Theory Reading Group: “Dynamic Wars”

Abreu and Gul: “Bargaining and Reputation” (Emca 2000)

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Summary

- Interested in bargaining outcomes as time between offers becomes arbitrarily small
- Incomplete information of particular form: each player may mimick a commitment type, so-called "irrational" type
- Study game where initially players pick a type and then enter a war of attrition where either continue to mimick or concede: uniqueness of sequential equilibrium outcome distribution
- Show that this is the unique continuous-time limit of any sequence of outcome distributions obtained from sequential equilibria of discrete-time bargaining games in which the time distance between offers approaches zero
- Comparative statics in terms of relative patience or irrationality or commitment opportunities (and limits thereof)

Outline

1. Continuous-time War of Attrition with complete information
2. Add one irrational type per player (reputation), uniqueness of sequential equilibrium, hence also outcome distribution
3. Consider discrete-time bargaining and support unique outcome distribution of previous game as limiting outcome of bargaining: irrespective of how the continuous-time limit is approached in terms of the bargaining protocols' details this unique equilibrium outcome distribution is the limiting distribution
4. (Most likely) Only mention generalisation, comparative statics, complete-rationality limit

Recall War of Attrition (1)

- Generalises idea of "chicken"
- Dynamic game with infinite horizon (discrete or continuous time)
- At any point in time each of two players chooses whether to concede or continue
- Once one player concedes the game ends (if other concedes before you then you "win")
- Payoffs are such that:
  - winning is best, and prefer to win sooner rather than later
  - continuation is costly (many variants, stationary and non-stationary)
- Pure strategy: a "stopping-time" (given the game is still on)

WoA (2)

- Equilibrium characterisation for continuous-time WoA under complete information (Hendricks et al. IER 1988):
  - at most one player concedes at time 0 with positive probability (i)
  - after time 0 each player concedes at a constant hazard rate which makes the opponent indifferent between concession and continuation (ii)
  - unless one player concedes at time 0 with probability one, there is no point in time until which one of the players will concede with certainty (iii)
- Applications: actual warfare, patent race, all-pay auction etc.

The Game

- Two players 1 and 2, continuous time so $t \in \mathbb{R}_+$
- Each player i may be "irrational" (probability z^i)
- Continuous-time WoA where at each t each i may concede or insist
    - rational types choose strategically, irrational types always insist
    - a strategy for a rational i is a cdf on the time domain $\mathbb{R}_+$: map it into unconditional $F^i(t) = \Pr(i \text{ concedes no later than } t)$ where $\lim_{t\to\infty} F^i(t) \leq 1 - z^i$
- Payoffs (interpret α^i as i's demand, assume $\alpha^1 + \alpha^2 > 1$):
    - if i concedes at t and before j: $\exp\left(-r^i t\right)\left(1-\alpha^j\right)$ and $\exp\left(-r^j t\right)\alpha^j$
    - if both concede at the same time t: $\exp\left(-r^i t\right)\frac{\alpha^i + \left(1-\alpha^j\right)}{2}$ for $\{i,j\} = \{1,2\}$

Special Case of Complete Information

- Consider first the case of $z^i = 0$ for both i, i. e. complete information, earlier characterisation (i)-(iii): $\left(F^1, F^2\right)$ is a sequential equilibrium if and only if

$$
\begin{aligned}
F^i(t) &= 1 - c^i \exp\left(-\lambda^i t\right) \\
c^i &= 1 - F^i(0) \\
\lambda^i &= r^j \frac{1-\alpha^i}{\alpha^j - (1-\alpha^i)} \\
0 &= \left(1-c^1\right)\left(1-c^2\right)
\end{aligned}
$$

- Note: λ^i is the constant hazard rate and F^i is increasing for $c^i > 0$

Proof

- Define $c^i \equiv 1 - F^i(0)$ as probability that i continues at 0
- (i) is then equivalent to $0 = F^1(0) F^2(0) = (1 - c^1)(1 - c^2)$
- Define the constant hazard rate of F^i as λ^i
- (ii) implies
  - $F^i(t) = 1 - (1 - F^i(0)) \exp(-\lambda^i t)$
  - $\lambda^i = r^j \frac{1-\alpha^i}{\alpha^j - (1-\alpha^i)}$
- (iii) implies that if $c^i > 0$ then F^i is increasing for all $t \in \mathbb{R}_+$ so support of F^i is entire $\mathbb{R}_+$ (no bite here)

Aside (1): Indifference and Hazard Rate

- Heuristic derivation of the hazard rate λ^i
- Indifference of j between conceding now and waiting another small amount $\Delta > 0$ of time and concede then
  - conceding now yields $1 - \alpha^i$ (current value)
  - postponing concession to Δ from now (current value)

$$\lambda^i \Delta \exp\left(-r^j \Delta\right) \alpha^j + \left(1 - \lambda^i \Delta\right) \exp\left(-r^j \Delta\right) \left(1 - \alpha^i\right)$$

  - equalising and rearranging

$$\lambda^i = \frac{\exp\left(r^j \Delta\right) - 1}{\Delta} \cdot \frac{1 - \alpha^i}{\alpha^j - \left(1 - \alpha^i\right)}$$

  - now $\lim_{\Delta \to 0+} \frac{\exp\left(r^j \Delta\right) - 1}{\Delta} = r^j$ using L'Hôpital's Rule
  - note that $\lambda^i > 0$

Aside (2): Constant Hazard Rate and Exponential Distribution

- Suppose cdf F has a constant hazard rate $\lambda > 0$, i. e.

$$\frac{F'(x)}{1-F(x)} = \lambda \quad \Leftrightarrow \quad F'(x) + \lambda F(x) = \lambda$$

$$\Leftrightarrow \quad \frac{d\left(F(x)\exp(\lambda x)\right)}{dx} = \lambda \exp(\lambda x)$$

- Integrating from 0 to t and rearranging

$$F(t) = 1 - (1 - F(0))\exp(-\lambda t)$$

Discussion

- Multiplicity: for c^i's, apart from them having to be probabilities only the last equation as restriction
- Examples:
  - $c^1 = 1$, $c^2 = 0$ and vice versa, so mixed ones no surprise
  - $c^1 = c^2 = 1$
- Welfare:
  - efficiency has one $c^i = 0$ (no delay)
  - i's equilibrium payoff $F^j(0)\alpha^i + \left(1 - F^j(0)\right)\left(1 - \alpha^j\right)$
  - payoffs when $c^1 = c^2 = 1$ are $1 - \alpha^i$ for each i (individually, same as if conceded immediately and other insisted)

Unique Equilibrium (1)

- In present game however have irrational types; however
  - (i) and (ii) carry over
  - (iii) is replaced by: there exists a time $T^0 < \infty$ at which the posterior probability of irrationality reaches one for both players simultaneously and concessions stop (iii')
- Proposition 1: this game has a **unique** sequential equilibrium given by $\left(\bar{F}^1, \bar{F}^2\right)$ characterised by

$$
\begin{aligned}
\forall t \leq T^0, \bar{F}^i(t) &= 1 - c^i \exp\left(-\lambda^i t\right) \\
\bar{F}^i\left(T^0\right) &= 1 - z^i \\
c^i &= 1 - \bar{F}^i(0) \\
\lambda^i &= r^j \frac{1-\alpha^i}{\alpha^j - \left(1-\alpha^i\right)} \\
0 &= \left(1-c^1\right)\left(1-c^2\right)
\end{aligned}
$$

Unique Equilibrium (2)

- Combining the first two lines, we obtain $c^i = z^i \exp\left(\lambda^i T^0\right)$ whence
$$\bar{F}^i(t) = 1 - z^i \exp\left(\lambda^i \left(T^0 - t\right)\right)$$
- Now note that $c^i < 1 \Leftrightarrow T^0 < -\frac{\ln\left(z^i\right)}{\lambda^i}$, define $T^i \equiv -\frac{\ln\left(z^i\right)}{\lambda^i}$
- From the last line, there is i such that $c^i = 1$ so from above $T^0 = T^i$
- Therefore $T^0 = \min\left\{T^1, T^2\right\}$, and $T^i < T^j$ implies $c^i = 1 > c^j$ so with some positive probability j concedes immediately

Some Intuition

- "This noise" picks unique equilibrium; crucial element: (iii')
- Intuition for necessity of (iii') given (i) and (ii):
  - if i known to be irrational earlier than j, say at τ^i, then j would surely concede at τ^i so j's conceding at τ^i would have positive probability
  - but then i would stop conceding at $\tau^i - \epsilon$ for sufficiently small $\epsilon > 0$, contradicting (ii) (constant hazard rate), hence $\tau^i = \tau^j \equiv T^0$
  - (i) says $c^i = 1$ for at least one i, this player i's probability of irrationality then reaches one at T^i which solves $F^i\left(T^i\right) = 1 - z^i \Leftrightarrow T^i = -\frac{\ln\left(z^i\right)}{\lambda^i} < \infty$
  - at most one player has $c^i < 1$ so $T^0 = \min\left\{T^1, T^2\right\}$

Proof Outline

- Sequential equilibrium will imply properties of strategies that uniquely pin them down to the above (uniqueness), below
- To finish the argument only need to verify that the proposed strategies constitute a sequential equilibrium (existence)
- For the second part:
  - pure strategies are a fixed time t_c of concession
  - given $\bar{F}^i$, player $j \neq i$ is indifferent among all pure strategies with $0 < t_c \leq T^0$
  - therefore any mixture on this support is a best response, in particular $\bar{F}^j$

Discussion (1)

- i's equilibrium payoff is $\bar{F}^j(0)\,\alpha^i + \left(1 - \bar{F}^j(0)\right)\left(1 - \alpha^j\right)$ so i "stronger" if
  - more patient (j concedes at faster rate)
  - more likely to be irrational ex ante (concede less often at 0)
- Why is e. g. $\left(c^1, c^2\right) = (1, 0)$ not an equilibrium?
  - 1's belief at $t > 0$ would be that 2 is irrational so concession probability jumps to one
  - 2's best response is not to concede at 0
  - note that as beliefs unavoidably change over time whenever not both always insist (not an equilibrium), non-stationarity guaranteed
- Stationary and sequential equilibrium:
  - complete-information equilibria stationary for $t > 0$ so sequentiality requirement no issue
  - here non-stationary equilibrium so what about beliefs and sequential rationality?

Discussion (2)

- Imagine in equilibrium the game has reached $t > 0$:
  - let $\omega^i(t)$ denote j's belief that i is irrational conditional on her not having conceded until t
  - then $\omega^i(t) = \frac{z^i}{1-F^i(t)}$ by Bayes' Rule so substituting for F^i gives $\omega^i(t) = \exp\left(-\lambda^i\left(T^0 - t\right)\right)$
  - replace z^i's with $\omega^i(t)$'s and compute sequential equilibrium as before (with hats); beliefs indeed guarantee that it coincides with the continuation strategies; check

$$\forall t > 0, -\frac{\ln\left(\omega^1(t)\right)}{\lambda^1} = -\frac{\ln\left(\omega^2(t)\right)}{\lambda^2} = T^0 - t$$

yields $\hat{T}^1 = \hat{T}^2 = T^0 - t$ and $\hat{c}^1 = \hat{c}^2 = 1$

Proving Uniqueness (1)

- Take any sequential equilibrium, say $\left(F^1, F^2\right)$, and define
  - $u^i(t)$ as i's expected utility if i deviated to pure strategy "insist until t and then concede with certainty"
  - $A^i \equiv \left\{t \in \mathbb{R}_+ \mid u^i(t) = \max_{s \in \mathbb{R}_+} \left\{u^i(s)\right\}\right\}$, note $A^i \neq \emptyset$
  - $\tau^i \equiv \inf \left\{t \in \mathbb{R}_+ \mid F^i(t) = \lim_{s \to \infty} F^i(s)\right\}$ where $\inf \emptyset \equiv \infty$

Proving Uniqueness (2)

- Steps, where $\{i, j\} = \{1, 2\}$:
  - $\tau^1 = \tau^2 \equiv \tau$ (a rational player will surely concede if she knows her opponent will never do so)
  - F^i jumps at $t \Rightarrow$
    - F^j does not jump at t (j would move any positive mass to slightly after t) and
    - F^j is constant on $(t - \epsilon, t)$ for some $\epsilon > 0$ small (j does not concede to not lose "discrete bonus")
  - F^i continuous at $t \Rightarrow u^j$ is continuous at t because

$$u^j(t) = \int_0^t \exp\left(-r^j x\right) \alpha^j dF^i(x) + \left(1 - F^i(t)\right) \exp\left(-r^j t\right) \left(1 - \alpha^i\right)$$

Proving Uniqueness (3)

- Further steps, where $\{i, j\} = \{1, 2\}$:
  - There are no (t', t'') with $0 \leq t' < t" \leq \tau$ such that both F^i's are constant on (t', t'') (if F^i constant then optimal to have F^j constant, but by continuity somewhat longer, true for both, cannot be)
  - $t' < t" < \tau \Rightarrow F^i(t") > F^i(t')$ (if one were constant then the other too, contradiction to previous)
  - F^i continuous at any $t > 0$ (if it jumped then F^j were constant on some $(t - \epsilon, t)$, contradiction to previous)
  - u^i constant on $(0, \tau]$ (from before A^i is dense in $[0, \tau]$ and u^i continuous for $t > 0$)
  - differentiate expression for u^i to obtain F^i with c^i undetermined (solve differential equation)
  - pin c^i down from $T^0 = \tau$ (argue as in intuition)

Bargaining

- Two players 1 and 2 decide on how to share a cake of size one
- Protocol (extensive form), where identify offer with 1's share:
  - $g : \mathbb{R}_+ \rightarrow \{0, 1, 2, 3\}$ where
    - $g(t) = 0$ means no one makes an offer at t
    - $g(t) = i$ for $i \in \{1, 2\}$ means player i makes an offer at t to which other player immediately responds by accepting or rejecting; game ends once respondent accepts
    - $g(t) = 3$ means both make simultaneous offers at t, game ends once the offers are compatible
  - defining $I^i \equiv \{t \in \mathbb{R}_+ \mid g(t) \in \{i, 3\}\}$, assume I^i infinite and for any $t < \infty$, $I^i \cap [0, t]$ finite (discrete time)
- Payoffs:
  - an outcome is x at t, or (x, t)
  - payoffs $\exp(-r^1 t)\, x$ and $\exp(-r^2 t)\,(1 - x)$

Irrational Types

- With probability z^i, player i is "irrational" and insists on a share α^i forever
- Assume $\alpha^1 + \alpha^2 > 1$ so the two irrational types never agree
- Reputation: by mimicking the irrational type a player may maintain a "tough" image
- Generalised to multiple irrational types of above behaviour
  - each identified with a share, $C^i \subset (0, 1)$ finite set of i's irrational types
  - distribution on C^i so $\pi\left(\alpha^i\right) = \Pr\left(i \text{ is type } \alpha^i \mid i \text{ is irrational}\right)$
  - assume $\max C^i + \min C^j > 1$

Continuous-Time Limit

- Take a sequence of discrete bargaining games $(g_n)_{n=1}^{\infty}$
- Say it converges to continuous time if $\forall \epsilon > 0 \exists N < \infty$ such that
$$\forall n \geq N, t \geq 0, i \in \{1, 2\} : i \in g_n([t, t + \epsilon])$$
- Let σ_n denote a sequential equilibrium of bargaining game g_n and θ_n the associated random variable that is its outcome (x, t)
- Denote by $\bar{\theta}$ the random variable which is the outcome of the unique sequential equilibrium
- Proposition 4: if $(g_n)_{n=1}^{\infty}$ converges to continuous time then any θ_n converges to $\bar{\theta}$ in distribution.

Main Ingredient

- "Coasian effect": when one player known to be rational and the other irrational with positive probability then there is no delay in the continuous-time limit (Myerson, Coase conjecture)
- Suppose i is known to be rational at time t; by above
  - either i gives in to j's demand so j obtains α^j
  - j concedes "right afterwards", revealing rationality as well, but to do this j must obtain at least α^j (if equilibrium then also at most)
- But then, by maintaining a belief in one's irrationality, a player obtains
  - her own preferred split with no delay if opponent reveals rationality
  - otherwise at least the opponent's preferred split (can guarantee that by conceding)
- This means revealing rationality turns into conceding, a WoA

WoA with Multiple Irrational Types

- Generalisation of the initial game
- At time 0, in sequential order
  - P1 chooses $\alpha^1 \in C^1$ and P2 updates beliefs
  - P2 ends game by agreeing, with payoffs α^1 and $1 - \alpha^1$, or P2 chooses $\alpha^2 \in C^2$ with $\alpha^1 + \alpha^2 > 1$ and P1 updates beliefs
  - P1 ends game by agreeing, with payoffs $1 - \alpha^2$ and α^2, or a WoA ensues
- Proposition 2: sequential equilibrium exists, and all sequential equilibria yield the same outcome distribution.
- Proposition 3: holding other things constant along the sequence
  - as both players become perfectly patient, if $\frac{r_i}{r_j} \to 0$ then i extracts all surplus (lim inf of i's equilibrium payoffs no less than $\left(1 - z^j\right) \max C^i$)
  - as probaility of i's irrationality approaches one

Comparative statics

- Proposition 3: take a sequence of such games B_n where v_n^i is the corresponding sequence of a **rational** i's (unique) equilibrium payoffs; holding other parameters constant along the sequence,
  - $\lim \frac{r_n^i}{r_n^j} = 0$ implies $\liminf v_n^i \geq \left(1 - z^j\right) \max C^i$ and $\limsup v_n^j \leq 1 - \left(1 - z^i\right) \max C^i$ (rational j concedes immediately so rational i demands $\max C^i$)
  - similarly, for $\lim z_n^i = 1$

Limit Result with Multiple Irrational Types

- The equilibrium outcome distribution is still unique
- Proposition 4 generalises provided that the same player moves first in every g_n of the sequence considered
- Remarks:
  - while, contrary to existing complete information theories, limit outcome independent of how details of the protocol such as intervals between offers approach the limit...
  - this is not true about the identity of the initial proposer
  - still, have very particular behavioural types

Complete Rationality

- "In the limit of complete rationality...
- Proposition 5: ...get close to efficiency, the closer the richer type spaces"
- Proposition 6: ..., generically, even when types spaces are not rich, there is no delay and thus efficiency is restored"

Conclusion

- Recap:
  - Add reputation incentives to obtain uniqueness for the continuous-time limit of a rather general sequence of discrete bargaining games
  - While reputation effects overwhelming in one-sided case (Myerson, Coase conjecture) **relative** patience still matters in two-sided case
- Highlight interesting relationship between bargaining and WoA!
- Comparison to "non-reputation models" of incomplete-information bargaining?
- Predictions?
- Richer type spaces?