Micro Theory Reading Group

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18 February 2011

Outline

1. **Today's Papers**
   - Relevance to Topic
   - Methodology Differences
2. **Takahashi - Community Enforcement with 1st Order Info**
   - Summary
   - Model
   - Equilibria
   - Discussion & Extensions
3. **Wolitzky - Repeated Public Good Provision**
   - Summary
   - Model
   - Results
   - Comparative Statics
   - Discussion & Extensions

Community Enforcement

- Repeated games ⇒ cooperation possible.
- When game played over a population, could extend "folk theorem" results.
- Monitoring is crucial.
- In large populations, public monitoring might be infeasible/impossible.
- Could "community" provide correct incentives?

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What They Ask and How They Ask

- Both papers use "Prisoner's Dilemma" situation stage games.
- In Takahashi, concentrate on equilibria construction.
  - Provide necessary & sufficient conditions (in terms of δ).
  - Check robustness and analyze properties of equilibria.
- In Wolitzky, however, look at equilibrium outcomes.
  - Which strategy within set of equilibria provide "maximal" outcomes?
  - How do these maximal outcomes change in parameters of game?

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- Repeated prisoner's dilemma played over matched pairs in a community.
- Results for continuum of players. (finite population as an extension)
- No network structure. (random matching)
- *Personal* vs. *community enforcement* (small vs. large community)
- *First-order information*: record of current partner's past play.
- Is first-order info is enough for cooperation in large community?

e.g. Consumer credit histories or online feedback (such as eBay reviews).

- Why no higher order info?
  1. Costly to store/transmit
  2. Cognitively more demanding to process

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Main Findings

	C	D
C	$1, 1$	$-l, 1+g$
D	$1+g, -l$	$0, 0$

$g, l > 0$

- Consider two classes of equilibria to sustain cooperation:

1. Strict Equilibria (grim-trigger strategy)
   - When strictly supermodular ($g < l$) then strict equil. sustains cooperation.
   - When submodular ($g \geq l$) then only D, D forever is strict.
2. Independent and Indifferent Equilibria (IIE)
   - Players choose actions independently of own records of play.
   - Players are indifferent between C and D at all histories.
   - With this equil. notion we can sustain cooperation for any $g, l > 0$.

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   - IIE notion from Piccione (2002) and Ely & Välimäki (2002).
   - Ely et. al (2005) extend construction to general games; *belief-free equil.*
   - IIE in repeated games with random matching $\overset{?}{\leftrightarrow}$ belief-free equil.
2. Repeated games with random matching:
   - Public monitoring: Kandori (1992), Dal Bó (2007)
   - Private monitoring: Ellison (1994), Deb (2008)

N.B. Common feature: $\underline{\delta}$ depends on n and $\lim_{n\to\infty} \underline{\delta} = 1$

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Setup and Definitions

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$g, l > 0 \quad A = \{C, D\}$

- Continuum of population $[0, 1]$.
- Matching $m : [0, 1] \rightarrow [0, 1]$ with $\forall i \in [0, 1]$, $m(m(i)) = i \neq m(i)$.
- m_t drawn "uniformly" and independently across time.
- i's total payoff is $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u(a_{it}, a_{m_t(i)}, t)$.
- $\delta \in (0, 1)$ is common discount factor.

Setup and Definitions

- Let $a_i^t = (a_{i1}, a_{i2}, \ldots, a_{it})$ be sequence of i's actions upto t.
- "*History repository*" **honestly** keeps track of all players' actions over time.
- At t after m_t is realized each $i \in [0, 1]$ has 3-fold information:
  1. She knows $(a_{is}, a_{m_s(i)}, s)$ for $s \leq t - 1$.
  2. She observes $a_{m_t(i)}^{t-1}$ from history repository for **free**.
  3. She knows $a_{m_s(i)}^s$ for all $s \leq t - 1$ where:
     - $a_{m_s(i),s}$ by own observation,
     - $a_{m_s(i)}^{s-1}$ from repository back in period s.

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Setup and Definitions

- WLOG assume that i uses only a_i^{t-1} and $a_{m_t(i)}^{t-1}$ for choosing action at t.
- $a_{m_s(i)}^s$ for $s \leq t-1$ not used due to indep. of $m_{t'}(i)$'s strategies for $t' \geq t$.
- For simplicity assume strategies are ex ante symmetric.
- Behavior strategy $\sigma_t : A^{2t-2} \to \Delta(A)$ where $\sigma_t(a^{t-1}, \bar{a}^{t-1})$.
- a^{t-1} is own record of play, $\bar{a}^{t-1}$ is current partner's record of play.

Setup and Definitions

- For $t \geq 1$ and $a^t \in A^t$, let $\mu_t(a^t)$ be fraction of players with records a^t.
- Given strategy σ, sequence of distributions of records $\mu = \{\mu_t\}$ is:

$$\mu_1(a_1) = \sigma_1(\emptyset)(a_1),$$
$$\mu_t(a^t) = \mu_{t-1}(a^{t-1}) \sum_{\bar{a}^{t-1} \in A^{t-1}} \mu_{t-1}(\bar{a}^{t-1})\sigma_t(a^{t-1}, \bar{a}^{t-1})(a_t)$$

- Along equilibrium path, players believe distribution of records of play equal μ_t with certainty.
- Assume same beliefs for off-equilibrium paths, due to “trembling hands”.

Setup and Definitions

- Continuation payoff for a player following σ when all other follow $\bar{\sigma}$ is:

$$U_t(\sigma, \bar{\sigma} | a^{t-1}, \bar{a}^{t-1}, \mu) = \sum_{a_t \in A} \sigma_t(a^{t-1}, \bar{a}^{t-1})(a_t) \Big((1-\delta) u(a_t, \bar{\sigma}_t(\bar{a}^{t-1}, a^{t-1})) + \delta \sum_{\bar{b}^t \in A^t} U_{t+1}(\sigma, \bar{\sigma} | a^t, \bar{b}^t, \mu) \mu_t(\bar{b}^t) \Big)$$

- By one-shot deviation principle, σ^* is equilibrium if $\forall t \geq 1$, every $a^{t-1}, \bar{a}^{t-1} \in A^{t-1}$ and every σ it holds that:

$$U_t(\sigma^*, \sigma^* | a^{t-1}, \bar{a}^{t-1}, \mu^*) \geq (1-\delta) u(a_t, \sigma_t^*(\bar{a}^{t-1}, a^{t-1})) + \delta \sum_{\bar{b}^t \in A^t} U_{t+1}(\sigma^*, \sigma^* | a^t, \bar{b}^t, \mu^*) \mu_t^*(\bar{b}^t) \tag{1}$$

Strict Equilibria

Definition 1

Equilibrium σ^* is **strict** if, at any history, each player strictly prefers the action prescribed by equilibrium to one-shot deviation; i.e. (1) holds with strict inequality whenever $a_t \neq \sigma_t^*(a^{t-1}, \bar{a}^{t-1})$.

Pairwise grim-trigger strat: $\sigma_t(a^{t-1}, \bar{a}^{t-1}) = \begin{cases} C & \text{if } a^{t-1} = \bar{a}^{t-1} = (C, \dots, C) \\ D & \text{otherwise} \end{cases}$

Strict Equilibria

Lemma

Pairwise grim-trigger strategy is a strict equilibrium iff $\frac{g}{(1+g)} < \delta < \frac{l}{(1+l)}$.

- If i and $m_t(i)$ have played C only, then PGTS prescribes C to both.
- In order to do so, need δ to be sufficiently large. (lower bound)
- If i played C only, but $m_t(i)$ played D in past, PGTS prescribes D to both.
- For i to do so, need δ to not be too high. (upper bound)

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Strict Equilibria

Proposition

1. If $g < l$ and $\delta > \frac{g(1+l)}{(1+g)l}$, then there exists a strict equil. w/ sym. payoff 1.
2. If $g \geq l$, then there is no strict equil. other than repetition of D.

- To show existence of strict equilibrium in 1st part, use Ellison trick.
- Divide game into subgames where in each use earlier lemma.
- For 2nd part, pursue contradiction by assuming a nontrivial strict equil.
- Conclude that $l > g$ has to hold contradicting with $g \geq l$ (submodularity).

Independent and Indifferent Equilibria (IIE)

Definition 2

σ^* satisfies:

- *Independence of own play* if $\sigma_t^*(a^{t-1}, \bar{a}^{t-1}) = \sigma_t^*(b^{t-1}, \bar{a}^{t-1})$ for all $t \geq 1$ and $a^{t-1}, b^{t-1}, \bar{a}^{t-1} \in A^{t-1}$
- *Indifference at all histories* if (1) holds with equality for all $t \geq 1$, $a^{t-1}, \bar{a}^{t-1} \in A^{t-1}$ and $a_t \in A$.

σ^* is IIE if σ^* satisfies both independence and indifference.

- By independence $U_t(\sigma^*, \sigma^* | a^{t-1}, \bar{a}^{t-1}, \mu)$ is independent of $\bar{a}^{t-1}$ and μ as $m_t(i)$ does not care about $\bar{a}^{t-1}$.
- But same reason implies i has no strict incentives to take $\bar{a}^{t-1}$ into consideration.
- For C to be played, need indifference between C and D at some histories.
- Author requires indifference at *all* histories for simplicity.

Independent and Indifferent Equilibria (IIE)

Proposition

Suppose that $\delta \geq max(\frac{g}{(1+g)}, \frac{l}{(1+l)})$. Then there is an IIE with symmetric payoff x iff $x \in [0, 1]$.

- If $g - l \leq 1$ then $[0, 1]$ is set of feasible payoffs under sym. strategies.
- Thus concentrating on IIE is WLOG then.
- If $g - l > 1$, however, $\exists$ other equil. alternating between (C, D) and (D, C) to sustain $x > 1$ in equil.
- So "only if" part applies exclusively to IIE when $g - l > 1$!

Independent and Indifferent Equilibria (IIE)

For any $x \in [0, 1]$, construct IIE by following algorithm:

- At t, for record a^{t-1} assign "target payoff" $V_t(a^{t-1})$. [Set $V_1(\emptyset) = x$]
- Given $m_t(i)$'s record $\bar{a}^{t-1}$, i chooses C w/ prob $p_t(\bar{a}^{t-1})$.
- $V_{t+1}(a^t)$ is computed recursively from $V_t(a^{t-1})$ and a_t.

To implement above use indifference condition:

$$V_t(a^{t-1}) \overbrace{=}^{\text{indif.}} (1-\delta)u(C, p_t(a^{t-1})) + \delta V_{t+1}(a^{t-1}, C) \qquad \text{if play } C$$
$$\underbrace{=}_{\text{indif.}} (1-\delta)u(D, p_t(a^{t-1})) + \delta V_{t+1}(a^{t-1}, D) \qquad \text{if play } D$$

Note above has 3 unknowns in 2 equations!

Independent and Indifferent Equilibria (IIE)

For any $x \in [0,1]$, construct IIE by following algorithm:

- At t, for record a^{t-1} assign "target payoff" $V_t(a^{t-1})$. [Set $V_1(\emptyset) = x$]
- Given $m_t(i)$'s record $\bar{a}^{t-1}$, i chooses C w/ prob $p_t(\bar{a}^{t-1})$.
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Note above has 3 unknowns in 2 equations!

Independent and Indifferent Equilibria (IIE)

Lemma

If $\delta \geq max(\frac{g}{(1+g)}, \frac{l}{(1+l)})$*, then* $\forall t \geq 1$*, every* $a^{t-1} \in A^{t-1}$ *and every* $V_t(a^{t-1}) \in [0,1]$*,* $\exists p_t(a^{t-1}) \in [0,1]$ *and* $V_{t+1}(a^{t-1}, C), V_{t+1}(a^{t-1}, D) \in [0,1]$ *s.t. algorithm construction possible.*

$$p_t(a^{t-1}) = V_t(a^{t-1})$$

$$V_{t+1}(a^{t-1}, C) = \frac{V_t(a^{t-1})}{\delta} - \frac{1-\delta}{\delta}u(C, p_t(a^{t-1}))$$

$$V_{t+1}(a^{t-1}, D) = \frac{V_t(a^{t-1})}{\delta} - \frac{1-\delta}{\delta}u(D, p_t(a^{t-1}))$$

IIE vs. Belief-free Equil

Definition 3

$(\tilde{\sigma}_1^*, \tilde{\sigma}_2^*)$ a *belief-free equil.* of 2-player repeated prisoner's dilemma with perfect monitoring if *i*'s continuation strategy $\sigma_i^*|(a_i^{t-1}, a_j^{t-1})$ is a BR to *j*'s continuation strategy $\sigma_j^*|(b_j^{t-1}, b_i^{t-1})$ for all $i \in \{1, 2\}$, $j \neq i$, $t \geq 1$ and $a_i^{t-1}, a_j^{t-1}, b_i^{t-1}, b_j^{t-1} \in A^{t-1}$.

IIE vs. Belief-free Equil

- Repository also stores second-order info: $(a_{m_s(m_t(i)),s})_{s=1}^{t-1}$
- Then $\hat{\sigma}_t(a^{t-1}, b^{t-1}, \bar{a}^{t-1}, \bar{b}^{t-1})$ is mixed action at t.
- a^{t-1} is own record of play.
- b^{t-1} past partners' play.
- $\bar{a}^{t-1}$ current partner's record.
- $\bar{b}^{t-1}$ current partner's past partners' play.
- $\hat{\sigma}$ satisfies independence if $\hat{\sigma}_t(a^{t-1}, b^{t-1}, \bar{a}^{t-1}, \bar{b}^{t-1})$ is independent of (a^{t-1}, b^{t-1}).

IIE vs. Belief-free Equil

Proposition

$(\tilde{\sigma}^*, \tilde{\sigma}^*)$ is a sym. belief-free equil. in 2-player repeated prisoner's dilemma w/ perfect monitoring iff $\hat{\sigma}^*$ is a continuum-population equil. w/ independence of own observations in the random matching repeated prisoner's dilemma w/ info up to 2nd order where $\forall t \geq 1$ and $a^{t-1}, b^{t-1}, \bar{a}^{t-1}, \bar{b}^{t-1} \in A^{t-1}$:

$$\hat{\sigma}^*_t(a^{t-1}, b^{t-1}, \bar{a}^{t-1}, \bar{b}^{t-1}) = \tilde{\sigma}^*(\bar{b}^{t-1}, \bar{a}^{t-1})$$

- Note above relationship holds when we have 2nd order info.
- For equivalence with only 1st order info, need strategies to have independence of own play in 2-player case
- Piccione maintains above $\Rightarrow$ all equil. in Piccione translated to Takahashi.

Linear IIE, Long-Run Stability

- In IIE algorithm shown, $p_t(a^{t-1})$ is linear in $V_t(a^{t-1})$.
- Linear IIE has nice qualitative properties.
- Fix an equil. with C forever and strategy σ^*.
- Ask if (small) positive mass mistakenly deviate, will it ruin cooperation?
- Add a shock at end of T ($\mu_T \neq \mu_T^*$) and from $T+1$, back to σ^*.

$$P_t = \sum_{a^{t-1}, \bar{a}^{t-1} \in A^{t-1}} \mu_{t-1}(a^{t-1}) \mu_{t-1}(\bar{a}^{t-1}) \sigma_t^*(a^{t-1}, \bar{a}^{t-1})(C)$$

- If $P_t \to 1$ as $t \to \infty$ then σ^* sustains cooperation in long-run.
- Letting σ^* be IIE w/ sym. payoff $x \in (0, 1]$:
    1. If $g < l$, then $P_t \to 1$ as $t \to \infty$
    2. If $g > l$, then $P_t \to 0$ as $t \to \infty$
    3. If $g = l$, then P_t is constant over t

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Finite Population

Proposition

If a continuum-population equil. strategy satisfies independence of own play, then strategy combined w/ any consistent belief system forms a sequential equil. of finite-population model of any size.

Table 1
Discount factor δ sufficient to sustain cooperation: $g = 1$.

Popul. size	Contagious eqm	Indep. & indiff. eqm		
	$l > 0$	$0 < l \leqslant 1$	$l = 2$	$l = 10$
2	0.50	0.50	0.67	0.91
4	0.68	0.50	0.67	0.91
10	0.79	0.50	0.67	0.91
100	0.89	0.50	0.67	0.91
1000	0.93	0.50	0.67	0.91

The column for contagious equilibria is taken from Ellison [8, Table 1].

- Matchings need not be uniform.
- $\underline{\delta}$ is independent of n.
- First-order info helpful if n large and/or l (relative to g) is small.

Noise, Bounded Records

- Noise could be in actions and/or in records. (due to mistakes etc.)
- Set of IIE payoffs changes continuously w.r.t. noise levels.
- What happens if strategy uses a bounded period length of records?
    1. If $g \neq l$, then $\nexists$ IIE w/ bounded records.
    2. $g = l$ and $\delta \geq \frac{g}{(1+g)}$, then for any $x \in [0, 1]$, $\exists$ IIE that has records of length 1 w/ sym. payoff x.
    3. If $g < l$ and $\delta \geq \frac{g(1+l)}{(1+g)l}$, then $\exists$ strict equil. that has records of length T w/ sym. payoff 1 where T satisfies $\delta^T \leq \frac{l}{(1+l)}$.
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Endogenous Asymmetry, Cheap Talk

- Suppose $g - l > 1$. For any $x \in [0, \frac{1+g-l}{2})$, $\exists\, \underline{\delta} < 1$ s.t. for any $\delta > \underline{\delta}$ $\exists$ equil. w/ sym. payoff x.
- To get above, construct an equil. alternating between (C, D) and (D, C).
- What happens if we allow cheap talk?
  1. If $g \geq l$, then only strict equil. is repetition of D, independent of messages.
  2. If $g \neq l$, then $\nexists$ IIE w/ bounded records, even if cheap talk allowed.

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What is That About?

- *N*-player repeated public good provision game w/ community enforcement.
- Stage game is like prisoner's dilemma. (0 contribution dominant)
- "All-or-nothing" monitoring. (not imperfect)
- Given common δ define maximum equil. level of public good (MELP).
- What strategies support MELP?
- How does MELP change (comparative statics exercise) in:
  - Type of public good (pure vs. divisible)
  - Group sizes
  - Monitoring technologies

e.g. Construction of infrastructure projects (repeated) in a village where each villager observes only contributions of her "neighbors".

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"All-or-nothing" Monitoring

- General representation of many monitoring scenarios.
- Also provides super tractability. (Characterize MELP for any $\delta < 1$)
- At all *t*, player *i* either perfectly observes *j*'s actions or not.

N.B. Not imperfect monitoring!

- Examples are:
  1. Uniform monitoring
  2. Quasi-public monitoring
  3. Random matching
  4. Arbitrary fixed network

Main Results

- MELP is sustained in “grim-trigger” strategies, σ^*.
- In particular reward schemes are not better.
- Symmetic σ^* under weak symmetry of monitoring. (“equal observability”)
- Under equal observability, incentives to contribute depend only on:
  1. Effective contagiousness: $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\# D(t, \Gamma)]$
  2. Rivalness of public good: $\alpha(\Gamma)$
- Comparative statics exercise by changing:
  - Group size (N) / monitoring structure (1st effect)
  - Public good type (2nd effect).
- Drop equal observability and assume fixed network.
- Contributions change in “centrality” of players and structure of network.
- Extend to local public goods and revisit results from equal obs.

Definitions

- N players every period simultaneously choose $x_i \geq 0$.
- $\alpha \sum_{j=1}^{N} x_j - c(x_i)$ where $\alpha \in (0, 1]$ is common benefit.
  - $c'(\cdot) > 0$, $c''(\cdot) > 0$
  - $c(0) = 0$, $c'(0) \in (\alpha, \alpha N)$
  - $\lim_{x \to \infty} c'(x) > \alpha N$.
- By above, one-shot game like prisoner's dilemma. ($x_i = 0$ is dominant)
- Common δ for all.

Definitions

- Define rand. var. $O(i, t)$ s.t. if $j \in O(i, t)$, then j monitors i at t.
- From realizations $O(i, t)$ for all i, create monitoring list at t, denoted $h_{j,t}$.
- $h_i^t \equiv (h_{i,0}, h_{i,1}, \ldots, h_{i,t-1})$ is i's history at t.
- Strategy $\sigma_i(h_i^t)$. (monitoring structure details captured in h_i^t)
- Define $D(\tau, t, i)$ to be set of players in period τ who have observed a player who observed a player who has observed... player i since time t.
- By assumed regularities, $D(\tau, t, i) = D(\tau - t, 0, i)$ for all i,t and τ.
- $D(\tau, i) = D(\tau, 0, i)$ is set of players who may learn about deviation within τ periods.
- Equal observability: $\mathbb{E}[\#D(\tau, j)] = \mathbb{E}[\#D(\tau, k)]$ for all j, k and τ.

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Definitions

- Consider set of sequential equilibria, Σ_{SE}.

Definition 1

MELP is:

$$X^* \equiv \sup_{\sigma \in \Sigma_{SE}} \alpha(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t \sum_{i=1}^{N} \sigma_i(h_i^t)\right]$$

Definition 2

Maximum equilibrium contribution of an individual player is:

$$\hat{x}_i \equiv \sup_{\sigma \in \Sigma_{SE}} (1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t \sigma_i(h_i^t)\right]$$

Definitions

Definition 3

σ is a *grim trigger strategy profile* if there are contributions $\{x_i^*\}_{i=1}^N$ s.t.

- $\sigma_i(h_i^t) = 0$ if i has ever observed j choose $x_j \neq x_j^*$ at h_i^t
- $\sigma_i(h_i^t) = x_i^*$ otherwise.

N.B. σ is symmetric grim trigger profile if $x_i^* = x^*$ for all i.

MELP Sustained in Grim-Trigger Strategies

Theorem 1

- There exists a unique grim-trigger profile, σ^* that sustains MELP.
- Any other equil. sustaining MELP has same equil. path w/ σ^*.
- σ^* also maxes x_i^*, so $x_i^* = \hat{x}_i$ and $X^* = \alpha \sum_i x_i^*$.
- $\forall i \in N$, condition that pins down x_i^*'s is:

$$\underbrace{c(x_i^*)}_{\text{Cost}} = \underbrace{\alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t \sum_{i=1}^{N} \mathbb{P}(j \in D(t,i))x_j^*}_{\text{Benefit}}$$

Symmetric Grim-Trigger Strategies to Sustain MELP

Theorem 2

- Under equal observability, $\exists$ a unique symmetric σ^* that sustains X^*.
- $X^* = \alpha N x^*$ where $x^* = \hat{x}_i$ for all i.
- If no equal observability, then set of δ's in $[0, 1]$ for which $X^* > 0$ and grim trigger σ^* to maintain X^* is symmetric has measure 0.

- Without equal observability, no symmetric equilibrium, except trivial one ($\hat{x}_i = 0$ for all i).

Assumption for Positive Contributions

$$\alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t\sum_{j=1}^{N}\mathbb{P}(j\in D(t,i)) > c'(0)$$

- Under above assumption, $\hat{x}_i > 0$.
- This allows us to make *strict* statements in comparative statics analysis.

Comparative Statics Under Equal Observability

- Γ satisfies equal observability.
- So symmetric grim-trigger strategy σ^* to sustain X^* exists.
- Let $\mathbb{E}[\#D(t,\Gamma)] \equiv \mathbb{E}[\#D(t,i,\Gamma)] = \mathbb{E}[\#D(t,j,\Gamma)]$ by equal obs.
- Then max per capita level of public good provision (also $\hat{x}_i$) is:

$$\alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]x - c(x) = 0$$

- Above expression on LHS is concave in x.
- So if $x^* > 0$ then $x^*(\Gamma) > x^*(\Gamma')$ if $\forall x$:

$$\alpha(\Gamma')(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma')]x - c(x) >$$

$$\alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]x - c(x)$$

Comparative Statics Under Equal Observability

- Then main result is:

Theorem 3

Given Γ' and Γ, two games, $x^*(\Gamma') > x^*(\Gamma)$ iff

$$\alpha(\Gamma') \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t,\Gamma')] > \alpha(\Gamma) \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t,\Gamma)]$$

- Result depends on two terms:
  1. "Rivalness" term, $\alpha(\Gamma)$.
  2. "Effective Contagiousness" term, $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t,\Gamma)]$.

Comparative Statics Under Equal Observability

- Assume game can be indexed by group size N:
  - $\alpha(\Gamma) \equiv \alpha(N)$
  - $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t,\Gamma)] \equiv \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t,N)]$
- Now for given δ can do comparative statics of MELP in N.
- Can also ask what is optimal N^* to max MELP?

Corollary

If public good is pure ($\alpha(N) = 1$), then $x^*(N)$ is strictly increasing when:

$$\forall t \qquad \frac{\partial \mathbb{E}[\#D(t,N)]}{\partial N} \geq 0 \qquad (>) \text{ for some } t.$$

Corollary

If public good is divisible ($\alpha(N) = 1/N$), then $x^*(N)$ is strictly increasing when:

$$\forall t \qquad \frac{\partial [\mathbb{E}[\#D(t,N)]/N]}{\partial N} \geq 0 \qquad (>) \text{ for some } t.$$

Group Size Exercises Under Equal Observability

- Perform comparative statics for different configurations:

Different All-or-nothing Configurations

1. **Uniform monitoring:** $\exists p \in (0,1]$ s.t. $j \in O(i,t)$ w/ p ind. across i, j, t.
2. **Quasi-public monitoring:** $\exists p(N) \in (0,1]$ s.t. $j \in O(i,t)$ for all j w/ $p(N)$ ind. across i, t. (Observe all players or no one)
3. **Random matching:** $\forall t$, all players randomly paired & $j \in O(i,t)$ iff i, j paired. (Kandori, Ellison Takahashi etc. setup w/ global benefits)
4. **Monitoring on a Circle:** All on a fixed circle and $\exists k \geq 1$ s.t. $j \in O(i,t)$ iff dist. between i, j is $\leq k$. (Equally observable fixed networks)

Different Public Good Types

1. **Pure Public Good:** $\alpha(N) = 1$
2. **Divisible Public Good:** $\alpha(N) = 1/N$

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Different All-or-nothing Configurations

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2. **Quasi-public monitoring:** $\exists p(N) \in (0, 1]$ s.t. $j \in O(i, t)$ for all j w/ $p(N)$ ind. across i, t. (Observe all players or no one)
3. **Random matching:** $\forall t$, all players randomly paired & $j \in O(i, t)$ iff i, j paired. (Kandori, Ellison Takahashi etc. setup w/ global benefits)
4. **Monitoring on a Circle:** All on a fixed circle and $\exists k \geq 1$ s.t. $j \in O(i, t)$ iff dist. between i, j is $\leq k$. (Equally observable fixed networks)

Different Public Good Types

1. **Pure Public Good:** $\alpha(N) = 1$
2. **Divisible Public Good:** $\alpha(N) = 1/N$

Group Size Exercises Under Equal Observability

- Perform comparative statics for different configurations:

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Group Size Exercises Under Equal Observability

- Results vary.
- Details are left for reading.

Monitoring Structure Exercises Under Equal Observability

- Now fix group size, N.
- $\alpha(N)$ is fixed since N fixed.
- We will change $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t, N)]$ in this exercise.
- What happens to MELP if we change monitoring structure?
- Which monitoring is better:

  Less reliable, more public vs. *more reliable, less public*

- Latter is less uncertain so SOSD former in $\#D(t, N)$ for all $i \in N$.
- Hence $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t, N)]$ is larger.
- Under broad conditions, MELP strictly higher when monitoring structure less uncertain.

N.B. Conditions cover 3 of 4 “all-or-nothing” structures (except circle).

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Monitoring on an Arbitrary Network

- Drop equal observability assumption.
- Ask what happens if we have a general (asymmetric) monitoring network.
- Introduce a new notion of “centrality”.

Theorem

If i is (strictly) more central than j then $\hat{x}_i (>) \geq \hat{x}_j$.

- Intuition is:
  - Defection by more central players leads to other central players to defect.
  - Hence central players are less inclined to deviate.
- Centrality measure calculation is arduous.
- Graph-theoretic tools to exploit/simplify “centralness” assessment.

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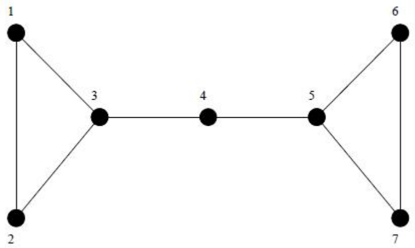

- To overcome difficulty of comparing 3 with 6 or 7, exploit symmetry.
- 3 and 5 similar just like 1,2,6 and 7 are.
- Letting $c(x) = x + x^3$ we have:

$$\text{If } \delta = 0.9 \qquad x_1^* \sim 2.167,\ x_3^* \sim 2.215,\ x_4^* \sim 2.225$$

$$\text{If } \delta = 0.4 \qquad x_1^* \sim 1.068,\ x_3^* \sim 1.182,\ x_4^* \sim 1.177$$

N.B. 3 and 4 are *NOT* more central than each other!

N.B. 3 has more dist-1 neighbors (low δ), 4 has more dist-2 neighbors (high δ).

Monitoring on an Arbitrary Network

- What is the impact of adding or removing links on MELP?
- Better connected societies provide more public good is verified.
- Additional link *ij* increases both players' contributions (by making defection costlier).
- In turn, all players path-connected to i, j contribute more in equil.

Theorem

Let L' and L be undirected networks s.t.

- $l_{k,k'} = l'_{k,k'}$ *for all* $(k, k') \neq (i, j)$
- $l'_{i,j} = 1$ *while* $l_{i,j} = 0$

Let C be connected component of L' containing i and j.
Then $\forall k \in C$, x^*_k is strictly higher under L' than under L.

Local Public Goods

- What if players benefit asymmetrically from each other's contributions?
- Generalize model for local goods. ("global" goods is a subcase)
- Benefits to be accrued only from observed players' contributions.
- Relevant for applications such as:
  - Cooperation in decentralized trade
  - Effort exertion in team projects for large organizations.
  - Pricing in differentiated market where subset of firms compete at a given t.
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Proposition

Fix $N \geq 4$ and even.

- With global public goods, $\hat{x}_i$ for all i is greater under random matching than fixed partnership.
- With local public goods as above, $\hat{x}_i$ is greater under fixed partnership than random matching.

- $\forall t$, $\mathbb{E}[\#D(t)]$ is higher under random matching than fixed partnerships.
- So defecting is more costly under random matching.
- This is desirable if good is global.
- But if good is local, then want to incentivize certain players.
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