Jumb-bidding and Budget constraints in All-Pay Auctions

Dekel, Jackson, Wolinsky, WP2006

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Theory Reading Group: Dynamic War of Attrition, 2012/01/20

Outline

Set-up

Discrete: ϵ is the smallest unit of money.

Two bidders, with valuations V_i. For simplicity, V_i is not a multiple of ϵ. Denote $[V_i]_\epsilon$ the maximum multiple of ϵ smaller than V_i.

Budgets: $B_i \in \mathcal{R}^+ \cup \{\infty\}$, bid b_i cannot exceed budget B_i.

Budgets and valuations are complete information.

Utility if winning the object $V_i - b_i$, if losing: $-b_i$ (all-pay auction)

General rules of the game

Discrete time $t \in \{1, 2, \dots\}$.

Bidders alternate at every period, starting with bidder 1 (so slightly assymmetric).

Game ends whenever bidder 1 cannot exceed bidder 2 last bid, or when 2 cannot match or exceed 1's last bid.

If the game does not end, payoff to each player is $-\infty$.

Two types of game

"Jump-bidding all-pay auction":
bidders can raise their bids by how much they want

"No-jump all-pay auction":
bidders can only raise their bids by ϵ

$\Rightarrow$ Discrete version of war of attrition

Paper compares jump/no-jump, with and without budget limits

Useful Lemma

Applies regardless of jump/no-jump, and budget limits.

Lemma 1

Consider a subgame starting with i. If i increases its bid with positive probability in equilibrium, then j drops out with positive probability (at any node that follows a bid by i).

Sketch of proof (for pure strategies):

- By contradiction, assume i raises bid at t, j stays in for sure at t'.
- There must be a node t'' where one bidder drops out for sure (assume i). Then her utility at t'' must be zero.
- This node is reached with probability one. Then i must have negative expected utility at node t since between t and t'' she will sink additional payments.

Useful lemma (cont.)

Consequences of Lemma 1:

⇒ In any equilibrium, either bidder has positive expected payoffs only at the first node.

⇒ If equilibrium path continues past first node, it must involve mixing or dropping out completely.

Benchmark:
no-jump, infinite budgets

Characterise the set of equilibrium paths:

Proposition 1

$\forall p \in [0,1]$ there exists an equilibrium in which 1 drops out with prob. p at the first node:

- *If $p = 1$, it is the end of the equilibrium path.*
- *If $p \in (0,1)$, in each subsequent node on the equilibrium path, bidder 2 drops with prob $\frac{\epsilon}{V_1}$ and bidder 1 drops with prob $\frac{\epsilon}{V_2}$*
- *If $p = 0$, bidder 2 drops with prob $q \geq \frac{\epsilon}{V_1}$,*
  - *If $q = 1$, the equilibrium path finishes*
  - *If $q < 1$, after that 1 and 2 drop respectively with prob $\frac{\epsilon}{V_2}$ and $\frac{\epsilon}{V_1}$*

Benchmark (cont.)

Sketch of proof:

- If mixing carries on after $t > 2$, it must be that players get zero incremental payoff.
- If (i) bids at $t > 2$, it must be that (j) dropped out with prob. $\geq \frac{\epsilon}{V_i}$ at $t-1$.
- if at $t-1$, (j) dropped out with prob. $> \frac{\epsilon}{V_i}$, then (i) would have stayed for sure, instead of mixing, at period $t-2$.

Comments on benchmark

Multiplicity of equilibria. In some equilibria, bidder with lowest value wins with higher probability. In one equilibrium, she wins for sure.

Literature uses reputation effects to select an equilibria:

- If introduce small probability of a crazy type who never drops out, get unique (mixed) equilibrium => unique outcome distribution
- Taking this probability to zero, equilibrium distribution approaches outcome where higer valuation bidder wins immediately.

Jump-bidding allows to select this as the unique equilibrium outcome, without adding noise.

Jump-bidding, infinite budgets

Now assume bidders are allowed to jump-bid, i.e. to raise their bids by multiples of ϵ.

Sequential + all-pay: all previous bids are sunk.

$\Rightarrow$ At every turn, bidder (i) is willing to bid up to $[V_i]_\epsilon$.

Proposition 2 (corrected)

- *If $[V_1]_\epsilon > [V_2]_\epsilon$, bidder (1) bids ϵ once, and bidder (2) drops out*
- *If $[V_1]_\epsilon < [V_2]_\epsilon$, bidder (1) drops out immediately.*

If $[V_1]_\epsilon = [V_2]_\epsilon$ there are multiple equilibria.

- E.g. one bidder always bids ϵ and the other always exits.

Comments

Jump-bids are not used in equilibrium but their threat precludes the lower value bidder to win.

Assume $[V_1]_\epsilon < [V_2]_\epsilon$. Can we get an equilibrium where 1 wins? No.

Not possible to have an equilibrium where 1 wins by bidding ϵ: bidder 2 could force 1 to drop out by bidding $[V_2]_\epsilon$, which is better for 2 than to drop out.

Comments

What if each bidder always raise by their maximum allowance, resp. $[V_1]_\epsilon$ and $[V_2]_\epsilon$?

$\Rightarrow$ At some point, 1 will have to raise by more than $[V_1]_\epsilon$ to stay in the race.

Example: $[V_1]_\epsilon = 4, [V_2]_\epsilon = 5$.

b_1	b_2
4	5
8	10
12	15
16	20
20	

By backwards induction, 1 would have rather dropped earlier: drops in the first round.

Useful lemma

Lemma 2

If one bidder has limited budget, consider any subgame: all equilibrium continuations are in pure strategies (on and off equilibrium paths).

Intuition:

Finite budget => any sugame is finite. At last node, assume i is mixing, then j drops for sure. Given V_i assumed not to be multiple of ϵ, then payoff of i bidding is strictly positive. Therefore i bids for sure.

Important consequence Lemma 1 and lemma 2 imply that with budget limits, in any subgame bidder i either drops out immediately, or bids enough to stay in auction and j drops out immediately afterwards.

$\Rightarrow$ Everything resolved in the first round.

No-jump, with limited budgets

At least one bidder has limited budget: either B_1 or $B_2 < \infty$.
Bidding is restricted to ϵ-increments.

Proposition 3

There is a unique equilibrium:

- *If $B_1 > B_2$, bidder 1 wins at price ϵ.*
- *If $B_1 \leq B_2$, bidder 2 wins at price 0 (bidder 1 drops out immediately).*

Sketch of proof:

Finite budget => at some node, lowest-budget player will want to drop out, the other to stay in. By backward induction, same is true ϵ before this node, etc.

$\Rightarrow$ Lowest-budget player drops out at first opportunity.

Comments on proposition 3

Highest-budgest bidder always wins, regardless of valuations.

In particula, this holds even when budget caps are much higher than valuations: budget caps still matter.
(They affect off-equilibrium play.)

Jump-bidding with limited budgets

With jump-bidding, one recovers *some* efficiency:

Proposition 4

1. *If* $\min\{B_1, [V_1]_\epsilon\} > \min\{B_2, [V_2]_\epsilon\}$, *bidder* 1 *wins.*
2. *If* $\min\{B_1, [V_1]_\epsilon\} \leq \min\{B_2, [V_2]_\epsilon\}$, *bidder* 2 *wins, except:*
3. *If* $B_1 > B_2$ *and* $[V_1]_\epsilon < [V_2]_\epsilon$ *and* $B_2 < k[V_1]_\epsilon$ *where* k *is the minimal integer such that* $k[V_1]_\epsilon \leq (k-1)[V_2]_\epsilon$, *then bidder* 1 *wins*

Points 1 and 2:
Bidder i cannot bid above B_i and does not want to bid above V_i. Therefore it makes sense that the winner is determined by the minimum of budgets and values.

Point 3 illustrate potential first-mover advantage

Explanation of case 3

Imagine the most agressive bidding, where each bidder bids $[V_i]_\epsilon$. Since $[V_1]_\epsilon < [V_2]_\epsilon$, at some point, 1 would have to bid more than $[V_1]_\epsilon$ to stay in the race.This happens at k^{th} offer by 1, where k is the minimal integer such that $k[V_1]_\epsilon \leq (k-1)[V_2]_\epsilon$.

Go back to our example: $[V_1]_\epsilon = 4, [V_2]_\epsilon = 5$.

	b_1	b_2
	4	5
	8	10
	12	15
	16	20
$k = 5$	20	

However if $B_2 < k[V_1]_\epsilon = 20$, then 2 would exhaust her budget before 1 would have to drop out (case 3).

Explanation of case 3 (cont.)

Even if the bidding race does not happen in equilibrium, its possibility causes 2 to drop out immediately in any subgame where it could happen (Lemma 3).

$\Rightarrow$ Bidder 1 can win with a preemptive bid that is lower than $\min\{[V_2]_\epsilon, B_2\}$.

Preemptive bid may need to be substantial

However bidder 1's preemptive bid may need to be greater than ϵ.

Example (from paper, corrected)
$[V_1]_\epsilon = 10, [V_2]_\epsilon = 13, B_2 = 15 < B_1 = 20, \epsilon = 1$
In all equilibria, $b_1 = 4$:

- If $b_1 \leq 3$, then bidder 2 could bid 13 and win.
- If $b_1 \geq 4$, then even maximum bid by 2 (13) still makes it worthwile for 1 to stay in the auction => 2 drops out

In short:

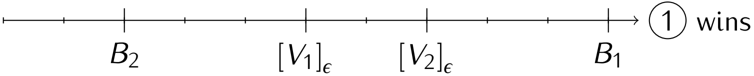
- Without jump-bidding, winner never has to pay more than ϵ.
- With jump-bidding, winner may have to pay substantial amount, even if there is complete information and a certain winner in all equilibria.

Further comments on proposition 4

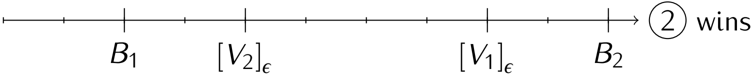
Efficiency

Proposition 4 does not exclude that the lowest-value bidder wins the auction. For instance:

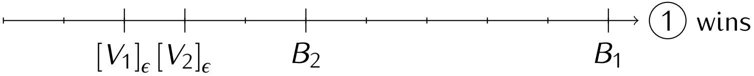
- Case 1:



- Case 2:



- Case 3:



General comments

Budget limits imply loss of efficiency

With or without jump-bidding, it may be the case that lowest-value bidder wins where there are budget limits. This problem is moderated by jump-bidding.

Timing:

Lemma 1 + Lemma 2 => Equilibria are in pure strategies => auction is always settled in the first round.

Incomplete information

The paper briefly discusses incomplete information with infinite budgets. There is an equilibrium in which:

- bidder 1 bids as in a sealed-bid all-pay auction;
- if V_2 smaller than 1's revealed valuation, then 2 drops;
- otherwise bid $b_2 = V_1$, then 1 drops.

This equilibrium is efficient so revenue equivalence holds.

Not clear that this equilibrium is unique.

Summary of the results

	No-jump	Jump-bidding
Infinite budgets	Anything goes	Highest-valuation wins at minimal cost
Limited budgets	Biggest budget wins at minimal cost	Not always efficient; may require substantial preemptive bid

Conclusion

Jump-bidding allows to refine the set of equilibria:
Without budget limits: retrieve efficiency

Once budget limits are introduced, efficiency breaks down.

- War of attrition with budgets: highest budget always win.
- With jump-bidding: lowest valuation may still win in some cases

Jump-bidding is never used on the equilibrium path, except in a specific case of preemptive bidding.
Jump-bidding allows to get rid of some inefficient equilibria.

All equilibria are in pure strategies: auctions are always resolved in the first round.

Application (Leininger 1991)

All these results were already shown in Leininger 1991 (JET)…(except he does not allow budgets to be smaller than valuations)

The context: patent race between an incumbent firm and a potential entrant. Calls "incumbent" whoever has a higher valuation (monopoly profit).

Even if incumbent has lower budget, if it moves first, it can deter entry by sinking a positive amount in R&D

- $\Rightarrow$ Potential explanation of why monopolists invest in R&D
- $\Rightarrow$ Threat of entry may lead monopolist to dissipate some of its rents.

Restoring efficiency in budget-constrained auctions

No auction can ensure generic efficiency when bidders are budget-constrained.

But Maskin (2000, EER):

- With two bidders, all-pay auction can be constrained-efficient if two bidders, and bidders are ex-ante symmetric (same distribution of valuations, same budgets) and at least one bidder is not budget-constrained
- With three bidders, all-pay auction can be constrained-efficient with potentially different bidders (same budgets but different distribution of valuations)

Loose intuition: amount to pay does not depend on whether winning or not => less incentives to distort reports of valuations.