^{-ax}$.

If so, $r_A(x) = -\frac{-a^2 e^{-ax}}{ae^{-ax}} = a$.

Notes

Concave Transformations I

Consider two decision makers, one with utility $u(\cdot)$, one with utility $v(\cdot)$.

Both $u(\cdot)$ and $v(\cdot)$ are strictly increasing.

Therefore there exists $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = \phi(v(x)) \quad \text{for any} \quad x \in \mathbb{R}_+.$$

It follows that

$$u'(x) = \phi'(v(x))\, v'(x) \quad \text{and}$$
$$u''(x) = \phi''(v(x))\left(v'(x)\right)^2 + \phi'(v(x))\, v''(x).$$

By eliminating $\phi'(v(x))$ from the two expressions obtain

$$u''(x) = \phi''(v(x))\left(v'(x)\right)^2 + \frac{u'(x)}{v'(x)} v''(x).$$

Notes

Concave Transformations II

The previous expression is equivalent to

$$\phi''\left(v(x)\right) = \frac{u''(x)v'(x) - v''(x)u'(x)}{\left(v'(x)\right)^3}.$$

Hence, $\phi\left(\cdot\right)$ is concave if and only if

$$-\frac{v''(x)}{v'(x)} \leq -\frac{u''(x)}{u'(x)}$$

Thus, $u(\cdot)$ is a **concave transformation** of $v(\cdot)$ if and only if $u(\cdot)$ has a larger coefficient of absolute risk aversion than $v(\cdot)$.

Absolute risk aversion is a **partial order** (not all preferences can be compared) over preferences that compares ranks preferences based on attitudes towards risk.

Notes

Relative Risk Aversion

Absolute risk aversion measures risk attitudes towards absolute changes in wealth at the current wealth level.

A different measure risk aversion can capture risk attitudes towards percentage changes in wealth at the current wealth level.

Define the **coefficient of relative risk aversion** as:

$$r_A(x) = -\frac{xu''(x)}{u'(x)}.$$

Absolute risk aversion is a **partial order** (not all preferences can be compared) over preferences that compares ranks preferences based on risk attitudes for percentage changes in wealth.

Aside: Decreasing absolute risk aversion has been used as a justification for income inequality...

Notes

Paradoxes & Advances: Choice under Uncertainty

Notes

Allais Paradox I

Consider outcome set $X = \{5 \text{ millions}, 1 \text{ millions}, 0 \text{ millions}\}$.

First consider the following two lotteries:

Outcome:	5 M$	1 M$	0 M$
Lottery A:	0%	100%	0%
Lottery B:	10%	89%	1%

Evidence suggests individuals choose A over B.

Such behavior requires

$$u(1) > 0.1u(5) + 0.89u(1) + 0.01u(0),$$
$$\Rightarrow 0.11u(1) > 0.1u(5) + 0.01u(0).$$

Notes

Allais Paradox II

Next consider the following two lotteries:

Outcome:	5 M$	1 M$	0 M$
Lottery C:	0%	11%	89%
Lottery D:	10%	0%	90%

Evidence suggests individuals choose D over C.

Such behavior requires

$$0.11u(1) + 0.89u(0) < 0.1u(5) + 0.9u(0),$$
$$\Rightarrow 0.11u(1) < 0.1u(5) + 0.01u(0).$$

Preferences violate the independence axiom (not linear in probabilities)!

In experiments individuals will correct this mistake over time.

Possible explanation comes from regret theory (choices are made to avoid disappointment from an outcome that did not materialize).

Notes

Subjective Probabilities

We have assumed that a decision maker ranks explicit and objective probability distributions over outcomes.

However, probabilities may be implicit and subjective.

For example, the decision maker may be asked to choose between:
(a) 100£ if Chelsea wins the Champions League;
(b) 100£ if Barcelona wins the Champions League.

Subjective probabilities may be revealed by the choices of decision makers.

If (b) is preferred to (a), we may conclude that the decision maker thinks that the probability of Barcelona winning the title is higher than the probability of Chelsea winning the title.

Notes

Ellsberg Paradox & Ambiguity Aversion

But choices may not consistent with "probabilistic" assessments.

Consider an urn containing 100 balls.

Some balls are red, while the rest are blue.

The decision maker is **not told** how many are red and how many are blue.

Extract a ball from this urn and consider the bets:

(a) £100 if the ball is red;

(b) £100 if the ball is blue;

(c) £99 with probability 50% – coin toss.

Many individuals would choose (c).

However, either $\Pr(red) \geq 1/2$ or $\Pr(blue) \geq 1/2$, regardless of whether probabilities are objective or subjective.

Notes

Maxmin Utility (Gilboa & Schmeidler)

Let S denote the set of possible states.

A **bet** is a function $x : S \longrightarrow \mathbb{R}_+$.

In particular, $x(s)$ is the monetary outcome in state s.

The set of "possible" probability distributions over the set of states is C.

Under reasonable axioms, preferences over bets x can always be represented by a utility function

$$\min_{F \in C} \int_S u(x(s))\, dF(s).$$

This approach can explain Ellsberg's Paradox. When individuals are pessimistic about their beliefs, they may prefer an objective lottery.

Notes

Limits to Risk-Aversion

Risk-aversion is a standard economic paradigm, but it imposes to much discipline when comparing **small** risks to **large** ones.

Rabin (2000) shows that:

- if an EU maximizer rejects a 50-50 gamble to win 11 or lose 10 **regardless of his wealth**, w,
- he rejects a 50-50 gamble to win y or lose 100, for any value of y!!

To show this observe that concavity implies:

$$u(w) - u(w-10) < +10u'(w-10)$$
$$u(w) - u(w+11) < -11u'(w+11)$$

Since he turns down the gamble, we also know:

$$u(w) > 0.5u(w-10) + 0.5u(w+11)$$

Combining all three inequalities yields:

$$u'(w+11) - u'(w-10) < -\frac{1}{11}u'(w-10)$$

Notes

Limits to Risk-Aversion

So gaining 21 causes the marginal utility of money to fall by 9%. Iterating this logic shows that small stakes compound quickly and cause marginal utility to plummet on large stakes.

According to EU, there is no amount we could offer him to accept the chance of losing 100.

EU was not designed to explain small-stakes gambles and requires individuals to be locally risk neutral.

Evidence suggests that the vast majority of individuals would reject the small bet, Barberis, Huang and Thaler (2006) and Arrow (1971).

A solution involves kinking the utility function around a reference point.

Notes

Prospect Theory (Kahneman & Tversky)

Subjective probabilities are **biased**:

- small probabilities are overestimated,
- large probabilities are underestimated.

The utility of an outcome may depend on a **reference point** r.

Let $(x_1, ..., x_N)$ denote a set of monetary outcomes.

Consider a lottery over outcomes $(p_1, ..., p_N)$.

Let $d_i(p_1, ..., p_N)$ denote the **distorted probability** of outcome x_i.

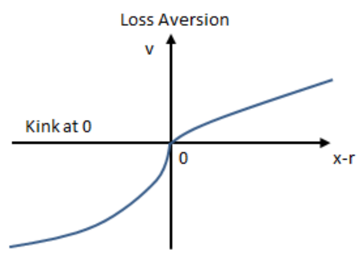
If so, under plausible axioms preferences can be represented as,

$$\sum_{i=1}^{N} d_i(p_1, ..., p_N)\, u(x_i|r).$$

Notes

Loss Aversion

If r is wealth and $u(x_i|r) = v(x_i - r)$, utility is defined on gains & losses.



Notes

Loss Aversion: Example

Suppose an agent has a piecewise-linear value function:

$$v(x) = \begin{cases} x & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases}$$

Let the reference point be her initial wealth $r = w$.

Does she accept a 50/50 gamble to win 11 or lose 10?

- If she turns it down, she gets:

$$v(w - r) = v(0) = 0$$

- If she accepts, she gets:

$$\frac{v(w + 11 - r) + v(w - 10 - r)}{2} = \frac{v(11) + v(-10)}{2} = -4.5$$

- So she turns it down

Notes

Loss Aversion: Example

What is the smallest value y for which she accepts a 50/50 gamble to win y or lose 100?

- If she turns it down, she gets:

$$v(w - r) = v(0) = 0$$

- If she accepts, she gets:

$$\frac{v(w + y - r) + v(w - 100 - r)}{2} = \frac{v(y) + v(-100)}{2} = \frac{y}{2} - 100$$

- So she will accept if $y \geq 200$.

Loss aversion also provides an explanation for the endowment effect.

Notes

Gambles on Gains and Losses

Most people are:

- risk-averse when thinking only of gains but
- risk-loving when thinking of losses.

Let (y, p) be a gamble that gives:

- y with probability p and
- 0 with probability $(1 - p)$.

Kahneman & Tversky 1979 shows that:

$(4000, 0.8)$ 20%	$(3000, 1)$ 80%	$(-4000, 0.8)$ 92%	$(-3000, 1)$ 8%
$(3000, 0.9)$ 86%	$(6000, 0.45)$ 14%	$(-3000, 0.9)$ 8%	$(-6000, 0.45)$ 92%

Rejects equal proportions choosing the risky option.

Notes

Diminishing Sensitivity

The second finding of Kahneman & Tversky 1979 is *diminishing sensitivity*.

Marginal sensitivity to changes from the reference point are smaller the farther the change is from the reference point.

KT argue this is also an extension of how we perceive things. For example, an object 101 feet away is indistinguishable from an object 100 feet away, but an object 2 feet away is easily perceived as very different from an object 1 foot away. Likewise, the relative impact of receiving (resp. losing) 10 instead of 0 is larger than the impact of receiving (resp. losing) 1,010 instead of 1,000.

This is not just risk-aversion: people will be risk-averse in the domain of gains but risk-seeking in the domain of losses.

Notes

Properties of Value Functions

Value Function is assumed to be:

1. $v(x)$ is continuous for any x and twice differentiable for $x \neq 0$.
2. $v(x)$ is strictly increasing, and $v(0) = 0$.
3. $v(y) + v(-y) < v(x) + v(-x)$ for $y > x > 0$.
4. $v''(x) \leq 0$ for $x > 0$, and $v''(x) \geq 0$ for $x < 0$.
5. $v'_-(0)/v'_+(0) > 1$.

These theories are convenient analytically and valuable guides to behavior. But occasionally fail to provide a unified approach to capture all biases displayed by human behavior.

Notes

Notes

Static Games

EC411 Slides 5

Nava

LSE

Michaelmas Term

Notes

Roadmap: Game Theory – Static Games

A game is a multi-person decision problem. Strategic thinking in such environments requires players to guess the behavior of others to decide on the best course of action.

The analysis of static considers two different classes of games:

- Complete Information Static Games
- Incomplete Information Static Games (extra)

Introduces fundamental notions of strategy and best response.

Different Solution Concepts are presented:

- Dominant Strategy Equilibrium
- Iterative Elimination of Dominated Strategies
- Nash Equilibrium
- Bayes Nash Equilibrium (extra).

Notes

Strategic Form Games:

Static Complete Information Games

Notes

Summary

Games of Complete Information:

- Definitions:
  - Game: Players, Actions, Payoffs
  - Strategy: Pure and Mixed
  - Best Response
- Solution Concept:
  - Dominant Strategy Equilibrium
  - Nash Equilibrium
- Properties of Nash Equilibria:
  - Non-Existence and Multiplicity
  - Inefficiency
- Examples

Notes

Introduction to Games

Any environment in which the choices of an individual affect the well being of others can be modeled as a game.

What pins down a specific game:

- Who participates in a game [Players]
- The choices that participants have [Choices]
- The well being of individuals [Payoffs]
- The information that individuals have [Rules of the Game]
- The timing of events and decisions [Rules of the Game]

Notes

Complete Information (Strategic Form) Game

A complete information game G consists of:

- A set of players:
  - N of size n
- An action set for each player in the game:
  - A_i for player i's
  - An action profile $\mathbf{a} = (a_1, a_2, ..., a_n)$ picks an action for each player
- A utility map for each player mapping action profiles to payoffs:
  - $u_i(\mathbf{a})$ denotes player i's payoff of action profile $\mathbf{a}$

$B \backslash G$	s	m
s	5,2	1,2
m	0,0	3,5

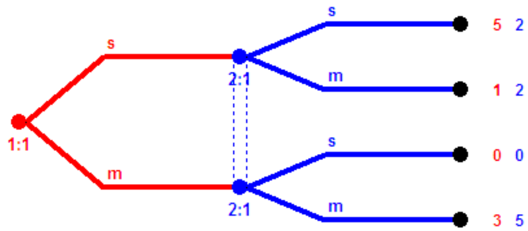
Notes

Representing Simultaneous Move Complete Info Games

Strategic Form

1\2	s	m
s	5,2	1,2
m	0,0	3,5

Extensive Form



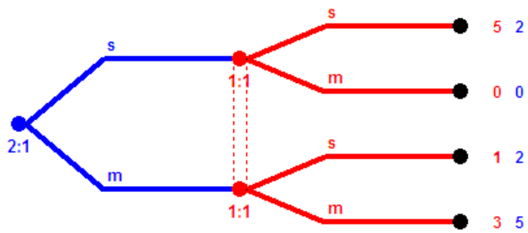
Notes

Representing Simultaneous Move Complete Info Games

Strategic Form

1\2	s	m
s	5,2	1,2
m	0,0	3,5

Extensive Form



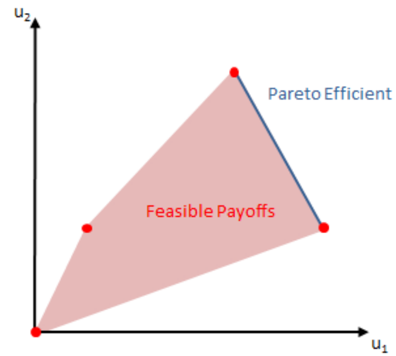
Notes

Feasible Payoffs and Efficiency

Strategic Form

1\2	s	m
s	5,2	1,2
m	0,0	3,5

Feasible Payoffs



Efficient payoffs are on the north-east boundary.

Notes

Information and Pure Strategies

A strategy in a game:

- is a map from information into actions
- it defines a plan of action for a player

In a complete information strategic form game:

- players have no private information
- players act simultaneously

In this context a strategy is any element of the set of actions

For instance a (pure) strategy for player i is simply $a_i \in A_i$

Notes

Notation

Define a profile of actions chosen by all players other than i by $\mathbf{a}_{-i}$:

$$\mathbf{a}_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$$

Define the set of possible action profiles for all players other that i as

$$A_{-i} = \times_{j \in N \backslash i} A_j$$

Define the set of possible action profiles for all players as

$$A = \times_{j \in N} A_j$$

Notes

Best Responses

The best response correspondence of player i is defined by:

$$b_i(\mathbf{a}_{-i}) = \arg\max_{a_i \in A_i} u_i(a_i, \mathbf{a}_{-i}) \quad \text{for any } \mathbf{a}_{-i} \in A_{-i}$$

Thus a_i is a best response to $\mathbf{a}_{-i}$ – i.e. $a_i \in b_i(\mathbf{a}_{-i})$ – if and only if:

$$u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a_i', \mathbf{a}_{-i}) \quad \text{for any } a_i' \in A_i$$

BR identifies the optimal action for a player given choices made by others

For instance:

$B \backslash G$	s	m
s	5,2	1,2
m	0,0	3,5

Notes

Strategic Form Games:

Dominance

Notes

Strict Dominance

- Strategy a_i **strictly dominates** a_i' if:

$$u_i(a_i, \mathbf{a}_{-i}) > u_i(a_i', \mathbf{a}_{-i}) \quad \text{for any } \mathbf{a}_{-i} \in A_{-i}$$

- a_i is **strictly dominant** if it strictly dominates any other a_i'
- a_i is **strictly undominated** if no strategy strictly dominates a_i
- a_i is **strictly dominated** (SDS) if a strategy strictly dominates a_i

In the following example s is strictly dominant for B:

$B\backslash G$	s	m
s	5,-	2,-
m	0,-	1,-

Notes

Weak Dominance

- Strategy a_i **weakly dominates** a_i' if:

$$u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a_i', \mathbf{a}_{-i}) \quad \text{for any } \mathbf{a}_{-i}$$
$$u_i(a_i, \mathbf{a}_{-i}) > u_i(a_i', \mathbf{a}_{-i}) \quad \text{for some } \mathbf{a}_{-i}$$

- a_i is **weakly dominant** if it weakly dominates any other a_i'
- a_i is **weakly undominated** if no strategy weakly dominates a_i
- a_i is **weakly dominated** (WDS) if a strategy weakly dominates a_i

In the following example s is weakly dominant for B:

$B\backslash G$	s	m
s	5,-	2,-
m	0,-	2,-

Notes

Dominant Strategy Equilibrium

Definitions (Dominant Strategy Equilibrium DSE)

A strict Dominant Strategy equilibrium of a game G consists of a strategy profile $\mathbf{a}$ such that for any $\mathbf{a}'_{-i} \in A_{-i}$ and $i \in N$:

$$u_i(a_i, \mathbf{a}'_{-i}) > u_i(a_i', \mathbf{a}'_{-i}) \quad \text{for any } a_i' \in A_i$$

- For weak DSE change $>$ with $\geq$ ...
- a profile $\mathbf{a}$ is a DSE **iff** $a_i \in b_i(\mathbf{a}'_{-i})$ for any $\mathbf{a}'_{-i}$ and $i \in N$
- Example (Prisoner's Dilemma):

$B\backslash S$	N	C
N	5,5	0,6
C	6,0	1,1

Notes

Example: The Hotelling Game

There are two parties are in an election – "Left" and "Right".

Political view points are represented by a number between -1 and 1.

Voter views are uniformly distributed on $[-1, 1]$.

The Hotelling Game:

- Each party chooses a policy position in the interval $[-1, 1]$:
  - The Right party chooses in $[0, 1]$;
  - The Left party chooses in $[-1, 0]$.
- Voters cast ballots in favor of the party closest to their ideal point.
- The party that gets a majority of votes wins.
- Parties care only about winning.

Where will each party position itself? Is there any dominant strategy?

Notes

Iterative Elimination of Dominated Strategies

If there is no DSE, we can still eliminate strictly dominated strategies,

... and by repeating the process we may rule out more strategies.

Consider the following example:

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

$\Rightarrow$

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

At the first instance only D is dominated for player 1.

No strategy is dominated a priori for player 2.

Strategies in green in the table are SDS and thus eliminated.

Notes

Iterative Elimination of Dominated Strategies

Once D has been eliminated from the game:

Strategy R is dominated for player 2.

No strategy is dominated for player 1.

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

$\Rightarrow$

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

Once R has been eliminated from the game:

Strategy T is dominated for player 1.

A final iteration yields (M, L) as the only surviving strategies.

Notes

Common Knowledge of Rationality

Definition

The fact F is a common knowledge if

- Every player knows F;
- Every player knows that every other player knows F;
- Every player knows that every player knows that every player knows F and so on.

Theorem

If, among players, there is a common knowledge of the game and of the fact that all players are rational, then the outcome of the game must be among those that survive iterative elimination of dominated strategies.

Strategies that survive iterative elimination of SDS are said to be **rationalizable**.

Notes

Dominance: Final Considerations I

Dominance and rationalizability are benchmarks of rationality.

The benefits of strict iterative elimination are that:

- the order of elimination is irrelevant;
- there is no need to know the other player's action;
- it all comes from rationality.

The limitations of strict iterative elimination are that:

- won't always reach solution;
- we must assume common knowledge of rationality and of the game;
- it often leads to inefficient outcomes.

Notes

Dominance: Final Considerations II

This is problematic in two ways:

- rationality must hold with probability 1;
- there must be unlimited "depth" of rationality.

Problem I: Rationality with Probability 1

Consider the following game for k very large:

1\2	L	R
U	$2, 5$	$3, 4$
M	$0, -k$	$2, k$

Iterative elimination leads for (U,L).

But is it likely player 2 will never play R for any value of k?

Notes

Dominance: Final Considerations III

Problem II: Unlimited Depth of Rationality

Consider a game in which:

- 10 players compete;
- each player writes a number between 1 & 1000;
- denote by $X = 1/2$ of the average of the players numbers;
- the player whose number is closest to X wins.

After iterative elimination of SDS, everyone quotes number 1.

But this does not match up with the empirical evidence.

A weaker notion of equilibrium may bypass some of these limitations.

Notes

Strategic Form Games:

Nash Equilibrium

Notes

Nash Equilibrium: Introduction

Dominance was the appropriate solution concept if players had no information or beliefs about choices made by others

The weaker notion of equilibrium that will be introduced presumes that:

- players have correct beliefs about choices made by others
- players choices are optimal given such beliefs
- the environment is common knowledge among players

Such model allows for tighter predictions when dominance has no bite

Notes

Nash Equilibrium

Definition (Nash Equilibrium NE)

A (pure strategy) Nash equilibrium of a game G consists of a strategy profile $\mathbf{a} = (a_i, \mathbf{a}_{-i})$ such that for any $i \in N$:

$$u_i(\mathbf{a}) \geq u_i(a_i', \mathbf{a}_{-i}) \quad \text{for any } a_i' \in A_i$$

- a profile $\mathbf{a}$ is a NE **iff** $a_i \in b_i(\mathbf{a}_{-i})$ for any $i \in N$

Properties:

- Strategy profiles are independent
- Strategy profiles common knowledge
- Strategies maximize utility given beliefs

Notes

Examples, Properties and Limitations

Games may have more NE's (Battle of the Sexes):

$B\backslash G$	s	m
s	5,2	1,2
m	0,0	3,5

Nash equilibria may not be efficient (Prisoner's Dilemma):

$B\backslash S$	N	C
N	5,5	0,6
C	6,0	1,1

Pure strategy Nash equilibria may not exist (Matching Pennies):

$B\backslash G$	H	T
H	0,2	2,0
T	2,0	0,2

Notes

Examples, Properties and Limitations

Games may have multiple NE (Battle of the Sexes):

$B\backslash G$	s	m
s	5,2	1,2
m	0,0	3,5

Nash equilibria may be inefficient (Prisoner's Dilemma):

$B\backslash S$	N	C
N	5,5	0,6
C	6,0	1,1

Pure strategy Nash equilibria may not exist (Matching Pennies):

$B\backslash G$	H	T
H	0,2	2,0
T	2,0	0,2

Notes

Example: Coordination Games

Consider a game played by two hunters, for $k \in [1,8]$:

$B \backslash G$	Stag	Hare
Stag	9,9	0,k
Hare	k,0	k,k

There are two NE:

- in one the hunters coordinate on hunting the stag;
- in the other they split and each hunts a hare on his own.

When compared to the hare-NE, the stag-NE entails:

- greater gains from coordination;
- greater risks of miscoordination.

If players communicate, they are likely to opt for Pareto dominating NE.

Nash equilibria can be viewed as **focal points**.

Notes

Example: Three Players

- A game with more than 2 players:

3	L		R	
1\2	A	B	A	B
T	1,1,0	0,0,0	0,1,1	0,2,1
D	0,1,1	1,2,0	1,0,0	2,1,1

- To find all NE check best reply maps:

3	L		R	
1\2	A	B	A	B
T	1,1,0	0,0,0	0,1,1	0,2,1
D	0,1,1	1,2,0	1,0,0	2,1,1

Notes

War of Attrition Example

Consider a game with two competitors involved in a fight:

- The set of players is $N = \{1, 2\}$.
- Competitors choose how much effort to put in a fight $A_i = [0, \infty)$.
- The value of winning the fight is for competitor $i \in N$ is v_i.
- The highest effort wins the fight and ties are broken at random.
- For each competitor the cost of fighting is simply $\min\{a_i, a_j\}$.
- The payoff of competitor i given their effort levels thus satisfy:

$$u_i(a_i, a_j) = \begin{cases} v_i - a_j & \text{if} \quad a_i > a_j \\ v_i/2 - a_j & \text{if} \quad a_i = a_j \\ -a_i & \text{if} \quad a_i < a_j \end{cases}$$

Notes

War of Attrition Example

As payoffs amount to:

$$u_i(a_i, a_j) = \begin{cases} v_i - a_j & \text{if} \quad a_i > a_j \\ v_i/2 - a_j & \text{if} \quad a_i = a_j \\ -a_i & \text{if} \quad a_i < a_j \end{cases}$$

Best response functions satisfy:

$$b_i(a_j) = \begin{cases} a_i > a_j & \text{if} \quad a_j < v_i \\ a_i = 0 \ \text{ or } \ a_i > a_j & \text{if} \quad a_j = v_i \\ a_i = 0 & \text{if} \quad a_j > v_i \end{cases}$$

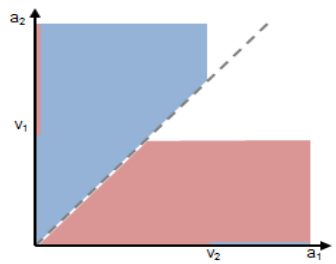
All Nash Equilibria of the game satisfy one of the following:

- $a_1 = 0$ and $a_2 \geq v_1$
- $a_2 = 0$ and $a_1 \geq v_2$

Notes

War of Attrition Example

An easy way to find the NE in such games is plotting BRs:



All Nash Equilibria of the game satisfy one of the following:

- $a_1 = 0$ and $a_2 \geq v_1$
- $a_2 = 0$ and $a_1 \geq v_2$

Notes

Example: Lobbying

Consider a game with two lobbyists petitioning over two versions of a bill:

- The value of having bill $i \in \{1, 2\}$ approved is v_i for i and zero for j.
- Lobbyists choose how many resources to invest $a_i \in [0, \infty)$.
- The probability that the policy preferred by lobbyist i is approved is

$$p_i = \frac{a_i}{a_i + a_j}$$

- The payoff of lobbyist i thus amounts to

$$u_i(a) = \frac{a_i}{a_i + a_j} v_i - a_i$$

- The payoff is concave and single peaked for any value a_j.

Notes

Example: Lobbying

Taking first order conditions yields

$$\frac{a_j}{(a_i + a_j)^2} v_i = 1 \quad \Leftrightarrow \quad a_i + a_j = (v_i a_j)^{1/2}$$

Thus, best response function necessarily satisfy

$$b_i(a_j) = \begin{cases} (v_i^{1/2} - a_j^{1/2}) a_j^{1/2} & \text{if} \quad a_j < v_i \\ 0 & \text{if} \quad a_j \geq v_i \end{cases}$$

Solving the two shows that the unique NE necessarily satisfies

$$a_i = v_j \left[\frac{v_i}{v_i + v_j} \right]^2$$

Notes

DSE implies NE

Fact

Any dominant strategy equilibrium is a Nash equilibrium

Proof.

If $\mathbf{a}$ is a DSE then $a_i \in b_i(\mathbf{a}'_{-i})$ for any $\mathbf{a}'_{-i}$ and $i \in N$.
Which implies $\mathbf{a}$ is NE since $a_i \in b_i(\mathbf{a}_{-i})$ for any $i \in N$. □

Notes

Nash Equilibrium: Final Considerations I

The most notable limitations of the NE solution concept are that:

- players have **correct beliefs** about other players' strategies;
- players are **rational** and choose the optimal action given these beliefs;
- NE are only immune to deviations by individual players, not groups.

The first requirement is very problematic, especially in games with more then one equilibrium.

Notes

Nash Equilibrium: Final Considerations II

NE is a good solution concept for:

- norms – everybody knows what side to drive on the road;
- preliminary talks – reaching to an agreement that is self-enforcing;
- stable solutions played over time – the game is played many times; players choose the action the best based on past experience (disregarding any strategic consideration); NE is a necessary condition for stability (but not sufficient and not necessarily easy converge to).

Notes

Strategic Form Games:

Mixed Strategies

Notes

Introduction to Mixed Strategies

A problematic aspect of the solution concepts discussed in the first two lectures was that equilibria did not always exist.

Intuitively, existence was not guaranteed because players had no device to conceal their behavior from others.

Formally, the reasons for the lack of existence were:

- Non-convexities in the choice sets
- Discontinuities of the best response correspondences

Next we introduce mixed strategies which solve both problems and guarantee existence of at least a Nash equilibrium.

Notes

Mixed Strategy Definition

Consider complete information static game $\{N, \{A_i, u_i\}_{i \in N}\}$

A mixed strategy for $i \in N$ is a probability distribution over actions in A_i

Thus σ_i is a mixed strategy if:

- $\sigma_i(a_i) \geq 0$ for any $a_i \in A_i$
- $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$

Intuitively $\sigma_i(a_i)$ is the probability that player i chooses to play a_i

E.G. $\sigma_1(B) = 0.3$ and $\sigma_1(C) = 0.7$ is a mixed strategy for 1 in:

1\2	B	C
B	2,0	0,2
C	0,1	1,0

Notes

Simplex

The set of possible probability distributions on a finite set B is:

- called **simplex**
- and denoted by $\Delta(B)$

It is straightforward to show that the simplex:

- Closed
- Bounded
- Convex

Notes

Notation

Define a profile of strategies for all players other than i by:

$$\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N)$$

Define a profile of strategies for all players by:

$$\sigma = (\sigma_1, ..., \sigma_N)$$

Notes

Payoffs from Mixed Strategies

Mixed strategy are chosen **independently** by players. Thus,

$$\Pr(\mathbf{a}|\sigma) = \prod_{j \in N} \Pr(a_j|\sigma) = \prod_{j \in N} \sigma_j(a_j)$$

The expected utility of player i from a mixed strategy profile σ is

$$\begin{aligned} u_i(\sigma) &= \textstyle\sum_{\mathbf{a} \in \mathbf{A}} \Pr(\mathbf{a}|\sigma) u_i(\mathbf{a}) = \\ &= \textstyle\sum_{\mathbf{a} \in \mathbf{A}} \prod_{j \in N} \sigma_j(a_j) u_i(\mathbf{a}) \end{aligned}$$

Example: If players follow $\sigma_1(B) = \sigma_2(B) = 0.3$ in the game:

Payoffs:

1\2	B	C
B	2,0	0,2
C	0,1	1,0

Probabilities:

1\2	B	C
B	9%	21%
C	21%	49%

The payoff to player 1 is: $u_1(\sigma_1, \sigma_2) = (.09)2 + (.49)1 + (.42)0 = 0.67$

Notes

Best Reply Correspondences

Denote the best reply correspondence of i by $b_i(\boldsymbol{\sigma}_{-i})$

The map is defined by:

$$b_i(\boldsymbol{\sigma}_{-i}) = \arg\max_{\sigma_i \in \Delta(A_i)} u_i(\sigma_i, \boldsymbol{\sigma}_{-i})$$

For instance consider the game:

1\2	s	m
s	5,2	1,2
m	0,0	3,5

If $\sigma_1(s) = 1$ then any $\sigma_2(s) \in [0,1]$ satisfies $\sigma_2 \in b_2(\sigma_1)$

If $\sigma_1(s) < 1$ then only $\sigma_2(s) = 0$ satisfies $\sigma_2 \in b_2(\sigma_1)$

Notes

Dominated Strategies

- Strategy σ_i **weakly dominates** a_i if:

$$u_i(\sigma_i, \mathbf{a}_{-i}) \geq u_i(a_i, \mathbf{a}_{-i}) \quad \text{for any } \mathbf{a}_{-i}$$
$$u_i(\sigma_i, \mathbf{a}_{-i}) > u_i(a_i, \mathbf{a}_{-i}) \quad \text{for some } \mathbf{a}_{-i}$$

- a_i is **weakly undominated** if no strategy weakly dominates it
- This allows us to rule out more strategies than before, eg:

1\2	L	C	R
T	6,6	0,2	0,0
B	0,0	0,2	6,6

- $\sigma_2(L) = \sigma_2(R) = 0.5$ strictly dominates C since:

$$u_2(\sigma_2, a_1) = 3 > u_2(C, a_1) = 2$$

Notes

Nash Equilibrium

Definition (Nash Equilibrium NE)

A Nash equilibrium of a game consists of a strategy profile $\boldsymbol{\sigma} = (\sigma_i, \boldsymbol{\sigma}_{-i})$ such that for any $i \in N$:

$$u_i(\boldsymbol{\sigma}) \geq u_i(a_i, \boldsymbol{\sigma}_{-i}) \quad \text{for any } a_i \in A_i$$

Implicit to the definition of NE are the following assumptions:

- Each agent chooses his mixed strategy **independently** of others
- Each agent **knows** exactly **the strategies** the others adopt
- Each agent chooses so to **maximize expected utility** given his beliefs

Notes

Nash Equilibrium: Computation Help

- A strategy profile σ is a Nash Equilibrium if and only if:

$$u_i(\boldsymbol{\sigma}) = u_i(a_i, \boldsymbol{\sigma}_{-i}) \quad \text{for any } a_i \text{ such that } \sigma_i(a_i) > 0$$
$$u_i(\boldsymbol{\sigma}) \geq u_i(a_i, \boldsymbol{\sigma}_{-i}) \quad \text{for any } a_i \text{ such that } \sigma_i(a_i) = 0$$

- These conditions are evocative of complementary slackness as:

$$[u_i(\boldsymbol{\sigma}) - u_i(a_i, \boldsymbol{\sigma}_{-i})]\, \sigma_i(a_i) = 0,$$
$$u_i(\boldsymbol{\sigma}) - u_i(a_i, \boldsymbol{\sigma}_{-i}) \geq 0, \;\; \sigma_i(a_i) \geq 0$$

- If a_i is strictly dominated, then $\sigma_i(a_i) = 0$ in any Nash equilibrium
- If a_i is weakly dominated, then $\sigma_i(a_i) > 0$ in a Nash equilibrium only if any profile of actions $\mathbf{a}_{-i}$ for which a_i is strictly worse occurs with zero probability

Notes

Examples: Classical Games

Games may have more NEs (Battle of the Sexes):

1\2	s	m
s	5,2	1,1
m	0,0	2,5

There are 2 PNE & a mixed NE in which $\sigma_1(s) = 5/6$ & $\sigma_2(s) = 1/6$:

$$u_1(s, \sigma_2) = 5\sigma_2(s) + (1 - \sigma_2(s)) = 2(1 - \sigma_2(s)) = u_1(m, \sigma_2)$$
$$u_2(m, \sigma_1) = \sigma_1(s) + 5(1 - \sigma_1(s)) = 2\sigma_1(s) = u_2(s, \sigma_1)$$

Games may have only mixed NE (Matching Pennies):

$B\backslash G$	H	T
H	0,2	2,0
T	2,0	0,2

There is a unique NE in which $\sigma_1(H) = \sigma_2(H) = 1/2$:

$$2(1 - \sigma_i(H)) = 2\sigma_i(H)$$

Notes

Example: A Continuum of NE

Games may have a continuum of NE:

1\2	L	R	C
T	4,1	0,2	3,2
D	0,2	2,0	1,0

The game has 1 PNE & a continuum of NEs, namely:

$$\sigma_1(T) = 2/3 \;\; \& \; \sigma_2(R) = 1 - \sigma_2(L) - \sigma_2(C)$$
$$\sigma_2(L) = 1/3 - (2/3)\sigma_2(C) \;\;\text{ for } \sigma_2(C) \in [0, 1/2]$$

These are all NE's since:

$$u_2(L, \sigma_1) = \sigma_1(T) + 2(1 - \sigma_1(T)) = 2\sigma_1(T) = u_2(R, \sigma_1) = u_2(C, \sigma_1)$$
$$u_1(T, \sigma_2) = 4\sigma_2(L) + 3\sigma_2(C) = 2\sigma_2(R) + \sigma_2(C) = u_1(D, \sigma_2)$$

Notes

Example: The Case of Kitty Genovese

Queens 1964. A woman is murdered. 38 neighbors hear. No one calls 911.

New York Time – March 27th, 1964:
www.garysturt.free-online.co.uk/The%20case%20of%20Kitty%20Genovese.htm

Model:

- n identical neighbors;
- 1 individual cost for calling;
- $x > 1$ utility of every player if **someone** calls the police.

In any PNE, exactly one neighbor calls. Improbable without coordination.

In a mixed NE, every neighbor calls with probability p, and is indifferent between calling and not calling:

$$U(\text{call}\ , p) = x - 1 = x\left(1 - (1-p)^{n-1}\right) = U(\text{no call}\ , p),$$

Notes

Example: The Case of Kitty Genovese

Therefore, the probability that a given neighbor calls 911 is

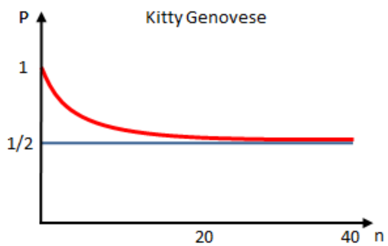
$$p = 1 - (1/x)^{1/(n-1)},$$

while the probability that at least one of the neighbors calls 911 is

$$P = 1 - (1-p)^n = 1 - (1/x)^{n/(n-1)}$$

For $n = 1$, we have $P = 1$. But as $n \to \infty$, P decreases to $1 - (1/x)$.

For $x = 2$, we have



Notes

Strategic Form Games:

NE Existence Theorems

Notes

Nash Equilibrium Existence

Theorem (NE Existence)

Any game with a finite number of actions possesses a Nash equilibrium.

Theorem (PNE Existence)

Any game with convex and compact action sets and with continuous and quasi-concave payoff functions possesses a pure strategy Nash equilibrium.

In both results, assumptions imply that best responses are convex-valued and upper-hemicontinuous by the Theorem of the Maximum.

As best responses map convex compact sets into themselves, existence holds by Kakutani Fixed Point Theorem.

Notes

Kakutani Fixed Point Theorem

The BR correspondence is of player i:

- **Upper-hemicontinuous** if for any a sequence $\{\sigma^k\}_{k=1}^{\infty}$ of strategy profiles such that $\sigma_i^k \in b_i(\sigma_{-i}^k)$ for any k we have that

$$\sigma^k \to \sigma \text{ implies } \sigma_i \in b_i(\sigma_{-i}).$$

- **Convex-valued** if $\sigma_i, \sigma_i' \in b_i(\sigma_{-i})$ implies

$$p\sigma_i + (1-p)\sigma_i' \in b_i(\sigma_{-i}) \text{ for any } p \in [0,1]$$

Theorem (Kakutani Fixed Point Theorem)

Let X be a non-empty, compact and convex subset of $\mathbb{R}^n$.

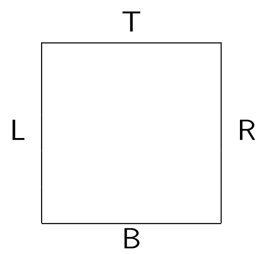
Let $b : X \rightrightarrows X$ be a non-empty, convex-valued, UHC correspondence.

Then b has a fixed point.

Notes

Nash Equilibrium Existence: 2x2 Intuition

Intuitively draw continuous BR maps in the chart:



Show that BR maps must intersect since:

- one map moves from the left to the right boundary;
- while the other from the top to the bottom boundary.

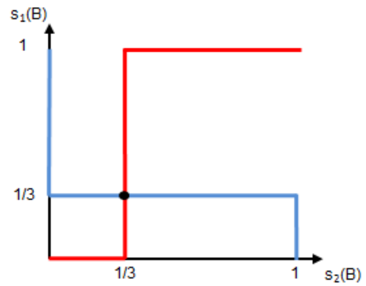
Usually BRs intersect an **odd number** of times.

Notes

UHC Best Responses Imply Existence

A game without PNE and with a single NE:

1\2	B	C
B	2,0	0,2
C	0,1	1,0



Notes

Bayesian Games:

Static Games with Incomplete Information

[**Not Examinable**]

Notes

Summary

Games of Incomplete Information:

- Definitions:
  - Incomplete Information Game
  - Information Structure and Beliefs
  - Strategies
  - Best Reply Map
- Solution Concepts in Pure Strategies:
  - Dominant Strategy Equilibrium
  - Bayes Nash Equilibrium
- Examples
- Mixed Strategies & Bayes Nash Equilibria

Notes

Bayesian Games

An static incomplete information game consists of:

- N the set of players in the game
- A_i player i's action set
- $\mathbb{X}_i$ player i's set of possible signals
  - A profile of signals $x = (x_1, ..., x_n)$ is an element $\mathbb{X} = \times_{j \in N} \mathbb{X}_j$
- f a distribution over the possible signals
- $u_i : A \times \mathbb{X} \to \mathbb{R}$ player i's utility function, $u_i(a|x)$

Notes

Example: Bayesian Game

Consider the following Bayesian game:

- Player 1 observes only one possible signal: $\mathbb{X}_1 = \{C\}$
- Player 2's signal takes one of two values: $\mathbb{X}_2 = \{L, R\}$
- Probabilities are such that: $f(C, L) = 0.6$
- Payoffs and action sets are as described in the matrix:

$1\backslash 2.L$	C	D
A	1,2	0,1
B	0,4	1,3

$1\backslash 2.R$	C	D
A	1,3	0,4
B	0,1	1,2

Notes

Information Structure

Information structure:

- X_i denotes the signal as a random variable
- belongs to the set of possible signals $\mathbb{X}_i$
- x_i denotes the realization of the random variable X_i
- $X_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$ denotes a profile of signals for all players other than i
- Player i observes only X_i
- Player i ignores X_{-i}, but knows f

With such information player i forms beliefs regarding the realization of the signals of the other players x_{-i}

Notes

Beliefs about Signals: Independence

If signals are independent:

$$f(x) = \prod_{j \in N} f_j(x_j)$$

This implies that the signal x_i of player i is independent of X_{-i}

Beliefs are a probability distribution over the signals of the other players

Any player forms beliefs about the signals received by the other players by using Bayes Rule

Independence implies that conditional observing $X_i = x_i$ the beliefs of player i are:

$$f_i(x_{-i}|x_i) = \prod_{j \in N \setminus i} f_j(x_j) = f_{-i}(x_{-i})$$

Notes

Beliefs about Signals: Interdependence

In the general scenario with interdependent signals, players form beliefs about the signals received by others by using **Bayes Rule**

Conditional observing $X_i = x_i$ the beliefs of player i are:

$$f_i(x_{-i}|x_i) = \Pr(X_{-i} = x_{-i}|X_i = x_i) =$$

$$= \frac{\Pr(X_{-i} = x_{-i} \quad \cap \quad X_i = x_i)}{\Pr(X_i = x_i)} =$$

$$= \frac{\Pr(X_{-i} = x_{-i} \quad \cap \quad X_i = x_i)}{\sum_{y_{-i} \in \mathbb{X}_{-i}} \Pr(X_{-i} = y_{-i} \quad \cap \quad X_i = x_i)} =$$

$$= \frac{f(x_{-i}, x_i)}{\sum_{y_{-i} \in \mathbb{X}_{-i}} f(y_{-i}, x_i)}$$

Beliefs are a probability distribution over the signals of the other players

Notes

Strategies

Strategy Profiles:

- A strategy consists of a map from available information to actions:

$$\alpha_i : \mathbb{X}_i \rightarrow A_i$$

- A strategy profile consists of a strategy for every player:

$$\alpha(X) = (\alpha_1(X_1), ..., \alpha_N(X_N))$$

- We adopt the usual convention:

$$\alpha_{-i}(X_{-i}) = (\alpha_1(X_1), ..., \alpha_{i-1}(X_{i-1}), \alpha_{i+1}(X_{i+1}), ..., \alpha_N(X_N))$$

Notes

Example: Bayesian Game

Consider the following game:

- Player 1 observes only one possible signal: $\mathbb{X}_1 = \{C\}$
- Player 2's signal takes one of two values: $\mathbb{X}_2 = \{L, R\}$
- Probabilities are such that: $f(C, L) = 0.6$
- Payoffs and action sets are as described in the matrix:

$1\backslash 2.L$	C	D
A	1,2	0,1
B	0,4	1,3

$1\backslash 2.R$	C	D
A	1,3	0,4
B	0,1	1,2

- A strategy for player 1 is an element of the set $\alpha_1 \in \{A, B\}$
- A strategy for player 2 is a map $\alpha_2 : \{L, R\} \rightarrow \{C, D\}$
- Player 1 cannot act upon 2's private information

Notes

Dominant Strategy Equilibrium

- Strategy α_i **weakly dominates** α_i' if for any a_{-i} and $x \in \mathbb{X}$:

$$u_i(\alpha_i(x_i), a_{-i}|x) \geq u_i(\alpha_i'(x_i), a_{-i}|x) \quad \text{[strict somewhere]}$$

- Strategy α_i is **dominant** if it dominates any other strategy α_i'
- Strategy α_i is **undominated** if no strategy dominates it

Definitions (Dominant Strategy Equilibrium DSE)

A Dominant Strategy equilibrium of an incomplete information game is a strategy profile α that for any $i \in N$, $x \in \mathbb{X}$ and $a_{-i} \in A_{-i}$ satisfies:

$$u_i(\alpha_i(x_i), a_{-i}|x) \geq u_i(\alpha_i'(x_i), a_{-i}|x) \quad \text{for any } \alpha_i' : \mathbb{X}_i \to A_i$$

- I.e. α_i is optimal independently of what others know and do

Notes

Interim Expected Utility and Best Reply Maps

The **interim stage** occurs just after a player knows his signal $X_i = x_i$

It is when strategies are chosen in a Bayesian game

The **interim expected** utility of a (pure) strategy profile α is defined by:

$$U_i(\alpha|x_i) = \sum_{\mathbb{X}_{-i}} u_i(\alpha(x)|x) f(x_{-i}|x_i) : \mathbb{X}_i \to \mathbb{R}$$

With such notation in mind notice that:

$$U_i(a_i, \alpha_{-i}|x_i) = \sum_{\mathbb{X}_{-i}} u_i(a_i, \alpha_{-i}(x_{-i})|x) f(x_{-i}|x_i)$$

The **best reply** correspondence of player i is defined by:

$$b_i(\alpha_{-i}|x_i) = \arg\max_{a_i \in A_i} U_i(a_i, \alpha_{-i}|x_i)$$

BR maps identify which actions are optimal given the signal and the strategies followed by others

Notes

Pure Strategy Bayes Nash Equilibrium

Definitions (Bayes Nash Equilibrium BNE)

A pure strategy Bayes Nash equilibrium of an incomplete information game is a strategy profile α such that for any $i \in N$ and $x_i \in \mathbb{X}_i$ satisfies:

$$U_i(\alpha|x_i) \geq U_i(a_i, \alpha_{-i}|x_i) \quad \text{for any } a_i \in A_i$$

BNE requires **interim optimality** (i.e. do your best given what you know)

BNE requires $\alpha_i(x_i) \in b_i(\alpha_{-i}|x_i)$ for any $i \in N$ and $x_i \in \mathbb{X}_i$

Notes

Example: Bayesian Game

Consider the following Bayesian game with $f(C, L) = 0.6$:

$1\backslash 2.L$	C	D
A	1,2	0,1
B	0,4	1,3

$1\backslash 2.R$	C	D
A	1,3	0,4
B	0,1	1,2

The best reply maps for both player are characterized by:

$$b_2(\alpha_1|x_2) = \begin{cases} C & \text{if} \quad x_2 = L \\ D & \text{if} \quad x_2 = R \end{cases} \qquad b_1(\alpha_2) = \begin{cases} A & \text{if} \quad \alpha_2(L) = C \\ B & \text{if} \quad \alpha_2(L) = D \end{cases}$$

The game has a unique (pure strategy) BNE in which:

$$\alpha_1 = A, \, \alpha_2(L) = C, \, \alpha_2(R) = D$$

DO NOT ANALYZE MATRICES SEPARATELY!!!

Notes

Transforming a Bayesian Game Example

Consider the following Bayesian game with $f(C, L) = p > 1/2$:

$1\backslash 2.L$	C	D
A	1,2	0,1
B	0,4	1,3

$1\backslash 2.R$	C	D
A	1,3	0,4
B	0,1	1,2

This is equivalent to the following complete information game:

$1\backslash 2$	CC	DC	CD	DD
A	$1, 3-p$	$1-p, 3-2p$	$p, 4-2p$	$0, 4-3p$
B	$0, 1+3p$	$p, 1+2p$	$1-p, 2+2p$	$1, 2+p$

You may then find BRs and BNEs in this modified table.

Notes

Relationships between Equilibrium Concepts

If α is a DSE then it is a BNE. In fact for any action a_i and signal x_i:

$$u_i(\alpha_i(x_i), a_{-i}|x) \geq u_i(a_i, a_{-i}|x) \quad \forall a_{-i}, x_{-i} \quad \Rightarrow$$

$$u_i(\alpha_i(x_i), \alpha_{-i}(x_{-i})|x) \geq u_i(a_i, \alpha_{-i}(x_{-i})|x) \quad \forall \alpha_{-i}, x_{-i} \Rightarrow$$

$$\textstyle\sum_{\mathbb{X}_{-i}} u_i(\alpha(x)|x) f_i(x_{-i}|x_i) \geq \sum_{\mathbb{X}_{-i}} u_i(a_i, \alpha_{-i}(x_{-i})|x) f_i(x_{-i}|x_i) \quad \forall \alpha_{-i} \Rightarrow$$

$$U_i(\alpha|x_i) \geq U_i(a_i, \alpha_{-i}|x_i) \quad \forall \alpha_{-i}$$

Notes

BNE Example I: Exchange

A buyer and a seller want to trade an object:

- Buyer's value for the object is 3\$
- Seller's value is either 0\$ or 2\$ based on the signal, $\mathbb{X}_S = \{L, H\}$
- Buyer can offer either 1\$ or 3\$ to purchase the object
- Seller choose whether or not to sell

$B \backslash S.L$	sale	no sale
3\$	0,3	0,0
1\$	2,1	0,0

$B \backslash S.H$	sale	no sale
3\$	0,3	0,2
1\$	2,1	0,2

- This game for any prior f has a BNE in which:

$$\alpha_S(L) = \text{sale}, \ \alpha_S(H) = \text{no sale}, \ \alpha_B = 1\$$$

- Selling is strictly dominant for $S.L$
- Offering 1\$ is weakly dominant for the buyer

Notes

BNE Example II: Entry Game

Consider the following market game:

- Firm I (the incumbent) is a monopolist in a market
- Firm E (the entrant) is considering whether to enter in the market
- If E stays out of the market, E runs a profit of 1\$ and I gets 8\$
- If E enters, E incurs a cost of 1\$ and profits of both I and E are 3\$
- I can deter entry by investing at cost $\{0, 2\}$ depending on type $\{L, H\}$
- If I invests: I's profit increases by 1 if he is alone, decreases by 1 if he competes and E's profit decreases to 0 if he elects to enter

$E \backslash I.L$	Invest	Not Invest
In	0,2	3,3
Out	1,9	1,8

$E \backslash I.H$	Invest	Not Invest
In	0,0	3,3
Out	1,7	1,8

Notes

BNE Example II: Entry Game

Let π denote the probability that firm I is of type L and notice:

- $\alpha_I(H) = \textit{Not Invest}$ is a strictly dominant strategy for $I.H$
- For any value of π, $\alpha_I(L) = \textit{Not Invest}$ and $\alpha_E = \textit{In}$ is BNE:

$$u_I(\textit{Not}, \textit{In}|L) = 3 > 2 = u_I(\textit{Invest}, \textit{In}|L)$$
$$U_E(\textit{In}, \alpha_I(X_I)) = 3 > 1 = U_E(\textit{Out}, \alpha_I(X_I))$$

- For π high enough, $\alpha_I(L) = \textit{Invest}$ and $\alpha_E = \textit{Out}$ is also BNE:

$$u_I(\textit{Invest}, \textit{Out}|L) = 9 > 8 = u_I(\textit{Not}, \textit{Out}|L)$$
$$U_E(\textit{Out}, \alpha_I(X_I)) = 1 > 3(1-\pi) = U_E(\textit{In}, \alpha_I(X_I))$$

$E \backslash I.L$	Invest	Not Invest
In	0,2	3,3
Out	1,9	1,8

$E \backslash I.H$	Invest	Not Invest
In	0,0	3,3
Out	1,7	1,8

Notes

Mixed Strategies in Bayesian Games

Strategy Profiles:

- A mixed strategy is a map from information to a probability distribution over actions
- In particular $\sigma_i(a_i|x_i)$ denotes the probability that i chooses a_i if his signal is x_i
- A mixed strategy profile consists of a strategy for every player:

$$\sigma(X) = (\sigma_1(X_1), ..., \sigma_N(X_N))$$

- As usual $\sigma_{-i}(X_{-i})$ denotes the profile of strategies of all players, but i
- Mixed strategies are independent (i.e. σ_i cannot depend on σ_j)

Notes

Interim Payoff & Bayes Nash Equilibrium

The interim expected payoff of mixed strategy profiles σ and (a_i, σ_{-i}) are:

$$U_i(\sigma|x_i) = \sum_{\mathbb{X}_{-i}} \sum_{a \in A} u_i(a|x) \prod_{j \in N} \sigma_j(a_j|x_j) f(x_{-i}|x_i)$$

$$U_i(a_i, \sigma_{-i}|x_i) = \sum_{\mathbb{X}_{-i}} \sum_{a_{-i} \in A_{-i}} u_i(a|x) \prod_{j \neq i} \sigma_j(a_j|x_j) f(x_{-i}|x_i)$$

Definitions (Bayes Nash Equilibrium BNE)

A Bayes Nash equilibrium of a game Γ is a strategy profile σ such that for any $i \in N$ and $x_i \in \mathbb{X}_i$ satisfies:

$$U_i(\sigma|x_i) \geq U_i(a_i, \sigma_{-i}|x_i) \quad \text{for any } a_i \in A_i$$

BNE requires **interim optimality** (i.e. do your best given what you know)

Notes

Computing Bayes Nash Equilibria

Testing for BNE behavior:

- σ is BNE if only if:

$$U_i(\sigma|x_i) = U_i(a_i, \sigma_{-i}|x_i) \quad \text{for any } a_i \text{ s.t. } \sigma_i(a_i|x_i) > 0$$

$$U_i(\sigma|x_i) \geq U_i(a_i, \sigma_{-i}|x_i) \quad \text{for any } a_i \text{ s.t. } \sigma_i(a_i|x_i) = 0$$

- Strictly dominated strategies are never chosen in a BNE
- Weakly dominated strategies are chosen only if they are dominated with probability zero in equilibrium
- This conditions can be used to compute BNE (see examples)

Notes

Example: Mixed BNE I

Consider the following example for $f(1, L) = 1/2$:

$1\backslash 2.L$	X	Y
T	1,0	0,1
D	0,1	1,0

$1\backslash 2.R$	W	Z
T	0,0	1,1
D	1,1	0,0

All BNEs for this game satisfy:

$$\sigma_1(T) = 1/2 \text{ and } \sigma_2(X|L) = \sigma_2(W|R)$$

Such games satisfy all BNE conditions since:

$$\begin{aligned} U_1(T, \sigma_2) &= (1/2)\sigma_2(X|L) + (1/2)(1 - \sigma_2(W|R)) = \\ &= (1/2)(1 - \sigma_2(X|L)) + (1/2)\sigma_2(W|R) = U_1(D, \sigma_2) \\ u_2(X, \sigma_1|L) &= \sigma_1(T) = 1 - \sigma_1(T) = u_2(Y, \sigma_1|L) \\ u_2(W, \sigma_1|R) &= (1 - \sigma_1(T)) = \sigma_1(T) = u_2(Z, \sigma_1|R) \end{aligned}$$

Notes

Example: Mixed BNE II

Consider the following example for $f(1, L) = q \leq 2/3$:

$1\backslash 2.L$	X	Y
T	0,0	0,2
D	2,0	1,1

$1\backslash 2.R$	W	Z
T	2,2	0,1
D	0,0	3,2

The mixed BNE for this game satisfy $\sigma_1(T) = 2/3$ and:

$$\sigma_2(X|L) = 0 \text{ (dominance) and } \sigma_2(W|R) = \frac{3 - 2q}{5 - 5q}$$

Such strategies satisfy all BNE requirements since:

$$\begin{aligned} U_1(T, \sigma_2) &= 2(1 - q)\sigma_2(W|R) = \\ &= q + 3(1 - q)(1 - \sigma_2(W|R)) = U_1(D, \sigma_2) \\ u_2(X, \sigma_1|L) &= 0 < 2\sigma_1(T) + (1 - \sigma_1(T)) = u_2(Y, \sigma_1|L) \\ u_2(W, \sigma_1|R) &= 2\sigma_1(T) = \sigma_1(T) + 2(1 - \sigma_1(T)) = u_2(Z, \sigma_1|R) \end{aligned}$$

Notes

Dynamic Games
EC411 Slides 6

Nava

LSE

Michaelmas Term

Notes

Roadmap: Game Theory – Dynamic Games

The analysis considers different classes of games:

- Extensive Form Games
- Infinitely Repeated Games
- Finitely Repeated Games

The notions of behavioral strategy, information set, and subgame are introduced.

Different Solution Concepts are presented:

- Nash Equilibrium
- Subgame Perfect Equilibrium

Notes

Extensive Form Games

Notes

Summary

Dynamic Games:

- Definitions:
  - Extensive Form Game
  - Information Sets and Beliefs
  - Behavioral Strategy
  - Subgame
- Solution Concepts:
  - Nash Equilibrium
  - Subgame Perfect Equilibrium
  - Perfect Bayesian Equilibrium (extra)
- Examples

Notes

Dynamic Games

- All games discussed in previous notes were static. That is:
  - players were taking decisions simultaneously, or
  - were unable to observe the choices made by others.
- Today we relax such assumptions by modelling the timing of decisions.
- In common instances the rules of the game explicitly define:
  - the order in which players move;
  - the information available to them when they take their decisions.
- A way of representing such dynamic games is in their Extensive Form.
- The following definitions are helpful to define such notion.

Notes

Basic Graph Theory

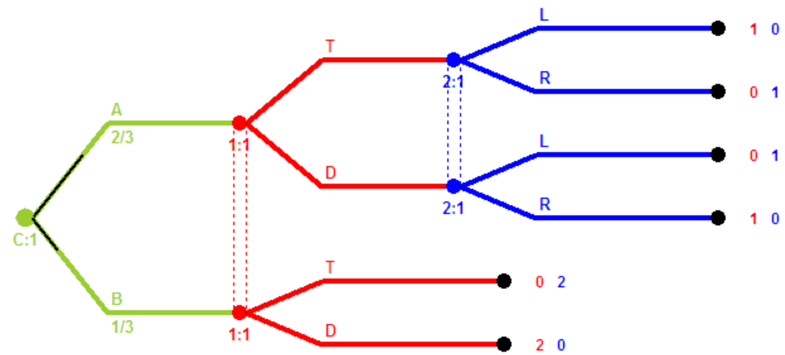
- A graph consists of a set of *nodes* and of a set of *branches*.
- Each branch connects a pair of nodes.
- A branch is identified by the two nodes it connects.
- A path is a set of branches:

$$\{\{x_k, x_{k+1}\} \,|\, k = 1, ..., m\}$$

  where $m > 1$ and every x_k is a different node of the graph.
- A *tree* is a graph in which any pair of nodes is connected by exactly one path.
- A *rooted tree* is a tree in which a special nodes designated as the *root*.
- A *terminal node* is a node connected by only one branch.

Notes

An Extensive Form Game



Notes

Extensive Form Games

An extensive form game is a rooted tree together with functions assigning labels to nodes and branches such that:

1. Each non-terminal node has a player-label in $\{C, 1, ..., n\}$
   - $\{1, ..., n\}$ are the players in the game
   - Nodes assigned to label C are *chance nodes*
   - Nodes assigned to label $i \neq C$ are *decision nodes* controlled by i

2. Each alternative at a chance node has a label specifying its probability:
   - *Chance probabilities* are nonnegative and add to 1

3. Each node controlled by player $i > 0$ has a second label specifying i's *information state*:
   - Thus nodes labeled $i.s$ are controlled by i with information s
   - Two nodes belong to $i.s$ **iff** i cannot distinguish them

Notes

Extensive Form Games

4. Each alternative at a decision node has *move* label:
   - If two nodes x, y belong to the same information set, for any alternative at x there must be exactly one alternative at y with the same move label

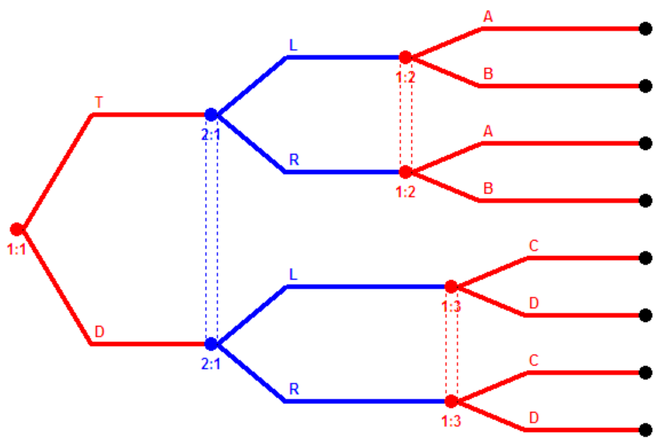
5. Each terminal node y has a label that specifies a vector of n numbers $\{u_i(y)\}_{i \in \{1,\dots,n\}}$ such that:
   - The number $u_i(y)$ specifies the *payoff* to i if the game ends at node y

6. All players have *perfect recall* of the moves they chose

Notes

Perfect Recall

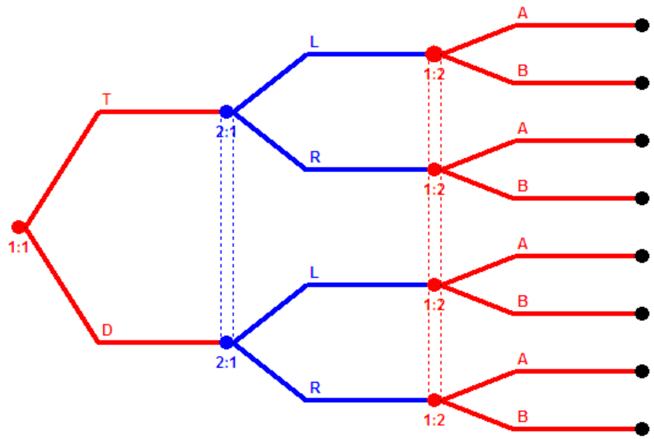
With perfect recall information sets 1.2 and 1.3 cannot coincide:



Notes

Without Perfect Recall

Without perfect recall assumption this is possible:

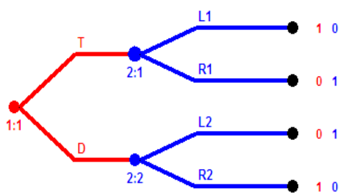


Notes

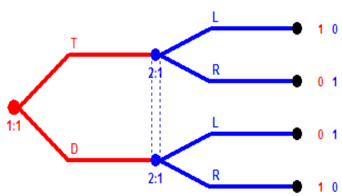
Perfect Information

An extensive form game has *perfect information* if no two nodes belong to the same information state

With Perfect Information



Without Perfect Information



Notes

Behavioral Strategies

Throughout let:

- S_i be the set information states of player $i \in N$;
- $A_{i.s}$ be the action set of player i at info state $s \in S_i$.

A *behavioral strategy* for player i maps information states to probability distributions over actions.

In particular $\sigma_{i.s}(a_{i.s})$ is the probability that player i at information stage s chooses action $a_{i.s} \in A_{i.s}$.

Throughout denote:

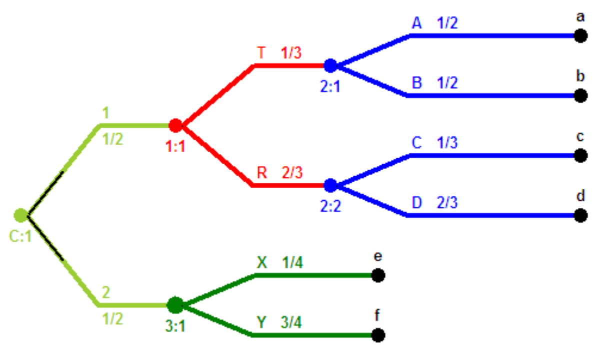
- a behavioral strategy of player i by $\sigma_i = \{\sigma_{i.s}\}_{s \in S_i}$;
- a profile of behavioral strategy by $\sigma = \{\sigma_i\}_{i \in N}$;
- the chance probabilities by $\pi = \{\pi_{0.s}\}_{s \in S_0}$.

Notes

Probabilities over Terminal Nodes

For any terminal node y and any behavioral strategy profile σ, let $P(y|\sigma)$ denote the probability that the game ends at node y.

E.g. in the following game $P(c|\sigma) = \pi_{0.1}(1)\sigma_{1.1}(R)\sigma_{2.2}(C) = 1/9$:



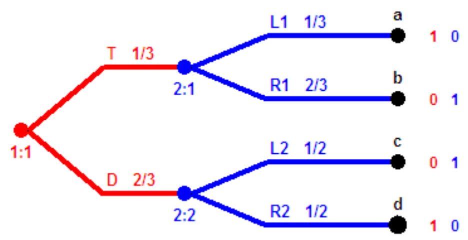
Notes

Expected Payoffs

If Ω denotes the set of end notes, the expected payoff of player i is:

$$U_i(\sigma) = U_i(\sigma_i, \sigma_{-i}) = \sum_{y \in \Omega} P(y|\sigma) u_i(y).$$

E.g. in the following game $U_1(\sigma) = 4/9$ and $U_2(\sigma) = 5/9$:



Notes

Extra: Exercise

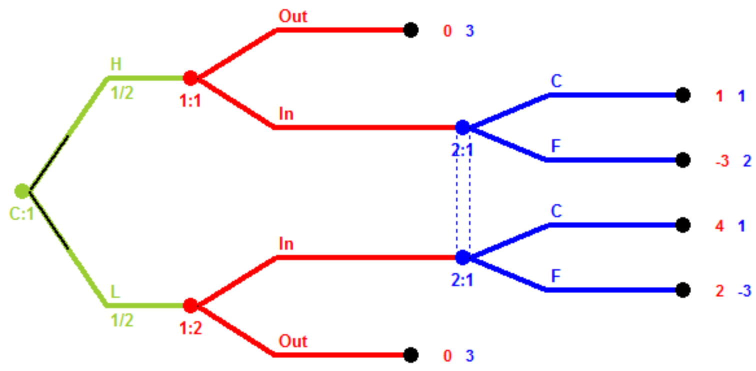
Draw the extensive form for the following two player game.

- Firms 1 has to decide whether to enter (in) or stay out (out) of a market in which it does not operate.
- Firm 2 is an old firm in the market and decides upon entry of Firm 1, whether to fight (F) or cooperate (C).
- Firm 2's costs are common knowledge.
- Firm 1's costs are random and **only** known to firm 1.
- Firm 1's costs are high (H) or low (L) with equal chance.
- Denote payoffs by (π_1, π_2), where π_i is the payoff of Firm i:

$$(\pi_1, \pi_2) = \begin{cases} (0,3) & \text{if } a_{1.1} = out \\ (-3,2) & \text{if } a_{1.1} = in, \quad a_{2.1} = F, \quad \text{cost are H} \\ (2,-3) & \text{if } a_{1.1} = in, \quad a_{2.1} = F, \quad \text{cost are L} \\ (1,1) & \text{if } a_{1.1} = in, \quad a_{2.1} = C, \quad \text{cost are H} \\ (4,1) & \text{if } a_{1.1} = in, \quad a_{2.1} = C, \quad \text{cost are L} \end{cases}$$

Notes

Extra: Answer



Notes

Extensive Form Games:

Nash Equilibrium

Notes

Nash Equilibrium Definition

Definition (Nash Equilibrium – NE)

A Nash Equilibrium of an extensive form game is any profile of behavioral strategies such that:

$$U_i(\sigma) \geq U_i(\sigma'_i, \sigma_{-i}) \quad \text{for any } \sigma'_i \in \times_{s \in S_i} \Delta(A_{i.s})$$

Recall that σ'_i is any mapping from information states to probability distributions over available actions.

The definition of NE does not change in the extensive form.

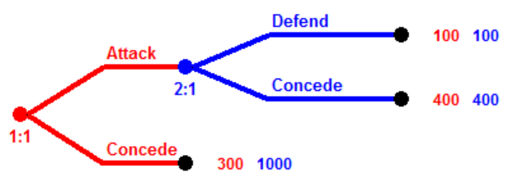
What differs is the notion of **strategy** that **is now defined for every information state**.

Notes

Example: NE of Selten's Horse

The following game has a continuum of NE:

- $\sigma_1(A) = 1$ and $\sigma_2(C) = 1$;
- $\sigma_1(A) = 0$ and $\sigma_2(C) \leq 2/3$.



Notes

Example: Transformation to Find NE

The corresponding strategic form game amounts to:

$1\backslash 2$	C	D
A	4,4	1,1
C	3,10	3,10

Strategy $\sigma_1(A) = 1$ and $\sigma_2(C) = 1$ is NE since:

$$U_1(A, C) = 4 > 3 = U_1(C, C)$$
$$U_2(C, A) = 4 > 1 = U_2(D, A)$$

Any strategy $\sigma_1(A) = 0$ and $\sigma_2(C) = q \leq 2/3$ is NE since:

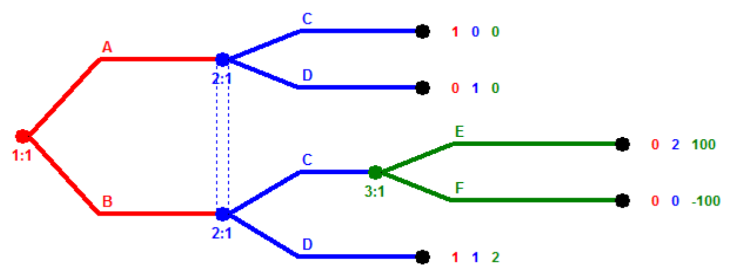
$$U_1(C, \sigma_2) = 3 \geq 4q + (1 - q) = U_1(A, \sigma_2)$$
$$U_2(C, C) = 10 = 10 = U_2(D, C)$$

Notes

Example: NE in the Extensive Form

A game may have many NEs:

- $\sigma_1(A) = 0$, $\sigma_2(C) = 0$, $\sigma_3(E) = 0$;
- $\sigma_1(A) = 1/2$, $\sigma_2(C) = 1/2$, $\sigma_3(E) = 1$.

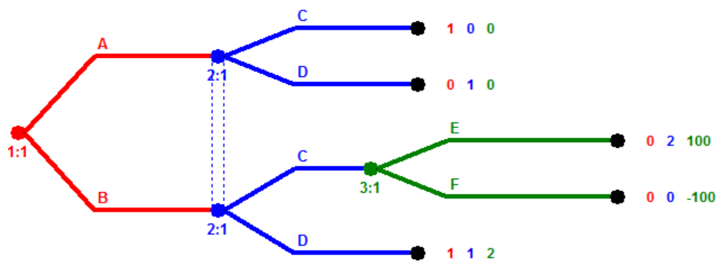


Notes

Example: NE in the Extensive Form

Profile $\sigma_1(A) = 0$, $\sigma_2(C) = 0$, $\sigma_3(E) = 0$ is NE:

$$U_1(B, \sigma_{-1}) = 1 > 0 = U_1(A, \sigma_{-1})$$
$$U_2(D, \sigma_{-2}) = 0 \geq 0 = U_2(C, \sigma_{-2})$$
$$U_3(F, \sigma_{-3}) = 2 \geq 2 = U_3(E, \sigma_{-3})$$

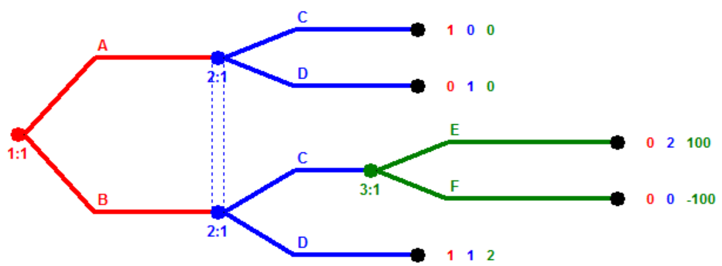


Notes

Example: NE in the Extensive Form

Profile $\sigma_1(A) = 1/2$, $\sigma_2(C) = 1/2$, $\sigma_3(E) = 1$ is NE:

$$U_1(A, \sigma_{-1}) = \sigma_2(C) = \sigma_2(D) = U_1(B, \sigma_{-1})$$
$$U_2(D, \sigma_{-2}) = \sigma_1(A) = \sigma_1(B)\sigma_3(E) = U_2(C, \sigma_{-2})$$
$$U_3(E, \sigma_{-3}) = \sigma_1(B)[2 + 98\sigma_2(C)] > \sigma_1(B)[2 - 102\sigma_2(C)] = U_3(F, \sigma_{-3})$$



Notes

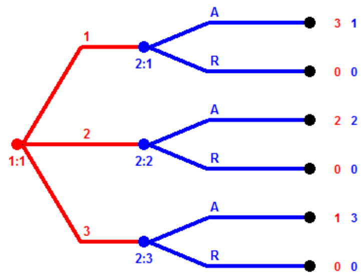
Example: NE of the Ultimatum Game

This game possesses three types of NE, namely for any $a_{2.2}, a_{2.3}$

NE1:
$\sigma_1 = [1]$ & $\sigma_2 = [A, a_{2.2}, a_{2.3}]$

NE2:
$\sigma_1 = [2]$ & $\sigma_2 = [R, A, a_{2.3}]$

NE3:
$\sigma_1 = [3]$ & $\sigma_2 = [R, R, A]$



In the table, σ_1 and σ_2 denote the behavioral (pure) strategies of the two players.

Notes

Example: NE of the Ultimatum Game

Strategy $\sigma_1 = [1]$, $\sigma_2 = [A, a_{2.2}, a_{2.3}]$ is NE since:

$$U_1(1, \sigma_2) = 3 > U_1(a_1, \sigma_2) \quad \text{for any } a_1 \in \{2, 3\}$$
$$U_2(\sigma_2, 1) = 1 \geq U_2(a_2, 1) \quad \text{for any } a_2 \in \{A, R\}^3$$

Strategy $\sigma_1 = [2]$, $\sigma_2 = [R, A, a_{2.3}]$, is NE since:

$$U_1(2, \sigma_2) = 2 > U_1(a_1, \sigma_2) \quad \text{for any } a_1 \in \{1, 3\}$$
$$U_2(\sigma_2, 2) = 2 \geq U_2(a_2, 2) \quad \text{for any } a_2 \in \{A, R\}^3$$

A similar argument works for the other proposed NE.

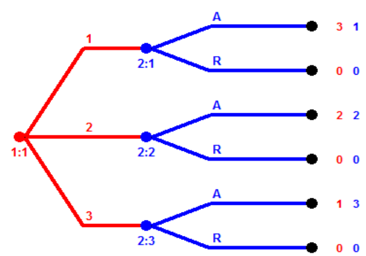
Only the first type of NE however, involves threats that are credible, since player 2 would never credibly reject an offer worth at least 1 when faced with an outside option of 0.

Notes

Example: Transformation to Find NE

The NE of the game can be found by looking at NE of the strategic form:

1/2	*AAA*	*RAA*	*ARA*	*AAR*	*RRA*	*RAR*	*ARR*	*RRR*
1	3, 1	0, 0	3, 1	3, 1	0, 0	0, 0	3, 1	0, 0
2	2, 2	2, 2	0, 0	2, 2	0, 0	2, 2	0, 0	0, 0
3	1, 3	1, 3	1, 3	0, 0	1, 3	0, 0	0, 0	0, 0



Notes

Extensive Form Games:

Subgame Perfect Equilibrium

Notes

Subgame Perfect Equilibrium

A *successor* of a node x is a node that can be reached from x for an appropriate profile of actions.

Definition (Subgame)

A *subgame* is a subset of an extensive form game such that:

1. It begins at a single node.
2. It contains all successors.
3. If a game contains an information set with multiple nodes then either all of these nodes belong to the subset or none does.

Definition (Subgame Perfect Equilibrium – SPE)

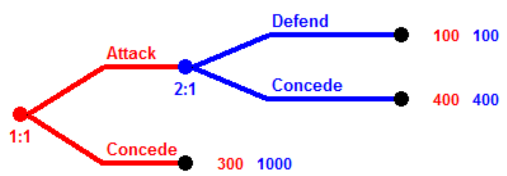
A *subgame perfect equilibrium* is any NE such that for every subgame the restriction of strategies to this subgame is also a NE of the subgame.

Notes

Unique SPE: Selten Horse

The following game had a continuum of NE, but only one SPE:

- $\sigma_{1.1}(A) = 1$ and $\sigma_{2.1}(C) = 1$ is unique SPE;
- $\sigma_{1.1}(A) = 0$ and $\sigma_{2.1}(C) \leq 2/3$ are all NE, but none SPE.

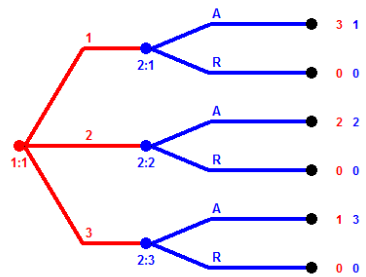


Notes

Unique SPE: Ultimatum Game

This following game had several NE, but only one SPE:

- $\sigma_{1.1}(1) = 1$ and $\sigma_{2.1}(A) = \sigma_{2.2}(A) = \sigma_{2.3}(A) = 1$.



Notes

Computing SPE – Backward Induction

Definition of SPE is demanding because it imposes discipline on behavior even in subgames that one expects not to be reached.

SPE however is easy to compute in perfect information games.

Backward-induction algorithm provides a simple way:

- At every node leading only to terminal nodes players pick actions that are optimal for them if that node is reached.
- At all preceding nodes players pick an actions that optimal for them if that node is reached knowing how all their successors behave.
- And so on until the root of the tree is reached.

A pure strategy SPE exists in any perfect information game.

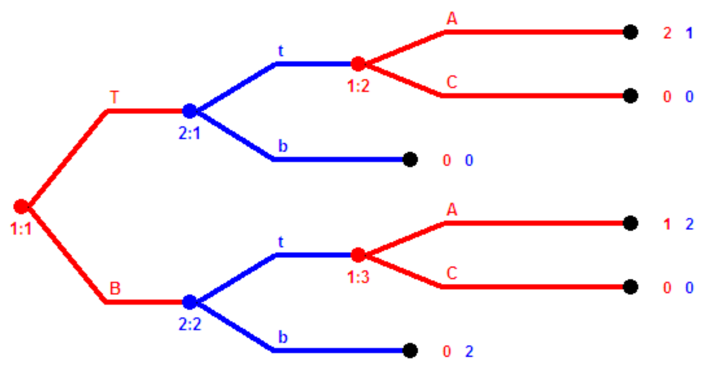
Notes

Example: Backward Induction

SPE: $\sigma_{1.2} = \sigma_{1.3} = [A]$, $\sigma_{2.1} = [t]$, any $\sigma_{2.2}$ and $\sigma_{1.1} = [T]$

NE but not SPE: $\sigma_{1.2} = \sigma_{1.3} = [A]$, $\sigma_{2.1} = [b]$, any $\sigma_{2.2}$ and $\sigma_{1.1} = [B]$

Again NE may be supported by empty threats.



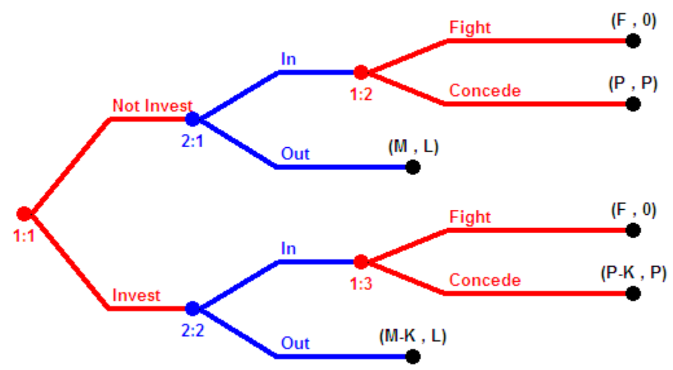
Notes

Example: Market Entry

Consider the following game between two producers.

Firm 1 is the incumbent and firm 2 is the potential entrant.

Assume $P > L > 0$ and $M > P > F$.



Notes

Example: Market Entry

To find an SPE with successful deterrence, notice that:

- If 1 does not invest, it prefers to concede given entry as $P > F$.
- Thus firm 2 prefers to enter if 1 does not invest as $P > L$.
- If 1 does invest it prefers to fight if entry takes place, provided that:

$$F > P - K$$

- If so firm 2 prefers to stay out if 1 has invested as $L > 0$.
- Thus firm 1 prefers to invest and deter entry if:

$$M - K > P$$

An SPE exists in which entry is effectively deterred if the cost satisfies:

$$M - P > K > P - F$$

Notes

Example: Trading Over Time

Consider a game played by 1 seller and 3 buyers:

- seller owns 3 identical objects and can trade in 2 periods;
- in every period $t \in \{0, 1\}$, the seller sets a price p_t;
- buyers observing the price decide whether to buy an object;
- buyers want to purchase a single object;
- v_i denotes its monetary value to buyer i where:

$$v_1 = 5, \ v_2 = 8, \ v_3 = 10.$$

The utility that i derives from purchasing an object at date t is:

$$(1/2)^t (v_i - p_t).$$

If n_t objects are sold at date t, the seller's payoff amounts to:

$$n_0 p_0 + (1/2) n_1 p_1.$$

Notes

Example: Trading Over Time (Subgame)

Let A be the set of buyers who have not bought at date 0.

In an SPE, when the active player set is A at date 1:

- buyer i buys an object if and only if $p_1 \leq v_i$;
- the seller sets the following prices so to maximize profits

$$p_1(A) = \begin{cases} 8 & \text{if} \quad A = \{1,2,3\} \\ 5 & \text{if} \quad A = \{1,2\} \\ 5 & \text{if} \quad A = \{1\} \end{cases} .$$

For instance, for $A = \{1,2,3\}$, this follows as

$$n_1 p_1 = 2 * 8 > \begin{cases} 3 * p_1 & \text{for} \quad p_1 \in [0,5] \\ 2 * p_1 & \text{for} \quad p_1 \in (5,8) \\ 1 * p_1 & \text{for} \quad p_1 \in (8,10] \end{cases} .$$

Notes

Example: Trading Over Time (Supergame)

If all units trade at date 0, then $p_0 \leq 5$.

When 3 **units trade at date** 0, the seller's best payoff amounts to

$$n_0 p_0 + (1/2) n_1 p_1(\emptyset) = 15 + 0 = 15.$$

When 0 **units trade at date** 0, the seller's best payoff amounts to

$$n_0 p_0 + (1/2) n_1 p_1(\{1,2,3\}) = 0 + 8 = 8.$$

If only buyer 3 buys at date 0, it must be that

$$v_3 - p_0 \geq (1/2)(v_3 - p_1(\{1,2,3\})) \quad \Leftrightarrow \quad p_0 \leq 9.$$

When 1 **unit trades at date** 0, the seller's best payoff amounts to

$$n_0 p_0 + (1/2) n_1 p_1(\{1,2\}) = 9 + 5 = 13.$$

Notes

Example: Trading Over Time (Supergame)

If only buyers 3 and 2 buy at date 0, it must be that

$$v_2 - p_0 \geq (1/2)(v_2 - p_1(\{1,2\})) \quad \Leftrightarrow \quad p_0 \leq 13/2.$$

When 2 **units trade at date** 0, the seller's best payoff amounts to

$$n_0 p_0 + (1/2) n_1 p_1(\{1\}) = 13 + 5/2 = 31/2.$$

In the unique SPE:

- $p_0 = 13/2$ and $p_1(A) = 5$;
- two units trade at date 0 and one at date 1;
- profits amount to 31/2.

If the seller could commit to prices, it would set $p_0 = p_1 = 8$ and raise a profit equal to 16.

Notes

Example: Consistent of Monetary Policy

First stage: Workers choose for a nominal wage, w.

Second stage: The central bank chooses an inflation rate, π.

The real wage is denoted by $w_r = w/(1+\pi)$.

The demand for workers denoted by $Q(w_r)$ and satisfies

$$Q(w_r) = 1 - w_r.$$

Worker's utility is determined by the total wage,

$$U(\pi, w) = w_r Q(w_r) = w_r(1 - w_r).$$

Central bank's objective function is

$$V(\pi, w) = \alpha Q(w_r) - \pi^2/2.$$

Notes

Example: Consistent of Monetary Policy

SPE of this game can be found using backward induction.

In the last stage given any wage w:

- The bank solves

$$\max_{\pi} V(\pi, w) = \max_{\pi} \alpha \left(1 - \frac{w}{1+\pi}\right) - \frac{\pi^2}{2}.$$

- First order conditions imply that

$$\alpha w = \pi (1+\pi)^2$$

- The solution $\pi(w)$ is an increasing function of αw.

Notes

Example: Consistent of Monetary Policy

In the first stage, the workers:

- Calculate $\pi(w)$ and take it as given.
- Solve the following problem:

$$\max_{w} U(\pi(w), w) = \max_{w} \frac{w}{1+\pi(w)} \left(1 - \frac{w}{1+\pi(w)}\right).$$

- FOC imply that:

$$w_r^* = \frac{w^*}{1+\pi(w^*)} = \frac{1}{2}.$$

The last condition reduces the bank's FOC at the optimum to:

$$\alpha/2 = \pi^* (1+\pi^*)$$

Notes

Example: Consistent of Monetary Policy

Conclusions:

- $w_r^* = Q(w_r^*) = 1/2$ independent of π^*;
- $\pi^* > 0$ positive inflation if $\alpha > 0$.

Including employment Q into the bank's objective function has no effect on employment Q or real wages, but leads to a positive inflation rate.

The bank wants to surprise workers with positive inflation.

But workers anticipate the bank's response, take positive inflation into account, and neutralize any effect of the bank's policy on real wages.

The bank cannot credibly commit not to change the inflation rate.

This argument favors restricting central bank policy to inflation targeting, leaving out any employment consideration.

Notes

Extensive Form Games:

Concluding Remarks

Notes

Limitations to NE and SPE Behavior

The examples showed that NE behavior was not immune to empty treats.

In a NE any player would:

- believe with certainty that all the other will comply with the equilibrium strategy;
- not consider the credibility (rationality) of opponents' behavior.

SPE behavior bypasses these limitations by imposing rationality on opponents' behavior, as only credible threats are believed in equilibrium.

SPE behavior however, is not immune to limitations:

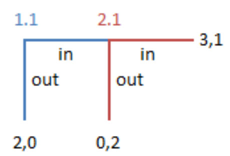
- Too much discipline on equilibrium behavior (centipede game);
- **Unbounded rationality** (chess).

Notes

Limitations to SPE Behavior: Centipede Game

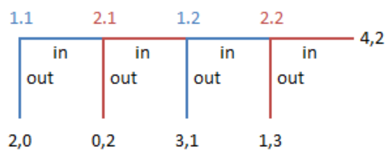
Rosenthal highlighted a key SPE complication in the **centipede game**.

Consider the following extensive form game:



In any SPE both players choose out.

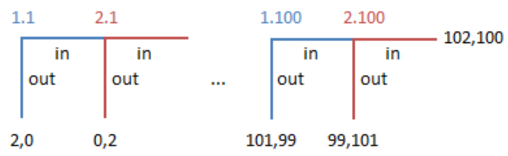
If we iterate the game as below, in any SPE players always choose out:



Notes

Limitations to SPE Behavior: Centipede Game

If we continue iterating the game:



In any SPE players keep choosing out in every period.

However, such a behavior appears implausible, as players sacrifice large future mutual gains in order to comply with rational behavior.

Notes

Limitations to SPE Behavior: Chess

Chess can be modeled as an extensive form game:

1. the tree is finite;
2. the game is zero sum game (one wins, the other loses);
3. it has complete information (information sets contain one node).

If we start at the end of the game apply backward induction at each step backwards, we obtain exactly one pair of payoffs (as the game is zero sum).

Chermelo 1912:

Chess is a game with a deterministic outcome:

- either white can guarantee victory;
- or black can guarantee victory;
- or each player can guarantee a tie.

This highlight the second major limitation of SPE behavior. Namely that it requires and **unbounded** level of **rationality** (computing power).

Notes

Extra: Dynamics and Uncertainty

If extensive form games without perfect information, subgame perfection cannot be imposed at every node, but only on subgames.

In such games a further equilibrium refinement may help to highlight the relevant equilibria of the game by selecting those which satisfy Bayes rule.

Definition (Perfect Bayesian Equilibrium – PBE)

A *perfect Bayesian equilibrium* of an extensive form game consists of a profile of behavioral strategies and of beliefs at each information set of the game such that:

1. strategies form an SPE given the beliefs;
2. beliefs are updated using Bayes rule at each information set reached with positive probability.

Notes

Repeated Games:

Infinite Horizon Games

Notes

Summary

Repeated Games:

- Definitions:
  - Feasible Payoffs
  - Minmax
  - Repeated Game
  - Stage Game
  - Trigger Strategy
- Main Result:
  - Folk Theorem
- Examples: Prisoner's Dilemma

Notes

Feasible Payoffs

Q: What payoffs are feasible in a strategic form game?

A: A profile of payoffs is feasible in a strategic form game if can be expressed as a weighed average of payoffs in the game.

Definition (Feasible Payoffs)

A profile of payoffs $\{w_i\}_{i\in N}$ is feasible in a strategic form game $\left\{N, \{A_i, u_i\}_{i\in N}\right\}$ if there exists a distribution over profiles of actions π such that:

$$w_i = \sum_{a\in A} \pi(a) u_i(a) \quad \text{for any} \quad i \in N$$

Unfeasible payoffs cannot be outcomes of the game.

Points on the north-east boundary of the feasible set are Pareto efficient.

Notes

Minmax

Q: What's the worst possible payoff that a player can achieve if he chooses according to his best response function?

A: The minmax payoff.

Definition (Minmax)

The (pure strategy) *minmax* payoff of player $i \in N$ in a strategic form game $\{N, \{A_i, u_i\}_{i \in N}\}$ is:

$$\underline{u}_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i})$$

Mixed strategy minmax payoffs satisfy:

$$\underline{v}_i = \min_{\sigma_{-i}} \max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})$$

The mixed strategy minmax is not higher than the pure strategy minmax.

Notes

Example: Prisoner's Dilemma

Minmax Payoff: $(1, 1)$

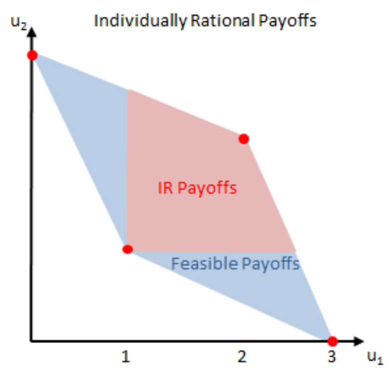
Feasible Payoffs: blue and red areas.

Individually Rational Payoffs: red area.

Stage Game

1\2	w	s
w	2,2	0,3
s	3,0	1,1

Payoffs



Notes

Example: Battle of the Sexes

Minmax Payoff: $(2, 2)$

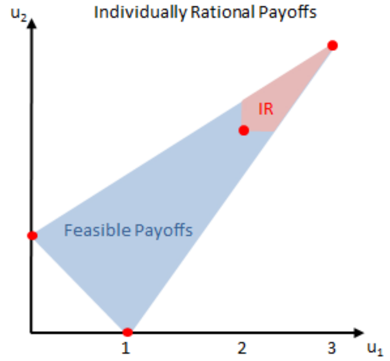
Feasible Payoffs: blue and red areas.

Individually Rational Payoffs: red area.

Stage Game

1\2	w	s
w	3,3	1,0
s	0,1	2,2

Payoffs



Notes

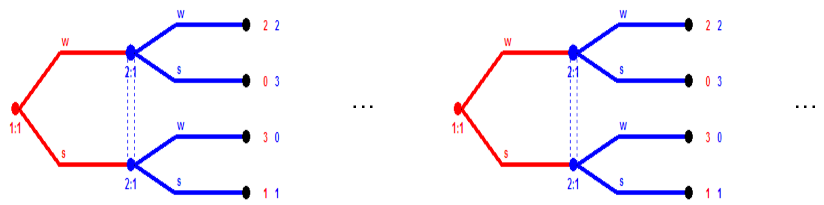
Repeated Game: Timing

Consider any strategic form game $G = \{N, \{A_i, u_i\}_{i \in N}\}$.

Call G the *stage game*.

An infinitely *repeated game* describes a strategic environment in which the stage game is played repeatedly by the same players infinitely many times.

Round 1 ... Round t ...



Notes

Repeated Games: Payoffs and Discounting

The value to player $i \in N$ of a payoff stream $\{u_i(1), u_i(2), ..., u_i(t), ...\}$ is:

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(t)$$

The term $(1-\delta)$ amounts to a simple normalization

... and guarantees that a constant stream $\{v, v, ...\}$ has value v.

Future payoffs are discounted at rate δ.

An infinitely repeated game can be used to describe strategic environments in which there is no certainty of a final stage.

In such view δ describes the probability that the game does not end at the next round which would result in a payoff of 0 thereafter.

Notes

Repeated Games: Perfect Information and Strategies

Today we restrict attention to perfect information repeated games.

In such games all players prior to each round observe the actions chosen by all other players at previous rounds.

Let $a(s) = \{a_1(s), ..., a_n(s)\}$ denote the action profile played at round s.

A *history* of play up to stage t thus consists of:

$$h(t) = \{a(1), a(2), ..., a(t-1)\}$$

In this context strategies map histories (ie information) to actions:

$$\alpha_i(h(t)) \in A_i$$

Notes

Prisoner's Dilemma Folk Theorem

Consider the prisoner's dilemma discussed earlier:

1\2	w	s
w	2,2	0,3
s	3,0	1,1

To understand how equilibrium behavior is affected by repetition, let's show why $(2, 2)$ is SPE of the infinitely repeated prisoner's dilemma.

Folk theorem shows that any feasible payoff that yields to both players at least their minmax value is a SPE of the infinitely repeated game if the discount factor is sufficiently high.

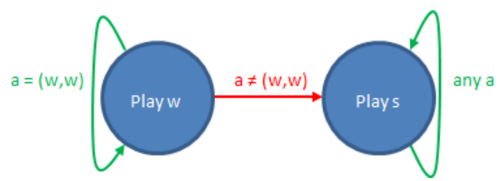
Notes

Grim Trigger Strategies

Consider the following strategy (known as *grim trigger strategy*):

$$a_i(t) = \begin{cases} w & \text{if } a(z) = (w, w) \text{ for any } z < t \\ s & \text{otherwise} \end{cases}$$

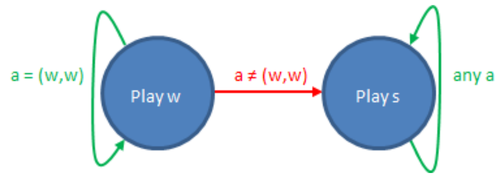
Such a strategy can be graphically represented as a 2-state automaton:



Notes

Grim Trigger SPE

Consider the grim trigger strategy:



If all players follow such strategy, no player can deviate and benefit at any given round provided that $\delta \geq 1/2$.

In subgames in which $a(t) = (w, w)$ no player benefits from a deviation as:

$$(1-\delta)\left(3+\delta+\delta^2+\delta^3+...\right) = 3-2\delta \leq 2 \quad \Leftrightarrow \quad \delta \geq 1/2$$

In subgames in which $a(t) = (s, s)$ no player benefits from a deviation as:

$$(1-\delta)\left(0+\delta+\delta^2+\delta^3+...\right) = \delta \leq 1 \quad \Leftrightarrow \quad \delta \leq 1$$

Notes

Folk Theorem

Theorem (SPE Folk Theorem)

In any two-person infinitely repeated game:

1. *For any discount factor δ, the discounted average payoff of each player in any SPE is at least his minmax value in the stage game.*
2. *Any feasible payoff profile that yields to all players at least their minmax value is the discounted average payoff of a SPE if the discount factor δ is sufficiently close to 1.*
3. *For any NE of the stage game, the infinitely repeated game possesses a corresponding SPE in which players' discounted average payoffs coincide with payoffs in the desired NE.*

Notes

Testing SPE in Repeated Games

Definition (One-Deviation Property)

A strategy satisfies the *one-deviation property* if no player can increase his payoff by changing his action at the start of any subgame in which he is the first-mover given other players' strategies and the rest of his own strategy.

Fact

A strategy profile in an extensive game with perfect information and infinite horizon is a SPE if and only if it satisfies the one-deviation property.

This observation can be used to test whether a strategy profile is a SPE of an infinitely repeated game as we did in the Prisoner's dilemma example.

Notes

Example: Different Equilibrium Behavior

Next show why $(1.5, 1.5)$ is also an SPE of the repeated PD as $\delta \to 1$.

Consider the following pair of strategies:

$$a(t) = \begin{cases} (w,s) & \text{if } t \text{ is even } \textbf{and } \; a(z) \notin \{(s,s),(w,w)\} \text{ for any } z < t \\ (s,w) & \text{if } t \text{ is odd } \textbf{and } \; a(z) \notin \{(s,s),(w,w)\} \text{ for any } z < t \\ (s,s) & \text{otherwise} \end{cases}$$

If $\delta \geq 1/2$, no player can profitably deviate from the strategy.

When $a(t) = (s,w),(w,s)$, no player benefits from a deviation as:

$$1 \leq (1-\delta)\left(3\delta + 3\delta^3 + 3\delta^5 + ...\right) = 3\delta/(1+\delta)$$
$$(1-\delta)\left(2+\delta+\delta^2...\right) = 2-\delta \leq (1-\delta)\left(3+3\delta^2+3\delta^4...\right) = 3/(1+\delta)$$

When $a(t) = (s,s)$, no player benefits from a deviation as:

$$(1-\delta)\left(0+\delta+\delta^2+\delta^3+...\right) = \delta \leq 1 \quad \Leftrightarrow \quad \delta \leq 1$$

Notes

Example: Different Punishments

Consider the following game – with minmax payoffs of $(1,1)$:

1\2	A	B
A	0,0	4,1
B	1,4	3,3

Two PNE: (A,B) and (B,A) with payoffs $(1,4)$ and $(4,1)$.

Always playing (B,B) is SPE of the repeated game for δ high enough.

Consider the following grim trigger strategy:

$$a(t) = \begin{cases} (B,B) & \text{if } \ a(s) = (B,B) \ \text{ for any } \ s < t \\ (B,A) & \text{if } \ a(s) = \begin{cases} (B,B) & \text{for } \ s < z \\ (A,B) & \text{for } \ s = z \end{cases} \quad \text{for } \ z \in \{0,..,t-1\} \\ (A,B) & \text{if } \ a(s) = \begin{cases} (B,B) & \text{for } \ s < z \\ (B,A) & \text{for } \ s = z \end{cases} \quad \text{for } \ z \in \{0,..,t-1\} \end{cases}$$

Notes

Example: Different Punishments

1\2	A	B
A	0,0	4,1
B	1,4	3,3

If all follow such strategy, no player can deviate and benefit if $\delta \geq 1/3$.

When $a(t) = (B,B)$, no player player benefits from a deviation if $\delta \geq 1/3$:

$$(1-\delta)\left(4+\delta+\delta^2+\delta^3+...\right) = 4-3\delta \leq 3 \quad \Leftrightarrow \quad \delta \geq 1/3$$

When $a(t) = (B,A)$, no player player benefits from a deviation as:

$$(1-\delta)\left(0+\delta+\delta^2+\delta^3+...\right) = \delta \leq 1 \quad \Leftrightarrow \quad \delta \leq 1$$
$$(1-\delta)\left(3+4\delta+4\delta^2+4\delta^3+...\right) = 3+\delta \leq 4 \quad \Leftrightarrow \quad \delta \leq 1$$

When $a(t) = (A,B)$, no one wishes to deviate for symmetric reasons.

Notes

Repeated Games:

Finite Horizon Games

Notes

Finitely Repeated Games

Now suppose that the stage-game G is repeated T times, for T finite.

As for infinitely repeated games, a strategy of a player is a plan that assigns a stage-game action in every period to any possible history.

The number of possible histories is now finite.

The payoff of player i given a sequence of stage-game actions $\{a^t\}_{t=1}^T$ is

$$\sum_{t=1}^T \delta_i^{t-1} u_i(a^t)$$

Theorem

Consider the finitely repeated prisoners' dilemma. Then, if **œ** *is subgame perfect, $\sigma_i(h) = s$ for any history h.*

Notes

Finitely Repeated Games

Proof.

Take any subgame in the last period T (for any previous history).

This game is played only for one period. Hence, any NE in this subgame implies that both players play s.

Consider now period $T-1$. We know that both players play s in period T regardless of what they do in period $T-1$.

Hence, no player plays w in period $T-1$ after any history as, by deviating to s, a player gains in period $T-1$ and suffers no losses in period T (the opponent plays s in period T regardless of the play in $T-1$).

Consider now period $T-2$, etc.... □

Adaptations of the proof show that this result extends:

- to any stage game with a unique NE;
- to uncertainty over finite time horizon T.

Notes

Repeated Games:

Comments on Private Monitoring

Notes

Imperfect Private Monitoring

Repeated games were initially studied under the assumption that players observe the actions chosen by all players in all the previous periods.

If so, players are able to coordinate their behavior to generate dynamic incentives capable of sustaining equilibrium outcomes which differ from static Nash behavior.

A more recent and vast literature has considered repeated games in which the actions of the players are monitored *privately* and *imperfectly*.

A recent result by Sugaya has established that a Folk theorem applies even in this setups under some additional assumptions.

Notes

Imperfect Private Monitoring

Consider the following Prisoner's Dilemma:

1\2	C	D
C	3,3	-1,4
D	4,-1	0,0

Suppose that this game is repeated an infinite number of times and that δ is the discount factor of both players.

Consider the "grim trigger" strategy:

- play C if (C, C) was played in every past period;
- play D otherwise.

If both players adopt the above strategy and δ is sufficiently high, we have an equilibrium in which (C, C) is played in every period.

Notes

Imperfect Private Monitoring

When actions are not observable, the construction of dynamic incentives is far more complicated.

Suppose that each player observes the actions of the opponents *privately* and *imperfectly*.

Examples:

Firms may not observe the price charged by competitors when prices determined in private negotiations.

Firms may only observe decreases in their sales, but may be unable to establish with certainty the cause:

- deviant behavior by their competitors (eg secret price cutting)
- their own behavior (eg low quality customer service).

Notes

Imperfect Private Monitoring

The grim trigger strategy cannot be an equilibrium with such monitoring:

- Suppose that player 1 expects player 2 to play C with certainty.
- If player 1 observes D (a fall in sales), he must conclude that this is not due to player 2 playing D (price cuts), but to other reasons.
- Then player one should not play D next period, as this would trigger player 2's punishment (the grim trigger strategy).
- Mutual deterrence evaporates: if player 1 continues to play C after observing D, player 2 does gain by playing D (secretly cutting prices).

Notes

On the Concavity of the Expenditure Function (EC411 – Nava)

Let $h(p,u)$ denote the $n \times 1$ vector of Hicksian demands.

The **law of compensated demand** states that for preferences that satisfy LNS and convexity, and for any two strictly positive $1 \times n$ price vectors $p, \bar{p} >> 0$, we have that

$$\underbrace{(p-\bar{p})}_{1\times n} \cdot \underbrace{(h(p,u)-h(\bar{p},u))}_{n\times 1} \leq 0.$$

This obtains because by the optimality of $h(p,u)$ in the expenditure minimization problem, as it implies both

$$\begin{aligned} p\cdot h(p,u) &\leq p\cdot h(\bar{p},u) \quad \& \\ \bar{p}\cdot h(\bar{p},u) &\leq \bar{p}\cdot h(p,u). \end{aligned}$$

By adding the two and rearranging the result the law of compensated demands follows.

Next we exploit the law of compensated demand to prove the concavity of the expenditure function. Note that the differential version of the law of demand states that

$$\underbrace{dp}_{1\times n} \cdot \underbrace{dh(p,u)}_{n\times 1} \leq 0.$$

Let $D_p h(p,u)$ denote the $n \times n$ Jacobian of $h(p,u)$ with respect to prices. Recall that that $dh(p,u) = D_p h(p,u) \cdot dp^T$. Substituting we obtain that

$$\underbrace{dp}_{1\times n} \cdot \underbrace{D_p h(p,u)}_{n\times n} \cdot \underbrace{dp^T}_{n\times 1} \leq 0,$$

and therefore that $D_p h(p,u)$ is negative semidefinite.

This immediately implies the **concavity of the expenditure function** since:

$$\begin{aligned} \frac{de(p,u)}{dp} &= h(p,u) \quad \text{and} \\ \frac{d^2e(p,u)}{dp^2} &= D_p h(p,u). \end{aligned}$$

More on Welfare (EC411 – Nava)

Consider the market for good 1, where $x_1(p, m)$ is demand for good 1. Suppose that the only price of good 1 goes from the old price p_1^O to the new price p_1^N. For convenience, throughout let $p = (p_1, p_{-1})$, let

$$u^O = v(p_1^O, p_{-1}, m) \quad \text{and} \quad u^N = v(p_1^N, p_{-1}, m).$$

As mentioned in lecture, CS measures the change in welfare normalized by the change in purchasing power of the consumer

$$\begin{aligned} CS &= \int_{p_1^O}^{p_1^N} \frac{\partial v(p,m)/\partial p_1}{\partial v\,(p,m)\,/\partial m} dp_1 = \int_{p_1^O}^{p_1^N} -x_1(p,m) dp_1 = \\ &= \int_{p_1^N}^{p_1^O} x_1(p,m) dp_1 = \int_{p_1^N}^{p_1^O} h_1(p, v(p,m)) dp_1 \\ &= \int_{p_1^N}^{p_1^O} \frac{\partial e(p, v(p,m))}{\partial p_1} dp_1. \end{aligned}$$

The first integral goes from p_1^O to p_1^N as prices go from p_1^O to p_1^N. Equalities above follows respectively from: Roy, change of order of integration, duality, and the envelope theorem. The order of integration makes sense as $CS > 0$ requires $p_1^O > p_1^N$, while $CS < 0$ requires $p_1^O < p_1^N$.

The other two welfare measures derived in lecture were

$$\begin{aligned} EV &= \int_{p_1^N}^{p_1^O} \frac{\partial e(p, u^N)}{\partial p_1} dp_1 = \int_{p_1^N}^{p_1^O} h_1(p, u^N) dp_1, \\ CV &= \int_{p_1^N}^{p_1^O} \frac{\partial e(p, u^O)}{\partial p_1} dp_1 = \int_{p_1^N}^{p_1^O} h_1(p, u^O) dp_1. \end{aligned}$$

In lecture we had also shown that

$$\begin{array}{lccc} \text{Normal:} & \dfrac{\partial x_1}{\partial m} > 0 & \Rightarrow & \dfrac{\partial h_1}{\partial u} > 0 \\ \text{Inferior:} & \dfrac{\partial x_1}{\partial m} < 0 & \Rightarrow & \dfrac{\partial h_1}{\partial u} < 0 \end{array}$$

since by LNS $\partial v / \partial m > 0$ and by duality

$$\frac{\partial x_1}{\partial m} = \frac{\partial h_1}{\partial u} \frac{\partial v}{\partial m}.$$

given that $x_1(p, m) = h_1(p, v(p, m))$.

To formally compare the 3 welfare criteria, keep the prices of all good, but for good 1 fixed. Then observe that if the price decreases $p_1^N \leq p_1 \leq p_1^O$, we obtain that

$$u^O \leq v(p,m) \leq u^N.$$

Consequently, if the good is normal $\partial h_1/\partial u > 0$, we know that

$$h_1(p,u^O) \leq x_1(p,m) = h_1(p,v(p,m)) \leq h_1(p,u^N).$$

which by integration immediately implies

$$CV \leq CS \leq EV.$$

If the good is inferior instead $\partial h_1/\partial u < 0$, we know that

$$h_1(p,u^O) \geq x_1(p,m) = h_1(p,v(p,m)) \geq h_1(p,u^N).$$

which by integration immediately implies

$$CV \geq CS \geq EV.$$

If the price increases instead $p_1^O \leq p \leq p_1^N$, we obtain that

$$u^N \leq v(p,m) \leq u^O.$$

Consequently, if the good is normal $\partial h_1/\partial u > 0$, we know that

$$h_1(p,u^N) \leq x_1(p,m) = h_1(p,v(p,m)) \leq h_1(p,u^O).$$

which by integration immediately implies

$$EV \leq CS \leq CV.$$

If the good is inferior instead $\partial h_1/\partial u < 0$, we know that

$$h_1(p,u^N) \geq x_1(p,m) = h_1(p,v(p,m)) \geq h_1(p,u^O).$$

which by integration immediately implies

$$EV \geq CS \geq CV.$$

Just as an FYI on Slutsky with discrete price changes, let me add that the substitution effect can be computed as

$$SE = h_1(p_1^N, p_{-1}, u^O) - h_1(p_1^O, p_{-1}, u^O).$$

The total effect can be computed as

$$\begin{aligned} TE &= x_1(p_1^N, p_{-1}, m) - x_1(p_1^O, p_{-1}, m) \\ &= h_1(p_1^N, p_{-1}, u^N) - h_1(p_1^O, p_{-1}, u^O), \end{aligned}$$

and consequently the income effect can be computed as

$$\begin{aligned} IE &= h_1(p_1^N, p_{-1}, u^N) - h_1(p_1^N, p_{-1}, u^O) \\ &= TE - SE. \end{aligned}$$

On the Convexity of the Production Set (EC411 – Nava)

Consider a firm producing 1 output with n inputs.

Let's show that if the production possibility set Y is convex, the production function $f : \mathbb{R}^n \to \mathbb{R}$ is concave.

Consider any two feasible production plans

$$(f(x'), -x'), (f(x''), -x'') \in Y,$$

and for $\lambda \in (0, 1)$, let

$$(y, -x) = \lambda(f(x'), -x') + (1 - \lambda)(f(x''), -x'').$$

As Y is convex $(y, -x) \in Y$. Moreover, by definition of the production function

$$f(x) \geq \bar{y}, \text{ for any } (\bar{y}, -x) \in Y.$$

This immediately delivers the result as the previous observations imply that

$$f(x) = f(\lambda x' + (1 - \lambda)x'') \geq \lambda f(x') + (1 - \lambda)f(x'').$$

On Concavity and Returns to Scale (EC411 – Nava)

Let $f(x)$ denote a concave production function for a vector of inputs x.

Concavity of the production function requires that for any $p \in [0, 1]$ and for any two input vectors x, x' we have

$$f(px + (1-p)x') \geq pf(x) + (1-p)f(x').$$

Further assume that $f(0) = 0$. If so, concavity implies that for any $p \in [0, 1]$

$$f(px) = f(px + (1-p)0) \geq pf(x) + (1-p)f(0) = pf(x).$$

This implies that the production function is subadditive as for any $t \geq 1$,

$$tf(x) = tf\left(\frac{1}{t}tx\right) \geq t\frac{1}{t}f(tx) = f(tx).$$

This implies decreasing or constant returns to scale. Strict concavity equivalently implies DRS.

On the Slope of the Supply Function (EC411 – Nava)

Let $y(p,w)$ denote the supply function of a firm and let $x(p,w)$ denote the vector of input demands.

Recall that $y(p,w)$ satisfies

$$y(p,w) = f(x(p,w)).$$

Thus, the slope of the supply function with respect to the output price is equal to

$$\frac{\partial y(p,w)}{\partial p} = \textstyle\sum_{i=1}^{n} \left[\frac{\partial f(x(p,w))}{\partial x_i}\frac{\partial x_i(p,w)}{\partial p}\right] = \underbrace{\left[\frac{\partial f(x(p,w))}{\partial x}\right]}_{1\times n}\underbrace{\left[\frac{\partial x(p,w)}{\partial p}\right]}_{n\times 1}.$$

Next, recall that first order conditions for profit maximization require

$$p\frac{\partial f(x(p,w))}{\partial x} = \underbrace{w}_{1\times n}.$$

By totally differentiating this equation with respect to the output price p, we further obtain that

$$\frac{\partial f(x(p,w))}{\partial x} + p\frac{\partial x(p,w)}{\partial p}^T \underbrace{\frac{\partial^2 f(x(p,w))}{\partial x^2}}_{n\times n} = 0$$

$$\Rightarrow \quad \frac{\partial f(x(p,w))}{\partial x} = -p\frac{\partial x(p,w)}{\partial p}^T \frac{\partial^2 f(x(p,w))}{\partial x^2}.$$

Exploiting the previous equation, we obtain that

$$\begin{aligned}\frac{\partial y(p,w)}{\partial p} &= \frac{\partial f(x(p,w))}{\partial x}\frac{\partial x(p,w)}{\partial p} = \\ &= -p\frac{\partial x(p,w)}{\partial p}^T \frac{\partial^2 f(x(p,w))}{\partial x^2}\frac{\partial x(p,w)}{\partial p} \geq 0,\end{aligned}$$

where the last inequality holds since the Hessian of the objective function, $\frac{\partial^2 f(x(p,w))}{\partial x^2}$, must be negative semidefinite at any maximum. This establishes that the supply function is upward sloping in the output price.

On the Continuity of the Cost Function (EC411 – Nava)

Let $x(w,y)$ denote the $n \times 1$ vector of conditional factor demands, and let $c(w,y)$ denote the cost function.

Suppose that $w' \geq w$, and observe that

$$c(w,y) = wx(w,y) \leq w'x(w,y),$$

as $w' \geq w$. Moreover, as costs are non-decreasing in factor prices,

$$c(w,y) = wx(w,y) \leq wx(w',y) \leq w'x(w',y) = c(w',y).$$

Finally by optimality of the cost function

$$c(w',y) = w'x(w',y) \leq w'x(w,y).$$

These observations together imply that

$$c(w',y) \in [wx(w,y), w'x(w,y)].$$

The cost function is then obviously continuous as

$$\lim_{w' \to w} c(w',y) \in \lim_{w' \to w} [wx(w,y), w'x(w,y)] = c(w,y).$$

More on Walras Law (EC411 – Nava)

By Walras Law we know that LNS implies $pz(p) = 0$. We want to show that it further requires $p_i z_i(p) = 0$ for any good i.

If $pz(p) = 0$, for any good i it must be that

$$p_i z_i(p) = -\sum_{k \neq i} p_k z_k(p).$$

Thus, suppose that for some good i we have that $p_i z_i(p) < 0$. As prices are are non-negative, this requires $p_i > 0$ and $z_i(p) < 0$. If so, $\sum_{k \neq i} p_k z_k(p) > 0$ by Walras Law, and therefore $p_k z_k(p) > 0$ for some k. But this would violate feasibility in the market of good k, as it requires $p_k > 0$ and $z_k(p) > 0$. Thus, Walras Law implies that $p_i z_i(p) = 0$ for any good i.

EC411 Classes Schedule

Week 2 MT – Consumption:
Problem Set 1 Solutions

Week 3 MT – Consumption:
Problem Set 2 Solutions

Week 4 MT – Production:
Problem Set 3 Solutions

Week 5 MT – Production:
Problem Set 4 Solutions

Week 6 MT – Partial Equilibrium:
Problem Set 5 Solutions

Week 7 MT – General Equilibrium:
Problem Set 6 Solutions & Some Hand-In Homework 1 Solutions

Week 8 MT – Uncertainty:
Problem Set 7 Solutions & Some Hand-In Homework 1 Solutions

Week 9 MT – Uncertainty and Game Theory:
Problem Set 8 Solutions

Week 10 MT – Game Theory:
Problem Set 9 Solutions

Week 11 MT – Game Theory:
Problem Set 10 Solutions & Some Hand-In Homework 2 Solutions

Please go over ALL mandatory problems and some of the additional problems.

Problem Sets

EC411 – LSE

Problem Set 1: Consumption Francesco Nava

EC411 – LSE MT 2019/2020

1. The preferences of a consumer are represented by the utility function

$$u(x, y) = x + y.$$

The price of a unit of good x is 2, the price of a unit of good y is p, and the income of the consumer is M. Moreover, the consumer is given a lump-sum subsidy of S that can only be used to purchase good y. Suppose that $p \neq 2$

   (a) Define the utility maximization problem of the consumer.

   (b) Find the optimal amounts of x and y as functions of p, M and S.

2. Answer the following questions:

   (a) A consumer has a utility function

$$u(x_1, x_2) = \sqrt{x_1^2 + x_2^2}.$$

   Find the Marshallian demand functions for both goods.

   (b) A consumer has a utility function

$$u(x_1, x_2) = \min\{x_1, 3 + x_2\}.$$

   Find the Marshallian demand functions for both goods.

   (c) True or false? In order to aid the poor, the government introduces a scheme whereby the first 1kg of butter a family buys is subsidised and the remaining amounts are taxed. Consider a family which consumes butter and is made neither better off nor worse off as a result of this scheme. If so, the total amount of tax it pays cannot exceed the subsidy it receives.

   (d) True or false? During a war, food and clothing are rationed. In addition to a money price, a certain number of ration points must be paid to obtain a good. Each consumer has an allocation of ration points which may be used to purchase either good, and also has a fixed money income. Suppose the money income of a consumer is raised and he buys more food and less clothing. If follows that clothing is an inferior good.

3. Assume that Marshallian demands are single valued – so that a unique bundle $x(p, m)$ is optimal for all $p >> 0$ and all $m > 0$. Prove that the indirect utility $v(p, m)$ is continuous if $U(x)$ is continuous.

Extra Problem:

1. A consumer is paid each week 4 units of x_1 and 4 units of x_2 which she may consume directly or trade with other customers at the going prices. In the first week she trades, and finally consumes 5 units of x_1 and 3 units of x_2. In the second week prices change, and she finally consumes 6 units of x_1 and 1 units of x_2. Assuming that her tastes remain unchanged for these two weeks:

   (a) Find the price ratio p_1/p_2 in the two weeks?

   (b) In which week is the consumer better off?

   (c) Is x_1 an inferior good for this consumer?

   (d) Is x_1 a Giffen good for this consumer?

Problem Set 2: Consumption Francesco Nava

EC411 – LSE MT 2019/2020

1. The preferences of a consumer are represented by the utility function

$$u(x, y) = xy.$$

Let p_x denote the price of good x and p_y denote the price of good y.

(a) Establish that utility is quasi-concave.

(b) Solve the utility maximization problem if the income of the consumer equals 12. Derive the Marshallian demand for each of the two goods.

2. Consider a strictly increasing and strictly concave map φ and a consumer with preferences

$$u(x_1, x_2) = x_1 + \varphi(x_2).$$

When do the Marshallian and Hicksian curves for x_2 coincide? What is the significance of this for the estimation of the welfare changes due to an increase in the price of x_2?

3. Consider a consumer with preferences

$$u(x_1, x_2) = x_1 \min\{x_2, 1\}.$$

Derive the expenditure function for this problem, and the Hicksian demand for each of the two goods. Next state and use the relevant parts of the duality theorem to derive the value function and the corresponding Marshallian demands.

Extra Problem:

1. The expenditure function of a consumer is denoted by $e(p, u)$, where p is the price vector and u the utility level. Show that if

$$\frac{\partial^2 e(p, u)}{\partial p_j \partial u} > 0,$$

commodity j cannot be an inferior good.

Problem Set 3: Welfare & DT Advances Francesco Nava

EC411 – LSE MT 2019/2020

1. The preferences of a consumer are represented by the utility function

$$u(x, y) = xy.$$

Suppose that the income of the consumer equals 12, and that the price of good y equals 3, while the price of good x increases from 1 to 2.

(a) Derive the income and substitution effect for such a change in prices.

(b) Derive the consumer surplus, the compensating variation, and the equivalent variation.

2. Consider a consumer with preferences

$$u(x_1, x_2) = x_1 - 1/x_2.$$

Suppose the price of x_1 is held at 1 and the price of x_2 rises from 1 to 4. Calculate the three welfare measures for $m \geq 2$. In general, which of our measures of welfare will be the largest?

3. Consider a consumer with reference dependent preferences deciding whether to buy a car. His consumption utility for money x_1 and cars x_2 is given by

$$v_1(x_1) = x_1 \quad \text{and} \quad v_2(x_2) = 60x_2.$$

whereas his universal gain loss function satisfies

$$\mu(v) = 3v \quad \text{for} \quad v \geq 0 \quad \text{and} \quad \mu(v) = 5v \quad \text{for} \quad v < 0.$$

Find the personal equilibria of this economy for any possible price p. What is the preferred personal equilibrium for any value of p?

Extra Problem:

1. Sketch the isoquants for the technologies defined by

$$\begin{aligned} \text{(i)} \quad y &= \left(x_1^{1/2} + x_2^{1/2}\right)^2; \\ \text{(ii)} \quad y &= \left(x_1^2 + x_2^2\right)^{1/2}. \end{aligned}$$

Comment on the economic significance of the differences between the technologies.

Problem Set 4: Theory of the Firm Francesco Nava

EC411 – LSE MT 2019/2020

1. Consider the following production function

$$y = x_1^{\alpha} x_2^{\beta}.$$

for $\alpha + \beta < 1$ and $\alpha, \beta > 0$.

   (a) Derive the factor demands $x_1(p, w)$ and $x_2(p, w)$.

   (b) Derive the supply function $y(p, w)$.

   (c) Find all of marginal price effects. Confirm the signs (and, where appropriate, relative magnitudes) that were derived in lectures.

   (d) Find the profit function $\pi(p, w)$. Confirm its properties.

2. Consider the following production function

$$y = x_1^{\alpha} x_2^{\beta}.$$

for any $\alpha, \beta > 0$.

   (a) Derive the conditional factor demands $h_1(w, y)$ and $h_2(w, y)$.

   (b) Find all of the marginal price effects. Confirm the signs (and, where appropriate, relative magnitudes) that were derived in lectures.

   (c) Find the cost function $c(w, y)$. Confirm its properties.

   (d) Sketch $c(w, y)$ as a function of y for each of the three cases:

$$\text{(i) } \alpha + \beta < 1; \;\; \text{(ii) } \alpha + \beta > 1; \;\; \text{(iii) } \alpha + \beta = 1.$$

   Also sketch the marginal and the average costs as functions of y, for each of the three cases. Why is it not necessary to assume $\alpha + \beta < 1$ for cost minimization?

   (e) Confirm that Shephard's Lemma holds.

3. Consider the problem of a beekeeper who sells honey at a price p_h per jar. It costs $\max\{h^2, 1\}$ to produce h jars of honey. Find the supply of honey of the beekeeper. Now, suppose that he learns how to produce wax. In particular, assume that for each jar of honey w grams of wax can be produced at no additional cost. If p_w denotes the price of wax, find the supply of wax of the beekeeper.

Extra Problem:

1. Consider the following production function

$$y = x_1^{\alpha} x_2^{\beta}.$$

for $\alpha, \beta > 0$.

(a) Sketch the production possibility set, Y, in each of the three cases:

$$\text{(i) } \alpha + \beta < 1; \ \text{(ii) } \alpha + \beta > 1; \ \text{(iii) } \alpha + \beta = 1.$$

For each case, draw the marginal product, average product and marginal rate of technical substitution as functions of x_1.

(b) Is the production function homothetic and/or homogeneous? In what arguments? To what degree? What returns to scale are there?

Problem Set 5: Partial Equilibrium Francesco Nava

EC411 – LSE MT 2019/2020

1. Suppose that every firm in the kitchen foil industry has a production function

$$Y = (A+T)^{1/2}L^{1/2},$$

where Y is the amount of foil, A is aluminium, T is tin, and L is labour. The production function of tin and the production function of aluminium exhibit constant returns to scale. In particular, 1 unit of tin can be produced with 1 unit of labour and 1 unit of aluminium can be produced with β units of labour, for some $\beta \in (0,2)$. Firms take prices as given in every market. The supply of labour is perfectly elastic at a wage equal to 4.

   (a) Find the cost function of each firm.

   (b) For all values of β in the interval $(0,2)$, determine how much tin and how much aluminium will be used in equilibrium for the production of one unit of foil.

   (c) For all values of β in the interval $(0,2)$, find the equilibrium price of foil.

2. In a competitive industry there are 200 firms. The cost function for 100 of these firms amounts to

$$c_1(y) = \begin{cases} 10y & \text{if} \quad y \leq 10 \\ y^2 - 10y + 100 & \text{if} \quad y > 10 \end{cases}$$

where y is the amount of output produced by the firm and $c(y)$ is its total cost in pence. The cost function for the remaining 100 firms amounts to

$$c_2(y) = \begin{cases} 12y & \text{if} \quad y \leq 10 \\ y^2 - 8y + 100 & \text{if} \quad y > 10 \end{cases}$$

   (a) Derive and sketch the industry's supply curve.

   (b) If the market demand curve is described by

$$y_d(p) = 2400 - 100p,$$

   where y_d is demand and p is the price in pence, find the price, the total output of the industry, and the outputs of the individual firms in equilibrium.

   (c) If the tax of 1 pence is imposed on sales of the good, what will happen to the price paid by consumers to the price received by firms and to output? What would be the effects of an increase in tax to 2 pence?

(d) Set out an analysis of the costs and benefits of the 2 pence tax.

3. Consider a competitive industry with downward-sloping demand and free-entry. The industry is composed of identical firms with U-shaped average total cost curves. Initially we are in long-run equilibrium.

   (a) The government announces that a substantial fixed annual licence fee will be imposed on all firms choosing to remain in the industry. How does each of the following change, (i) industry output, (ii) output per firm, and (iii) numbers of firms active in the industry.

   (b) Suppose that instead of the licence fee, the government imposes a per unit sales tax. Relative to the original situation with no tax, what is the long-run effect on: (i) output per firm, and (ii) number of firms active in the industry?

Problem Set 6: General Equilibrium Francesco Nava

EC411 – LSE MT 2019/2020

1. Consider a pure exchange economy with two commodities, 1 and 2, and two agents, A and B. Their utility functions are

$$U^A(x^A) = \log(x_1^A) + \log(x_2^A);$$
$$U^B(x^B) = \log(x_1^B) + 2\log(x_2^B).$$

Agent A has an initial endowment of 1 unit of good 1 and 4 units of good 2. Agent B has 6 units of good 1 and 3 units of good 2.

(a) Find the competitive equilibrium price ratio, and the equilibrium allocation of commodities.

(b) Repeat part (a), but assuming that B's preferences satisfy

$$U^B(x^B, x^A) = \log(x_1^B) + 2\log(x_2^B) + 2\log(x_1^A).$$

(c) Is the competitive equilibrium that you found in part (b) Pareto efficient? Prove your answer. What is the economic significance of the final term in B's utility function in part (b) and what effects does it have on the competitive equilibrium and on welfare?

2. Consider an economy with two consumption goods, 1 and 2, and one factor of production, 3, with prices denoted by p_1, p_2, and p_3, respectively. There are two types of agents, A and B. The utility function of a type i agent, for $i \in \{A, B\}$ satisfies

$$U^i(x^i) = \min\left\{x_1^i, x_2^i\right\}$$

where x_j^i denotes i's consumption of good j. Each type A agent owns a firm that produces good 1 using good 3 via a technology characterised by the profit function

$$\pi^A(p) = p_1^2/4p_3.$$

Each type B agent owns a firm that produces good 2 using good 3 via a technology characterised by the profit function

$$\pi^B(p) = p_2^2/p_3.$$

The initial endowments of goods 1 and 2 are equal to 0 and the initial endowment of good 3 is equal to 10 for both types of agents. The proportion of type A agents in this economy is 1/2. All the agents are price-takers.

(a) Find the supply and the input demand functions of types A and B.

(b) Find the competitive equilibrium prices.

3. Consider a production economy with two goods (call them good 1 and 2), two consumers (call them A and B), and one firm (call it F). Consumer A is the sole owner of firm F, and is endowed with 10 units of good 1 and with none of good 2. Consumer B owns only 26 units of good 2. The firm transforms combinations of good 1 and 2 into good 1, and its production function satisfies

$$f\left(x^F\right) = 3\left(x_1^F\right)^{1/3}\left(x_2^F\right)^{1/3}.$$

The preferences of the two consumers respectively satisfy

$$U^A(x^A) = \log(x_1^A) + 3\log(x_2^A);$$
$$U^B(x^B) = x_1^B + 2(x_2^B)^{1/2}.$$

(a) State and solve the profit maximization problem of the firm.

(b) State and solve the utility maximization problem of each consumer.

(c) Find the Walrasian equilibrium.

(d) Consider the economy you have analysed in parts (a), (b) and (c) of this exercise, but suppose that there is no firm and that consumer A is only endowed of 10 units of good 1. Define and characterize the contract curve and the core of this exchange economy.

Extra Problem:

1. Consider an exchange economy with two goods (call them good 1 and 2), two consumers (call them A and B). Consumer A is endowed with 10 units of good 1 and with 5 units of good 2. Consumer B owns only 5 units of good 2. The preferences of the two consumers respectively satisfy

$$U^A(x^A) = \max\{x_1^A, x_2^A\};$$
$$U^B(x^B) = x_1^B + 2(x_2^B)^{1/2}.$$

(a) State and solve the utility maximization problem of each consumer.

(b) Does this economy possess a competitive equilibrium? If yes, find it. If not, prove why.

(c) Define and characterize the contract curve and the core of this economy.

(d) Does the second welfare theorem apply to this setting.

Problem Set 7: Uncertainty Francesco Nava

EC411 – LSE MT 2019/2020

1. A consumer who has an initial wealth of 36 pounds must choose between two options, X and Y. Option X is a lottery which gives 0 with probability $1/4$ and 28 pounds with probability $3/4$; option Y instead is worth 13 pounds with probability equal to 1. The consumer is an expected utility maximizer with a utility function $u(w)$, where w denotes the consumer's wealth in pounds. Utility $u(w)$ is continuous and strictly increasing in w.

   (a) Suppose that the consumer is risk-averse. Can you determine which option she will choose?

   (b) Suppose that someone who knows the outcome of the lottery in option X is willing to sell the information to the consumer. If the consumer is risk-neutral, how much would she be willing to pay to know the outcome of the lottery before making her choice between X and Y?

   (c) Suppose that the consumer is risk-loving and, as in (b), she can buy information about the outcome of the lottery prior to making her choice between X and Y. Will the consumer pay a positive amount to know the outcome of the lottery? Explain carefully.

2. A decision maker is considering how much to invest in two assets. The first asset yields no return, whereas the second asset is risky. The price of the first asset is 1, whereas p denotes the price of one unit of the risky asset. At maturity the a unit of risky asset is worth 1.20 dollars with probability $1/2$, but is worth only 0.90 cents with the complementary probability. The preferences of the decision maker over future monetary holdings x are determined by the function $u(x) = \log(x)$. Derive the demand for the risky asset if the decision maker has M dollars to invest.

3. A farmer can grow wheat, or potatoes, or both. If the weather is good, an acre of land yields a profit of £$2,000$ if devoted to wheat, of £$1,000$ if devoted to potatoes. Should the weather be bad, an acre of wheat yields £$1,000$ and of potatoes £$1,750$. Good and bad weather are equally likely.

   (a) Assuming that the farmer has utility function $\log(m)$ when his income is m, what proportion of his land should he turn over to wheat?

   (b) Suppose the farmer can buy an insurance policy which pays, for every £1 of premium, £2 if the weather is bad, and nothing if the weather is good. How much insurance will he take out and what proportion of his land will he devote to wheat? What would the answers be if the policy paid only £1.50 to compensate for bad weather?

Problem Set 8: Uncertainty & Static Games Francesco Nava

EC411 – LSE MT 2019/2020

1. In a portfolio selection model: let W denote initial wealth; let I denote final wealth; let B denote amount invested in a "risky asset"; let M denote amount of wealth held as money where money is a "sure asset" with a zero return; let r denote the (random) return on the risky asset with $E(r) > 0$ – meaning that when £B are placed in the risky asset, a random amount £$(1 + r)B$ will be realised. Suppose that the investor has a Bernoulli utility function

$$u(I) = \beta I - \alpha I^2,$$

for $\alpha, \beta > 0$.

   (a) Graph this utility function and show that it exhibits risk aversion. Interpret α and β.

   (b) Write out the choice problem which determines the utility-maximising division of initial wealth between the risky asset and money.

   (c) Compute the optimal investment B and prove that it increases with β. Interpret the result.

2. Identify all the Nash Equilibria of the following games, and find dominant strategies:

(i)

$1\backslash 2$	a	b
A	$0,1$	$2,2$
B	$3,0$	$0,3$

(ii)

$1\backslash 2$	a	b
A	$2,1$	$2,2$
B	$3,1$	$1,0$

(iii)

$1\backslash 2$	a	b
A	$0,2$	$2,2$
B	$3,0$	$0,3$

3. Consider the following Matching pennies game

$1\backslash 2$	a	b
A	$1,0$	$0,1$
B	$0,1$	$1,0$

   (a) Show that the game below has no Nash equilibrium in pure strategies.

   (b) Suppose player 1 *tosses a coin* to decide which action to take; i.e. he has probability p of playing A and probability $(1-p)$ of playing B. Similarly, suppose that 2 plays a with probability q and b with probability $(1-q)$. Plot best responses and find values of p and q such that these strategies form a Nash equilibrium.

   (c) Can you see any property which this equilibrium shares with game (iii) of Problem 2?

Extra Problems:

1. Consider an expected utility maximizer facing lotteries, p, over two outcomes, $p = (p_1, p_2)$. Suppose that the player solves the following maximization problem:

$$\max_{a \in A} p_1 u(a + x) + p_2 u(a),$$

where A denotes a compact choice set and $u(\cdot)$ a continuous utility function. Let $U(p)$ denote the value function of this problem. Is $U(p)$ convex in p, is it concave in p, or neither? Hint: Think of the properties of the profit functions and of how we proved those properties.

2. Write down your preferences in the following two situations. Situation 1:

Outcome:	5 M$	1 M$	0 M$
Lottery A:	0%	100%	0%
Lottery B:	10%	85%	5%

Situation 2:

Outcome:	5 M$	1 M$	0 M$
Lottery C:	0%	15%	85%
Lottery D:	10%	0%	90%

Having made your two choices, write down the restrictions which these imply on your utility function of income if they are to be consistent. Are they consistent?

1. Consider an auction with two buyers participating and a single object for sale. Suppose that each buyer knows the values of all the other bidders. Order players so that values decrease, $x_1 > x_2$. Consider a 2$^{\text{nd}}$ price sealed bid auction. In such auction: all players simultaneously submit a bid b_i; the object is awarded to the highest bidder; the winner pays the second highest submitted bid to the auctioneer; the losers pay nothing. Suppose ties are broken in favour of player 1. That is: if $b_1 = b_2$ then 1 is awarded the object.

   (a) Characterize the best response correspondence of each player.

   (b) Characterize all the Nash equilibria for a given profile (x_1, x_2).

   (c) Consider the Nash equilibrium in which both players bid their value – that is, $b_i = x_i$ for $i \in \{1, 2\}$. Is this a dominant strategy equilibrium?

2. Four patients have to undergo surgery and rehabilitation in one of two hospitals. Hospital A specializes in surgery. But its elite surgery unit is small. The likelihood of successful surgery, p_A^S, depends on the number of patients treated in the surgery unit, n, as follows:

$$p_A^S(n) = \begin{cases} 17/20 & \text{if} \quad n = 1 \\ 15/20 & \text{if} \quad n = 2 \\ 11/20 & \text{if} \quad n = 3 \\ 7/20 & \text{if} \quad n = 4 \end{cases}.$$

The rehabilitation unit of hospital A is large, but conventional. The likelihood of successful rehabilitation $p_A^R = 1/2$ is independent of the number of patients treated. Hospital B specializes in rehabilitation. But its elite rehabilitation unit is small. The likelihood of successful rehabilitation, p_B^R, depends on the number of patients treated by the unit, n, as follows:

$$p_B^R(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 16/20 & \text{if} \quad n = 2 \\ 14/20 & \text{if} \quad n = 3 \\ 12/20 & \text{if} \quad n = 4 \end{cases}$$

The surgery unit of hospital B is large, but conventional. The likelihood of successful surgery $p_B^S = 1/2$ is independent of the number of patients treated. Surgery outcomes are independent of rehabilitation outcomes. A patient's payoff is 1 if both treatments are successful, and 0 otherwise. A patient is classified as recovered, only if both treatments are successful. [Hint: Recall that if events A and B are independent $\Pr(A \cap B) = \Pr(A)\Pr(B)$].

(a) First suppose that the existing health regulations require all patients to undergo surgery and rehabilitation at the same hospital. Patients can only choose in which of the two hospital to get both treatments. Set this problem up as a strategic-form game. Find a Nash equilibrium of this game. What is the average probability of recovery among the four patients in this equilibrium?

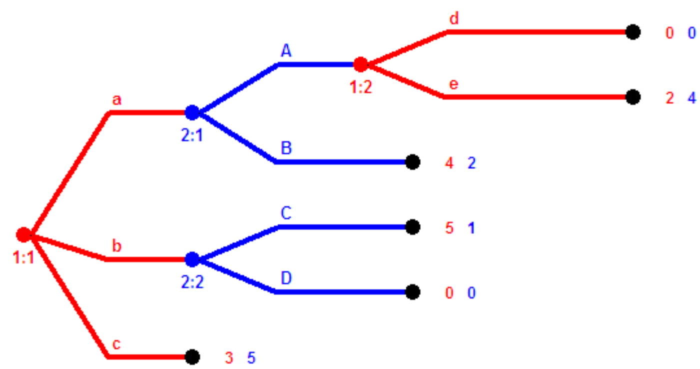
(b) In an attempt to increase the average recovery probability regulators decide to lift the ban on having surgery and rehabilitation at different hospitals. Now patients are free to choose in which hospital to get either treatment. Set this problem up as a strategic-form game. Find a Nash equilibrium of this game. What is the average probability of recovery among the four patients in this equilibrium?

(c) Still unsatisfied about the average recovery probability, regulators decide to try a third policy, in which one of the four patients is randomly selected and sent to the two elite units (surgery in A and rehabilitation in B), while the remaining three are sent to the larger conventional units (surgery in B and rehabilitation in A). What is the average probability of recovery among the four patients with this policy in place? Compare the average probabilities of recovery under the three regulations. Give an intuitive explanation to the observed change in recovery probabilities.

3. Consider three players choosing whether or not to incur a cost c for the provision of a public good. The public good is provided if and only if at least two players incur the cost. If the public good is not provided, contributions are lost. If the good is provided, all players enjoy a benefit v from the public good. Further suppose that the payoff of a player simply amounts to the benefit from the public good, if it is provided, minus the cost incurred, if any. For $c = 3$ and $v = 8$, find all the mixed strategy Nash equilibria of the game in which all players incur the cost with equal probability. Are these equilibria Pareto efficient?

Extra Problems:

1. Consider the following extensive form game:



   (a) Find the unique Subgame Perfect equilibrium of this game.

   (b) Find a pure strategy Nash equilibrium with payoffs $(3, 5)$.

   (c) Find a pure strategy Nash equilibrium with payoffs $(4, 2)$.

2. Consider the following game:

$1\backslash 2$	a	b
A	$6, 4$	$3, 5$
B	$5, 3$	$2, 2$

   (a) In the game shown below, identify the unique Nash Equilibrium in pure strategies.

   (b) Now consider a "sequential" adaptation with the same payoffs as the game below: player 1 first chooses a strategy, and then player 2, knowing 1's choice, chooses his action. What are the pure strategy Nash equilibria in this game? What are the pure strategy subgame perfect equilibria?

1. Consider two firms trying to gain the status of market leader. Becoming market leader is worth v dollars to each competitor. Each firm can decide how much effort to devote to this process. The effort chosen by each firm belongs to the interval $[0, 1]$. Given the effort decisions of the two firms,e_A and e_B, the likelihood of Firm A becoming market leader is given by $e_A(1 - e_B)$, whereas the likelihood of Firm B becoming leader is given by the complementary probability. Effort is costly, and the monetary cost of effort depends on the resources devoted by both competitors. In particular, for both firms the effort cost coincides with $ke_A^2 e_B$. Suppose that $v < k$.

   (a) Derive the best responses for each of the two firms in this game. Plot them and find all the pure strategy Nash equilibria of this game.

   (b) Prior to the beginning of the game Firm A can choose how much to invest in developing the market. In particular, suppose that Firm A can set v to any number in the interval $[0, k]$ at a cost $c(v) = v^3/3k$. What investment would the firm choose in a Subgame Perfect equilibrium, if both firms exert effort at the competition stage?

2. Consider the following complete information extensive form game played by one seller and four buyers. The sellers owns four identical objects. There are two periods in which objects can be traded. In every period $t \in \{1, 2\}$, the seller moves first and sets a price p_t. Upon observing this price, buyers then decide whether to purchase or not one of the objects in that period. Each buyer wishes to purchase a single object and leaves the market upon purchasing an object. Let v_i denote the monetary value of the object to Buyer i, and suppose that

$$v_1 = 1,\ v_2 = 3,\ v_3 = 5,\ v_4 = 7.$$

   If so, the utility that Buyer i derives from purchasing the object at date $t \in \{1, 2\}$ at price p_t simply amounts to

$$\delta^{t-1}(v_i - p_t).$$

   Let n_t denotes the number of objects sold at date $t \in \{1, 2\}$. If so, the payoff of the seller simply amounts to

$$n_1 p_1 + \delta n_2 p_2.$$

   Throughout, assume that $\delta \in (0, 1)$.

   (a) Let A denote the set of active buyers, namely those who have not purchased an object at date 1.

Let $t = 2$ and consider the following four possible scenarios: $A = \{1\}$, $A = \{1, 2\}$, $A = \{1, 2, 3\}$, and $A = \{1, 2, 3, 4\}$. Find the Subgame Perfect equilibrium price p_2 set by the seller at date $t = 2$ and the demand of every buyer in each of the four possible scenarios. What is the equilibrium payoff of the seller in each of these subgames?

(b) Given the equilibrium behavior outlined in part (a), find the highest price p_1 at which only Buyer 4 purchases the good in the Subgame Perfect equilibrium. Show that at such a price no other buyer is willing to buy at date 1 in the SPE.

(c) Find the highest price p_1 at which only Buyers 3 and 4 purchase the good in the Subgame Perfect equilibrium. Show that at such a price only Buyers 1 and 2 are not willing to buy at date 1 in the SPE.

(d) Find the highest price p_1 at which only Buyer 1 does not purchase the good in the Subgame Perfect equilibrium. Show that at such a price Buyer 1 is not willing to buy at date 1 in the SPE.

(e) Use the observations in parts (a-d) to find the Subgame Perfect equilibrium prices, demands and payoffs for all possible values of δ.

(f) If the seller were able to set at date 1 both prices p_1 and p_2, would he be able to increase his payoff? If so, can you argue why?

3. Consider the following asymmetric Prisoner's Dilemma:

$1\backslash 2$	C	D
C	$3, 4$	$1, 6$
D	$4, 0$	$2, 2$

(a) Find the minmax values of this game. Consider the infinitely repeated version of this game in which all players discount the future at the same rate δ. The following is a "tit for tat" strategy: any player chooses C provided that the other player never chose D; if at any round t a player chooses D, then the other player chooses D in round $t + 1$ and continues playing D until the player who first chose D reverts to C; if at any round t a player chooses C, then the other player chooses C in round $t + 1$. Write this strategy explicitly.

(b) Find the unique value for the common discount factor δ for which the strategy of part (a) sustains always playing C as a SPE of the infinitely repeated game.

(c) Then, consider the following "trigger" strategy: any player chooses C provided that no player ever played D; otherwise any player chooses D. Write the two incentive constraints that would have to be satisfied for such a strategy to be a NE. Then, write the two additional incentive constraints that would have to be satisfied for such a strategy to be a SPE. What is the lowest discount rate for which such strategy satisfies all the constraints.

Extra Problem:

1. Consider the following game:

$1\backslash 2$	L	R
T	$0, 0$	$4, 1$
D	$1, 4$	$3, 3$

Find all the Nash equilibria of this static game. Next consider the infinite repetition of the game, and a strategy prescribing to play (D, R) so long as no player has ever deviated and to play (T, L) otherwise. Is this strategy a Nash equilibrium for the infinite repetition? Is it a Subgame Perfect equilibrium?

Hand-In Homework 1 Francesco Nava

EC411 – LSE MT 2019/2020

Please submit your homework to your class teacher by Monday week 7.

Question 1 (35 Marks)

Consider a consumer whose utility function is

$$u(x_1, x_2) = -\exp(-ax_1x_2)$$

Can you take first order conditions to solve the utility maximization problem? Explain your argument. Next solve the utility maximization problem, and derive Marshallian demands and the indirect utility function. Given your calculations, state and use the duality theorem to find the expenditure function and Hicksian demands.

Question 2 (40 Marks)

A price-taking firm has a production function given by

$$y = 3(x_3)^{1/3} (\max\{x_1, 8x_2\})^{1/3}$$

where x_1, x_2 and x_3 are inputs and y is output. Let w_1, w_2 and w_3 denote input prices, and let p denote the output price. Solve the profit maximization problem and the corresponding cost minimization problem. While solving these problems motivate your approach.

Question 3 (25 Marks)

Consider a simple exchange economy with two goods and two consumers. The endowments of the two consumers satisfy respectively $\omega^1 = (5, 1)$ and $\omega^2 = (1, 9)$, while their preferences satisfy:

$$U^1(x^1) = \min\{2x_1^1, x_2^1 + 2\} \text{ and } U^2(x^2) = 2\sqrt{x_1^2 x_2^2}.$$

Find Marshallian demands for each of the two consumers. Explain your approach. Solve for the Walrasian equilibrium of this economy.

Hand-In Homework 2 Francesco Nava

EC411 – LSE MT 2019/2020

Please submit your homework to your class teacher before the end of week 10.

Question 1 (40 Marks)

Consider an exchange economy with two goods (call them good 1 and 2) and two consumers (call them A and B). The preferences of the two consumers respectively satisfy:

$$u^A(x^A) = -\max\{|x_1^A - 5|, |x_2^A - 5|\},$$
$$u^B(x^B) = 2\log x_1^B + \log x_2^B.$$

Consumer A is endowed with $k \in (0, 10)$ units of good 1 and with k units of good 2. Consumer B owns only $20 - k$ units of good 1 and with $10 - k$ units of good 2.

1. State and solve the utility maximization problem of each consumer. (10 Marks)
2. Find the competitive equilibria of this economy for all $k \in (0, 10)$. (12 Marks)
3. Define and characterize the contract curve and the core of this economy. (12 Marks)
4. State the first welfare theorem. Then, exploit observations in parts (b) and (c) to find a violation of the first welfare theorem. (6 Marks)

Question 2 (10 Marks)

Bob is considering whether to buy health insurance. If Bob is healthy, his wealth amounts to W. However if he falls ill, it only amounts to $V < W$. The likelihood of eventually falling ill is 1/3. Bob's preferences depend only on realized wealth x, and his utility function is determined by

$$u(x) = \log(x).$$

Assume that $W = 30$, that $V = 10$, and that the unit-price of insurance coverage is equal to $r \in [1/3, \infty)$. Find the expected value of purchasing S units of coverage. Find Bob's expected utility from purchasing S units of coverage. Find Bob's demand for coverage as a function of r.

Question 3 (20 Marks)

Consider a game played by two lobbyists 1 and 2 sponsoring two distinct versions of a bill. The value of having version $i \in \{1, 2\}$ approved equals v_i to lobbyist i and equals 0 to lobbyist j, for some $v_i > 0$. Lobbyists choose how many resources, $r_i \in [0, \infty)$, to invest to sway the parliament to support their preferred version. The probability that the policy preferred by lobbyist i is approved is given by the following function of the resources invested by the two lobbyists

$$p_i(r_i, r_j) = \begin{cases} 1/2 & \text{if } \; r_i = r_j \\ r_i/(r_i + r_j) & \text{if } \; r_i \neq r_j \end{cases}.$$

Resources are however costly and the payoff of lobbyist i amounts to

$$u_i(r_i, r_j) = p_i(r_i, r_j)v_i - r_i r_j$$

The payoff is concave and single peaked in r_i for any value r_j. Find the best response for each of the two lobbyists. Find the Nash equilibria of this game. Is there always an efficient equilibrium?

Question 4 (30 Marks)

Suppose that two players have to split a cake of size 10 according to an "I cut, you choose" protocol. In particular, assume that Player 1 cuts the cake into two parts and that Player 2 gets to choose which of the two slices to consume (while the other slice is consumed by Player 1). Suppose that the cake is continuously divisible.

1. First, assume that the cake is homogeneous, so that both players value all its parts alike, and care only about the size of the slice that they get to consume. Set this problem up as an extensive form game. Find the Subgame Perfect equilibria of this game. Write the behavioural strategy for both players, and check that no deviation is profitable. (11 Marks)

2. Now suppose that 3/5 of the cake is chocolate flavoured, while 2/5 of the cake is vanilla flavoured. Player 1 only likes chocolate, and only cares about the amount of chocolate in the slice he consumes; while 2 likes all the parts of the cake alike, and cares only about the size of the slice consumed. Also, assume that Player 1 can cut slices with any composition of chocolate and vanilla. Find a Subgame Perfect equilibrium of this game. Write the behavioural strategy for both players, and check that no deviation is profitable. (11 Marks)

3. Does the game in part (a) possess a pure-strategy Nash equilibrium in which a player receives a slice different in size from the one that you have characterized in part (a)? Explain. (8 Marks)