Inflating away the public debt? 
An empirical assessment

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Abstract

This paper proposes and implements a new method to measure the impact of future inflation on the real value of outstanding public debt. It produces a distribution of debt debasement that relies on a forward-looking approach and is based on two inputs: (1) the distribution of privately-held nominal debt by maturity, for which we provide new estimates for the United States, (2) the distribution of risk-adjusted inflation dynamics over time, for which we provide a novel econometric copula estimator using data from options contracts. Applying the method, we find that it is unlikely that inflation will lower the U.S. fiscal burden significantly. This is because of the interaction of debt concentrated at short maturities, and perceived inflation shocks that have little short-run persistence and low volatility on impact. If instead inflation is combined with financial repression, both together can result in substantial debt debasement.

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1 Introduction

Higher-than-expected inflation can have some benefits, and one of its most celebrated is to erode the real value of outstanding debt. Public debt is at record high levels across many countries, after years of steady growth (Reinhart, Reinhart and Rogoff, 2012) and a sharp jump with the 2020 pandemic recession (International Monetary Fund, 2020). This constrains monetary and fiscal policy and puts a drag on economic activity because of expected higher future distortionary taxes. A common way that sovereigns pay for high public debt is by having high, and sometimes even hyper, inflation (Reinhart and Rogoff, 2009), and this may happen again following the pandemic recession (Pastor, 2020). Whether this is feasible or likely in the future is an open question. How likely will inflation be sufficiently high to substantially alleviate current heightened debt levels? What are the characteristics of the debt and the stochastic properties of inflation that make substantial debasement more or less likely?

In this paper, we propose a method to quantify the likelihood of future inflation substantially eroding the real value of current public debt. We measure the effect of inflation on the fiscal burden by constructing the distribution of inflation-driven debt debasement. We combine characteristics of the debt (its maturity structure, holders, and nominal or real payouts) and of future inflation (its persistence and the likelihood of large increases) to quantify the probability of inflation-driven debt reduction of various sizes. We then apply our method to the United States on annual data from 2009 to 2017 and thereby inform the debate on how the Unites States may deal with its outstanding public debt. Finally, we show how to map the distribution of debt debasement into central objects in theories of inflation and its effects.

Both our general goal of quantifying the fiscal impact of inflation and the concrete application to the United States require that a debt debasement measure must have four properties. First, it must be forward looking, using present data. We ask whether inflation can erode the debt in the future, not if it did so in the past. Second, it must be accurate, using the actual maturity and private holdings of the public debt. We do not provide back-of-the-envelope calculations, but precise estimates that net out holdings of public debt by government agencies and are based on the exact maturity structure. Third, the calculations must rely on plausible scenarios for inflation that are supported by data and reflect the fact that inflation is stochastic. We ask whether quantitatively it is likely that public nominal
debt can be inflated away, not if this is theoretically possible.\footnote{Of course it is, just drive the price level to infinity!} Fourth, the calculation must take investors’ expectations into account, and how these are reflected in prices, otherwise the measure will dramatically overstate what government can do. We use market prices for inflation in different scenarios extracted from options prices.

Focused on different questions, previous work on this topic has instead looked at the historical experience, it has used approximations for the debt maturity structure, and it has not emphasized using plausible and data-driven distributions of future inflation paths. Considering the historical distribution, Hall and Sargent (2011) characterize the contribution of inflation for the evolution of U.S. public debt in the past.\footnote{Reinhart and Sbrancia (2015) conduct a similar exercise for other developed countries, while Berndt, Lustig and Yeltekin (2012) and Chung and Leeper (2007) use vector autoregressions to deal with expectations of spending shocks, and Giannitsarou and Scott (2008) link spending shocks with governments choosing to inflate.} A different part of the literature studies the many theoretical links between inflation and the public debt, but typically assumes that public debt has a one-period maturity.\footnote{For example Cochrane (2011) and Davig, Leeper and Walker (2011) link high levels of U.S. debt to future higher inflation.} Finally, a third set of papers provides back-of-the-envelope calculations on how much inflation will lower the debt, but makes ad hoc and implausible assumptions on the dynamics of inflation and expectations.\footnote{See Aizenman and Marion (2011) and Bohn (2011).} Our inflation-driven debt debasement distribution is forward looking, accurate, and based on plausible future inflation scenarios.

We start by developing the theory underlying our measure. Based on the law of motion for debt and the intertemporal budget constraint, we derive a simple forward-looking formula showing that the real value of debt is equal to a weighted average of the payments due at different horizons, with the weights given by the expected inverse of compounded risk-adjusted inflation. The formula makes transparent how the distribution of future risk-adjusted inflation paths interacts with the maturity of debt to map into the distribution of inflation-induced debt debasement. Because the theory behind our measure requires only the existence of a government budget constraint and of a stochastic discount factor to price government bonds, it can easily be embedded into equilibrium macroeconomics models, providing new insights into discussions of inflation targeting, the fiscal theory of the price level, and the maturity of government debt.\footnote{See Cochrane (2017); Del Negro and Sims (2015); and Bianchi and Melosi (2017); and Missale and Blanchard (1994); Persson, Persson and Svensson (2006); Debortoli, Nunes and Yared (2017), respectively.}

Calculating debasement distributions requires two main inputs. First, detailed data on
holders and maturity of nominal public debt. Second, the joint distribution of future risk-adjusted inflation at different horizons. This is needed to simulate future inflation paths, which in turn allows us to quantify the risk-adjusted probability of debasement of various sizes. We propose new measures of both. Our main purpose in assembling these data is to shed light on the effect of inflation on the public debt. Our new empirical estimates of the evolution of the maturity structure of privately-held U.S. debt are an independent contribution of interest, as they imply a significantly shorter distribution than commonly appreciated. Likewise, the econometric methods we propose to construct joint distributions of future inflation from option prices are of independent interest, as they will be useful in other contexts.

The debt maturity structure takes into account the identity of the holders of the debt, and whether it is nominal or real. We find that nominal privately-held debt makes up a little less than half of total outstanding debt on average in 2009-17. The majority of that debt is held by foreigners, who would bear the costs of possible debasement. Another large share is owned by the Federal Reserve. While debasing it would not loosen the government fiscal burden, it does raise issues about the solvency of the central bank (Hall and Reis, 2013). We find that the private domestic sector holds a disproportionate share of the debt of shorter maturities, and a smaller share of longer maturity debt, resulting in low probabilities of large debasement. In contrast, the Federal Reserve is more exposed to higher-than-expected inflation since its portfolio is composed of longer maturity debt.

The second input required for a forward-looking measure that is based on plausible inflation scenarios is the joint distribution of future inflation. We construct this in two steps. First, we use data on inflation caps and floors with payoffs that depend on the realizations of the CPI at different maturities, building on the small literature that has used options data to infer the density functions for cumulative inflation. However, knowledge of the cumulative distributions constructed in previous work—for example, the probability of average inflation over the next two years lying below various cutoffs—is not enough. We complement these data with data on forward distributions, that is probabilities of inflation in a given future year lying below various cutoffs. Only using the information which is embedded in both sets of derivatives can one obtain a joint distribution of inflation at various future horizons.

Our second step is to develop a new method to estimate this joint distribution. It re-

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6Using data on cumulative inflation distributions, Kitsul and Wright (2013) look at the response of the density around monetary policy announcements, while Fleckenstein, Longstaff and Lustig (2017) assess the risk of deflation in the United States. We instead study the effect of future inflation on the real value of debt.
lies on the theory of copulas, and uses only the marginal densities of both cumulative and one-year forward inflation at a given point in time. It allows the researcher to extract information about the co-movement of inflation over time and from there to simulate future inflation paths. These distributions are under the forward measure for various maturities (Jamshidian, 1989). While there are many ways to extract objective and subjective probability forecasts for inflation, including financial prices, surveys, and economic and statistical models, these methods tend to focus on forecasting the mean while placing less emphasis on higher moments. In addition, and crucially, they are not appropriate for pricing. Our goal is to measure the market value of the debt under different scenarios, so we need the risk-adjusted probabilities that are relevant for pricing the government debt. Using these probabilities also allows us to embed reactions of market prices into the debasement distribution. Analyzing our data with our method, we find that risk-adjusted inflation is estimated to be generally close to stationary but that the process has long memory, reflected in a slow decay of the autocorrelation function.

Combining our approach and these two inputs, we measure the probability of debasement exceeding cutoffs of various magnitudes for the United States for annual data from 2009 to 2017. These are value-at-risk measures of the likelihood of debt debasement due to inflation. In 2017, the risk-adjusted probability of debasement lying below a mere 3% of GDP is almost 95%. Any loss above 3.7% has less than a 1% probability. Moreover, most of the losses fall on foreign holders of the government debt and the Federal Reserve is also hit. Our results are robust to using real probabilities, instead of risk-adjusted ones: depending on the approach used to move from one measure to the other, debasement cutoffs either represent an upper bound of debasement or the results are approximately unchanged.

What is behind these modest effects of inflation on the fiscal burden? We conduct a series of stress tests, or counterfactuals, disciplined by the data on inflation distributions. We show that our main results are due to two complementary factors. First, the maturity of the relevant U.S government debt, which is sensitive to inflation and held privately, is quite low. Second and related, over short horizons of a few years, market participants place a very low probability on persistently high U.S. inflation. In the near term, there is much debt but little extra inflation, and for longer horizons, there could be significant inflation, but there is little debt. As a result, the total effect is small.

While this interaction roughly explains the broad conclusion on the U.S. application,

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7We refer to these probabilities, which are appropriate for valuation, as risk-adjusted or risk-neutral probabilities.
our goal is to supply methods that provide precise and accurate numerical answers across applications at different times and for different countries. Starting from our accurate calculations, we are able to identify the main drivers of debasement – debt duration, inflation volatility and inflation persistence. This leads us to propose a new rule of thumb calculation that combines these three measures. A different concept that is sometimes used as a rule of thumb is duration (the sensitivity of debasement to a marginal shock to inflation). By itself, any measure of sensitivity cannot deliver a distribution. It says nothing about the likely magnitude of debasement because it ignores the volatility of inflation and so it cannot provide a probability of inflation being high enough to erode the debt, for instance by more than 1%. More importantly, duration provides an incorrect answer to the question we pose because it does not take into account the stochastic properties of inflation, since it assumes a permanent increase in inflation that is parallel across the whole distribution, when in fact, actual and expected inflation persistence is much lower. Our proposed accurate calculations and associated rule of thumb take into account both volatility and persistence, which together with the maturity of the debt captures most of the variation in debasement over time in the U.S. data.

Our paper focuses on inflation-driven debt debasement. However, the real value of debt as a percentage of GDP may also decline because of an increase in GDP. Such an increase may be more likely during periods of high inflation, combining to deliver a large decline in the real debt burden. To account for this, we complement our measures with estimates of the joint distribution of output from an estimated vector autoregression. Realizations of higher inflation are associated with higher-than-expected output growth, because of the Phillips curve, resulting in a significantly higher reduction of the debt. A third driver of debt to GDP dynamics is the effect of changes in the real term structure of interest rates. We propose a novel way of decomposing annual changes in the debt-to-GDP ratio into the inflation surprise, the growth surprise, and the real interest rate surprise. All of these components combine and, at different times from 2009 to 2017, significantly affect the debt-to-GDP ratio. These results connect our findings with the estimates by Hall and Sargent (2011) of the reduction in the U.S. debt-to-GDP ratio after World War II.

Finally, we apply our method to both a policy question and to a set of theoretical questions. Financial repression is often used in developing countries. Such a policy provides revenue directly, which is easy to estimate and, more interesting for this paper, it substantially increases the scope for inflationary debasement in a way that our new method can assess. We show that extreme financial repression, where bondholders are paid with reserves
at the central bank that pay zero interest, yet must be held for a fixed number of periods, is equivalent to ex post extending the maturity of the debt. Under such circumstances, inflation has a much larger impact, so that if repression lasts for a decade, permanently higher inflation that previously lowered the real value of debt by 2.3% now lowers it by 18.7% of GDP. We also discuss several theoretical applications. Many models assume one-period debt for convenience, but then struggle to adequately calibrate the link between the fiscal burden and inflation in the actual data. Our estimates provide the correct targets for these calibrations. In particular, our estimates measure the extent of fiscal backing of an inflation-targeting central bank, they provide the joint distribution of fiscally-driven inflation, and they provide the key elasticity of the debt burden to inflation that is needed to calibrate models of tax smoothing and sovereign default.

To conclude, it is useful to point out also what we do not do. We study the effect of higher-than-expected inflation on debt debasement. There are many other links through which inflation may affect the state of public finances. This focus allows us to make a minimal set of assumptions of no arbitrage in the government debt market, without having to use a full equilibrium model. Also, there are many ways in which inflation leads to redistributions of wealth (Doepke and Schneider, 2006; Berriel, 2013). We focus solely on the redistribution from private holders of debt, domestic or foreign, to the fiscal authority. Finally, our goal is positive, not normative, so we do not describe whether it is desirable to use inflation given the current maturity of the debt (Faraglia et al., 2013; Leeper and Zhou, 2013) or its holders.

The rest of the paper is organized as follows. Section 2 provides the theoretical basis for the construction of the debasement distribution, section 3 assembles the debt and inflation data and develops a new approach, based on the theory of copulas, to calculate joint inflation distributions. Section 4 presents the results: value-at-risk measures of debasement, data-driven stress tests, a new rule of thumb measure, the effect of adding GDP dynamics, and the decomposition of shocks to the debt burden. Section 5 discusses applications: financial repression and the link from our estimates to different classes of theoretical models. Section 6 concludes.

8For instance, higher inflation raises seignorage (Hilscher, Raviv and Reis, 2018), and affects tax revenues (Krause and Moyen, 2016).
2 Theory: the debt burden and risk-neutral densities

Our goal is to quantify the effect of future inflation on the real value of debt. This is not straightforward since there are many future inflation paths and outstanding debt securities. In this section, we start by providing a general but workable definition of the debt burden and its evolution, allowing for a rich maturity structure of both nominal and real debt. We show that assuming no arbitrage and taking a present value approach can aggregate the numerous ways in which future inflation can affect the value of debt. Our approach takes into account the stochastic nature of inflation, reflects market expectations of possible future policy changes, and makes minimal assumptions. The result, a mapping from future inflation to the present value of debasement, forms the basis for constructing inflationary debasement distributions.

2.1 The public debt

The real market value of government debt at date $t$ is equal to the present discounted value of the full maturity structure of future promised payments:

$$W_t = \sum_{j=0}^{\infty} \frac{H_t^j B_t^j}{P_t} + \sum_{j=0}^{\infty} Q_t^j K_t^j. \quad (1)$$

$B_t^j$ is the par value of zero-coupon nominal debt held at the beginning of period $t$ (carried over from period $t - 1$) that has a maturity of $j$ years, so that at date $t$ the government expects to pay $B_t^j$ dollars at date $t + j$. $K_t^j$ is the par value of real debt held at date $t$ that has a maturity of $j$ years, referring, for example, to Treasury inflation protected securities (TIPS). $H_t^j$ is the market price (or inverse-yield) of nominal debt with a maturity of $j$ years at date $t$. Likewise, $Q_t^j$ is the price (or inverse-yield) of real liabilities (TIPS) with a maturity of $j$ years at date $t$. Finally, $P_t$ is the price level, and we will use the notation $\pi_{t,t+j} = P_{t+j}/P_t$ to denote gross cumulative inflation between two dates. The following normalizations apply: $H_t^0 = Q_t^0 = 1$ and $P_0 = 1$.

Modeling the government debt this way involves a few simplifications. First, the government often has a wide variety of outstanding debts. The implicit assumption above is that their prices are the same as those of all-else-identical traded debt securities. Second, it assumes that coupon-paying bonds can be priced as portfolios of zero-coupon bonds. In this way, we limit the set of debt instruments issued by the government and simply consider promised payments (either principal or coupon payments) at each point in time. Third,
unfunded real liabilities of the government like Social Security can be included in $K^j_t$. Theoretically, they pose no problem. In practice, measuring any of these precisely is a challenge. However, since they are real liabilities, they are unaffected by inflation and therefore do not affect our debt debasement calculation. We therefore focus on nominal public debt instruments issued by the government.\textsuperscript{9}

If all debt were due right away, then the expression in equation (1) would reduce to $B_t^0/P_t + K_t^0$. The simple rule of thumb that an increase in $P_t$ lowers the debt burden proportionately to the privately-held nominal debt is accurate. However, with longer maturities, higher future price levels affect yields and so also the value of debt. In addition, there is a whole sequence of future price levels that have the ability to reduce the real value of debt. We therefore next consider the evolution of debt.

2.2 The law of motion for debt

Each period, to pay for the debt, the government must either collect a real fiscal primary surplus of $s_t$, or borrow more from the private sector:

$$W_t = s_t + \sum_{j=0}^{\infty} \frac{H_t^{j+1} B_{t+1}^j}{P_t} + \sum_{j=0}^{\infty} Q_t^{j+1} K_{t+1}^j.$$  \hfill (2)

Combining the previous two equations provides a law of motion for debt. Looking forward from date 0 for $t$ periods, we can write it as:

$$W_0 = W_{t+1} \prod_{\tau=0}^{t} Q_t^\tau + \sum_{i=0}^{t} \prod_{\tau=0}^{i} Q_t^\tau s_i + \sum_{i=0}^{t} \prod_{\tau=0}^{i} Q_t^\tau \sum_{j=0}^{\infty} (H_i^{j+1} - H_i^j H_{i+1}^j) \frac{B_{t+1}^j}{P_i} + \sum_{i=0}^{t} \prod_{\tau=0}^{i} Q_t^\tau \sum_{j=0}^{\infty} (Q_i^{j+1} - Q_i^j Q_{i+1}^j) K_{i+1}^j + \sum_{i=0}^{t} \prod_{\tau=0}^{i} Q_t^\tau \left( \frac{H_i^j P_{i+1}}{P_i} - Q_i^j \right) \sum_{j=0}^{\infty} \frac{H_i^{j+1} B_{i+1}^j}{P_{i+1}}.$$  \hfill (3)

This equation makes it clear that, to make progress, we need some way to aggregate the effect of future inflation realizations. Inflation can affect almost every term on the right-hand side, without a clear way to decompose them. Worse, to judge how a particular path for inflation $\{\pi_{0,i}\}_{i=0}^{t}$ affects the fiscal burden, we would need to know how inflation will change the slope of the yield curve at every maturity (the $H_i^{j+1} - H_i^j H_{i+1}^j$ term), the composition of

\textsuperscript{9}As we discuss in the next section, most of the non-marketable debt is held by the government so that debasement of it would not affect the fiscal burden. We also discuss the maturity structure of TIPS and predicted Social Security liabilities.
the debt by maturity, and the future real/nominal split (the $B_{i+1}^j$ and $K_{i+1}^j$ terms). Likewise, we would need to know the link between inflation and the real yield curve (the $Q_{i+1}^j - Q_1^1 Q_{i+1}^j$ term) as well as the ex post differences between nominal and real returns (the $H_{1}^1 P_{t+1} / P_t - Q_1^1$ term). Finally, recall that this expression holds for every possible path of inflation as well as for every realization of uncertainty in the economy. Without aggregation, there is an unwieldy large number of possible measures of how much the fiscal burden will change in the future.\footnote{Hall and Sargent (2011) overcome these problems by using a version of this equation that looks backwards, instead of forward. Given debt in the present ($W_{t+1}$) and in the past ($W_0$), there are historical data on most of the terms above. Our purpose is to look forward to figure out how debt depends on future, not past, inflation.}

### 2.3 Looking forward: the intertemporal budget constraint

We take a present value approach and assume that there is a stochastic discount factor to price all of these government liabilities. It is well understood that this is equivalent to requiring the absence of arbitrage. Note that all we require is that there is no arbitrage between Treasuries with different maturities, not across asset classes.\footnote{Our construction of risk-neutral distributions in section 4 also requires no arbitrage in the inflation derivatives market.} U.S. government bond markets are among the most liquid in the world, have fewer restrictions on short-selling, and serve as the fundamental asset for many traded derivatives. We are also allowing for varying risk premia or excess profits across different maturities, as in models of segmented markets or preferred habitats, so we are not ruling out possible liquidity premia or the effectiveness of quantitative easing policies.\footnote{See Vayanos and Vila (2009).}

The stochastic discount factor at date $t$ for a real payoff at date $t + j$ is denoted by $m_{t,t+j}$, and the conventional pricing equations for $j$-period bonds are:

$$1 = \mathbb{E}_t \left( \frac{m_{t,t+j}}{Q_t^j} \right) = \mathbb{E}_t \left( \frac{m_{t,t+j} P_t}{H_t^j P_{t+j}} \right).$$

A nominal discount bond costs $H_t^j / P_t$ in real units at date $t$, and pays off $1 / P_{t+j}$ real units in $j$ periods; its return times the stochastic discount factor has to have an expectation of 1. The same applies for a real bond. The absence of arbitrage over time implies that the stochastic discount factors across any two maturities, $n$ and $j$, are linked by: $m_{t,t+j} = m_{t,t+n} m_{t+n,t+j}$ for $1 \leq n \leq j$. 

\footnote{Hall and Sargent (2011) overcome these problems by using a version of this equation that looks backwards, instead of forward. Given debt in the present ($W_{t+1}$) and in the past ($W_0$), there are historical data on most of the terms above. Our purpose is to look forward to figure out how debt depends on future, not past, inflation.}

\footnote{Our construction of risk-neutral distributions in section 4 also requires no arbitrage in the inflation derivatives market.}

\footnote{See Vayanos and Vila (2009).}
Multiplying by stochastic discount factors at different dates and taking expectations of equation (3), while taking the limit as time goes to infinity, and imposing that the government cannot run a Ponzi scheme, we get the following result:\footnote{We could allow for bubbles or Ponzi schemes by the government. As long as their value does not depend on inflation, then proposition 1 below is unchanged.}

\[ W_0 = \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( \frac{B_0^t}{P_t} \right) \right] + \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} K_0^s \right] = \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} s_t \right]. \] (5)

The first equality provides a workable measure of the time zero debt burden. The government liabilities are fixed at time zero, so this expression depends only on real discount factors looking into the future and on the distribution of all future price levels. Moreover, it shows that because the real payments do not depend on inflation, we can focus on the nominal debt to assess the effect of inflation.

The second equality shows that we can interpret our measure as saying how much fewer taxes the government can collect by lowering the debt burden. Higher inflation may not only lower the real payments on the outstanding nominal debt, but also change primary fiscal surpluses.\footnote{In Section 5 we discuss the relation between our results and theoretical macro models that satisfy the intertemporal budget constraint.} In companion work (Hilscher, Raviv and Reis, 2018), we measure one of these effects through the seignorage revenues that higher inflation generates. Our purpose in this paper is to measure the effect of inflation on the outstanding public debt, and expression (5) shows that this depends on outstanding nominal debt alone.

### 2.4 Inflationary debasement: A formula for the debt burden as a weighted average

The price level is the only source of uncertainty in determining how much the outstanding nominal debts that mature in \( t \) periods will pay in real terms. Therefore, even though the stochastic discount factor depends, in principle, on all sources of uncertainty in the economy, only its marginal distribution with respect to inflation will lead to non-zero terms once we multiply by inverse inflation and take expectations. Therefore:

\[ \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( \frac{B_0^t}{P_t} \right) \right] = \sum_{t=0}^{\infty} B_0^t \mathbb{E} \left( \frac{m(\pi_{0,t})}{\pi_{0,t}} \right). \] (6)
where $\pi_{0,t} = P_t/P_0$ and $P_0 = 1$. Next, recall the standard definition of a risk-neutral density $f(.)$, which is equal to the product of the physical density times the stochastic discount factor divided by the expectation of this product. Recall as well that $R_t = \mathbb{E}(m(\pi_{0,t}))^{-1}$ is the real risk-free return between 0 and $t$ from the perspective of date 0. By definition, it does not depend on future realizations of inflation. Then, it follows that:

$$
\mathbb{E} \left( \frac{m(\pi_{0,t})}{\pi_{0,t}} \right) = R_t^{-1} \int \left( \frac{f(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t}.
$$

Combining all the results gives our formula for the debt burden as a function of inflation.

**Proposition 1.** The debt burden is a weighted average of the nominal payments that the government must make at all present and future dates:

$$
\sum_{t=0}^{\infty} \omega_t B^t_0
$$

with weights given by:

$$
\omega_t = R_t^{-1} \int \left( \frac{f(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t}.
$$

Our new measure, the distribution of inflationary debt debasement, relies on this formula. The current market value of debt is the expected value of future promised payments. The distribution of debasement results from the dispersion around this mean, specifically the probability of extreme positive shocks to inflation. Recalling the four desired properties of a measure stated in the introduction, this formula can satisfy them all.

First, the formula is forward looking, linking future inflation to the debt burden today, and it provides a way to aggregate all the possible different paths for inflation. It forms the basis for computing a value-at-risk measure of debt debasement: the probability that the reduction of debt value via inflation lies above a certain cutoff.

Second, it shows that what matters are the promised nominal payments at different horizons to the private holders of debt. The maturity of the debt may change in the future in arbitrary ways, but from today’s perspective, the ability to inflate current debt depends only on today’s distribution, which is included without any approximations. The next section explains how we measure it for the United States.

Third, the formula explicitly takes into account that inflation is stochastic and not perfectly controlled by policy. However, the foreseen paths for inflation are not arbitrary, but
rather are disciplined by market expectations as revealed by the joint distribution of inflation which reflects its time-series properties. The probabilities depend both on shocks to fundamentals, as well as on potential changes in the policy regime as perceived by the markets.

Fourth, it depends on inflation alone, as all of its relevant effects on prices are captured in the inflation densities. Beyond the direct effect of inflation on the real payments, this also includes the possible effect of inflation on real interest rates through Fisher effects, or changes over time in the compensation for inflation risk, or in liquidity premia. The relevant densities are for risk-adjusted inflation, so factors that affect bondholders’ views and market prices are captured.

3 Data: U.S. Treasuries by holder and maturity and distributions of future inflation

The two inputs required by proposition 1 are the maturity structure of privately-held nominal government debt and the joint risk neutral distribution of future inflation. We next provide data and methods to construct these two.

The Treasury reports the total U.S. federal debt at the end of 2017 as $20.5 trillion, or 103.6% of GDP. Yet, this number does not distinguish between nominal and indexed bonds, nor does it include information about the maturity structure. Moreover, it includes debt held by different branches of the U.S. government. We want to measure government liabilities held by the private sector since any gains of the Treasury at the expense of another government account, will sooner or later have to be covered by the Treasury.

3.1 Holdings of U.S. Treasuries at different maturities

The appendix describes our multiple data sources. The main source of data is the Center for Research on Security Prices (CRSP) that reports the private holdings of all outstanding marketable government notes and bonds at the end of 2017.\footnote{We build on Hall and Sargent (2011) and significantly extend their work, to a monthly rather than annual distribution of maturities, to consider different claimholders, and to use more sources of data.} We measure monthly total nominal payments, using both face value and coupons at each maturity. We adjust the size of payments to exclude holdings by state and local governments. We then use the nominal yield curve to calculate market values.
At the end of 2017, the total market value of privately-held marketable nominal Treasury securities was $9.6 trillion, or 49.6% of GDP.\(^{16}\) This baseline number is much smaller than the $20.5 trillion headline number reported by the Treasury because it does not include holdings by other branches of government and real securities.\(^{17}\) The headline number includes non-marketable securities ($6.0 trillion), the vast majority of which are part of intragovernment holdings ($5.7 trillion). The largest single such holding ($2.8 trillion) is the Federal Old-Age and Survivors Insurance Trust Fund (Social Security). Excluding non-marketable debt therefore does not materially affect our results since we would be excluding most of it on account of it being intragovernmental. The headline number also includes the amount of outstanding TIPS ($1.3 trillion), which are almost exclusively held by the private sector, but which will retain their real value in the face of inflation. We also exclude debt held by state and local governments, especially in state and local pension funds.\(^{18}\) Holdings are approximately 7.6% of total privately held debt. Privately-held debt also does not include holdings of the Federal Reserve. Any losses on the portfolio of the central bank will map directly into smaller seignorage payments to the Treasury (Hall and Reis, 2013), so the same argument that excludes Social Security holdings applies to Fed holdings ($2.3 trillion face value).

We break up total private holdings into domestic and foreign by maturity. The market value of foreign held debt is $6.3 trillion or 32.4% of GDP.\(^{19}\) Depending on the size and maturity breakdown of foreign and domestic holdings, debasement will affect the two groups differentially.

Though not included in our baseline debt number, we assemble data on Fed holdings. Debasement of these is of independent interest due to the unprecedented size of its balance sheet. The Federal Reserve Bank of New York keeps the information on each bond held by the Federal Reserve in its System Open Market Account (SOMA). We use this information to also obtain detailed holdings for the Federal Reserve at each maturity. At 12.6% of GDP, the value of the Fed portfolio is a little smaller than domestically held debt in private hands.\(^{20}\)

\(^{16}\)All market values are calculated using the zero coupon yield curve provided by Gürkaynak, Sack and Wright (2007). We ensure that the link between the maturity structure of nominal yields, inflation distributions, and the real yield curve holds by constructing a real yield curve according to equation 9. Using instead real yields provided by Gürkaynak, Sack and Wright (2010) does not affect our results.

\(^{17}\)In the following we focus on face values as reported by the Monthly Statement of the Public Debt.

\(^{18}\)We do not include debt issued by state and local governments, estimated by the Census to be $3.0 trillion in 2016. There is a large variety of these debt instruments, and no good source that reports their private holdings. We therefore restrict ourselves to federal debt.

\(^{19}\)This probably over-states how much foreigners ultimately own, since they include holdings in offshore financial centers, some of which may held by American citizens.

\(^{20}\)We do not include reserves issued by the central bank, not because they are not government debt, which
Figure 1 shows the maturity structure of privately-held nominal debt ($B_0^b$) using monthly data. A noticeable feature of the distribution is how concentrated it is on the short end. The maturity distribution decays very quickly in the first few years, then flattens between 5 and 10 years, becomes close to zero between 10 to 20 years partly because of the lack of issuance of 30-year bonds between 2001 and 2006, and then picks up starting at 20 and especially from 25 to 30 years. The figure shows that simple approximations, like assuming a single bond with a maturity equal to the average, or assuming an exponential distribution, are not appropriate. The average maturity of the U.S. government debt, weighted by present value of private holdings, is 5.1 years according to our calculations. The maturity structure and average maturity imply that there is scope for debasement but the concentration at short maturities means that inflation in the near term will have the highest impact.

Domestic and foreign holdings as well as Fed holdings have different maturity structures and will, therefore, have different exposures to inflation. Figure 2 aggregates to the annual frequency, and plots the foreign holdings (so domestic is the difference between private and foreign), together with the Fed’s holdings. Private domestic holdings of U.S. Treasuries are concentrated in low maturities, with little holdings above 7 years. Foreigners account for most of the holdings above 5 years, and the Federal Reserve holds a disproportionate share of maturities above 5 years partly due to quantitative easing policies. For comparison, we also assemble data on those real liabilities that are possible to measure with at least some accuracy.

Figure 3 plots annual outstanding TIPS and Social Security liabilities up to 2030, so 13 years out. TIPS liabilities follow a similar pattern to nominal liabilities and are much smaller. Social Security liabilities combine Old-Age and Survivors Insurance (OASI) and Disability Insurance (DI) projected shortfalls, the difference between direct program revenues, thus excluding trust fund interest payments, and costs. These numbers are put together by the Social Security Administration (SSA) in order to project forward what will happen to the trust fund. If Social Security benefits are not reduced, these real liabilities may become quite large, especially at longer maturities. However, the liabilities are real since benefits are indexed to inflation, and therefore cannot be inflated away.

3.2 The marginal densities of expected inflation

We now turn to assembling estimates of inflation distributions. We first construct the maturity structures of marginal risk-adjusted distributions of cumulative and one-year forward
Figure 1: Government payments due to the private sector by maturity ($B_0^t$)

Figure 2: Breakdown of annual payments by holder of debt, by maturity
inflation. We construct these distributions using data from the growing market for inflation derivatives.\textsuperscript{21}

### 3.2.1 Data on zero-coupon and year-on-year inflation options

The market for over-the-counter U.S. dollar (USD) inflation options emerged in 2002 and it has grown at a very fast rate, especially after 2008. Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2017) use data from these markets as well, and argue that since 2009 the market has been liquid enough to reliably reflect market expectations of inflation.\textsuperscript{22} We use daily data of caps and floors on the CPI for January 9 of 2018 from Bloomberg.\textsuperscript{23}

Like Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2017), we use data on zero-coupon caps and floors that pay off if cumulative inflation between the start

\textsuperscript{21}The appendix contains more details on the data, the method to estimate the distributions, and several robustness checks.

\textsuperscript{22}Kitsul and Wright (2013) write that by 2011, trading in the inter-dealer market was close to $22 billion, while J.P. Morgan (2013) estimates the annual trading volume in inflation derivatives at the start of 2013 was $50 billion.

\textsuperscript{23}The appendix looks at dates nearby to ensure there was nothing special about this particular date.
of the contract and its maturity lies above or below the annually compounded strike price. The strike price ranges from $-2\%$ to $3\%$ (floors) and $1\%$ to $6\%$ (caps), in $0.5\%$ increments. We have data for all maturities between 1 and 10 years, together with data for 12 and 15 years, which we will use to check our estimates.

Unlike Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2017), we also use data on year-on-year inflation caps and floors. These contracts are portfolios of caplets and floorlets that pay off at the end of each year if inflation during that year is above or below the strike price. Strike prices range from $-3\%$ to $6\%$. These contracts thus depend on inflation realizations during one-year periods in the future. There seems to be no market demand for instruments with strike prices outside of this range, and contract prices imply that risk-adjusted probabilities of lying outside of these limits are close to zero. We use year-on-year options with maturities that range from 1 to 10 years.

### 3.2.2 Recovering the risk-adjusted probability density of inflation

As the classic work of Breeden and Litzenberger (1978) noted, given a rich enough set of option contracts with observable prices, it is possible to recover non-parametrically the risk-neutral distribution of inflation without making any specific distributional assumptions about inflation or its link to other asset prices. The formula linking the risk-adjusted distribution of inflation and the price ($a_0$) and strike ($\pi_{0,t}$) of an option with maturity of $t$ years is:

$$f(\pi_{0,t}) = \pi_{0,t} R_t \left( \frac{\partial^2 a_0}{\partial \pi_{0,t}} \right).$$

(10)

Intuitively, if the price of a call option declines quickly with the strike price, then the outcomes around this strike price are more likely. Because our data give us many option prices for different strike prices and at different intervals, we can estimate this partial derivative by using the differences in these prices.

Instead of point expectations of future inflation, we recover risk-adjusted distributions for future inflation. That is, the distributions reflect the likelihood of different values of inflation, the risk associated with them, and the market price of this risk. This distinguishes risk-neutral distributions from many of the common measures of inflation expectations. Unlike opinion surveys, we are extracting risk-neutral rather than subjective expectations, and we do so from observing profit-making behavior. Unlike the break-even rate of inflation from comparing real and nominal yields, we have a whole distribution for inflation instead of a single number. Moreover, we do not need to worry about the price of the embedded floor
which ensures that TIPS always pay back at least par value. Finally, unlike models of the term structure that use the yield curve to extract market-based inflation expectations, these densities do not rely on the associated (often strong) identifying assumptions.

The distributions for expected inflation recovered from options reflect the expectations of participants in this market. These are mostly pension funds, life insurance companies, and banks, who need to hedge the inflation risk of many of their liabilities, together with speculators placing bets on inflation. These are also some of the major holders of the public debt, whose positions would be inflated away. Therefore, theirs are the relevant expectations for our valuation question. As with any measure of financial prices, one may worry about trading and liquidity frictions driving a wedge between prices and expectations. Applying our method to high-frequency questions, such as how would the debasement distribution change from one day or week to the next in the midst of a financial crisis could be problematic. At the same time, for low-frequency questions such as the one we pose here, any sluggishness of adjustment of prices should not be too important, and we have checked that our results are robust to using options prices from one month before and after. Moreover, we have checked that distributions react to news at daily frequencies at least in the right directions.

In order to construct distributions, the option data require considerable amount of cleaning. Aside from measurement error, we face the difficulty that the options in general are not traded simultaneously, resulting in option pricing functions that are not always well behaved. To screen out such data, we first drop option prices from the data if they contain simple arbitrage opportunities: (i) if the call (put) premium does not monotonically decrease (increase) in the strike price, (ii) if the call option premium does not increase monotonically with maturity, and (iii) if butterfly spreads that correspond to Arrow-Debreu securities do not have positive prices. Next, before we take differences of the data, we transform prices to implied volatility space: for each maturity, we calculate Black and Scholes (1973) implied volatilities for all strike prices, smooth them, convert back to option prices, and construct distributions according to equation (10). We provide additional details and discussion in the appendix.

3.2.3 Zero-coupon and year-on-year distributions

Using our data on zero-coupon floors and caps, we extract the density $f(\pi_{t,t+j})$, with $j = 1, \ldots, 10$. This gives a term structure of the cumulative risk-adjusted inflation distributions.

\footnote{Note that the implied volatility smoothing using Black and Scholes does not mean that we are using this model to price the options. It is simply a non-linear transformation of the pricing function.}
In turn, using data on year-on-year contracts, we construct one-year forward risk-adjusted distributions \( f(\pi_{t+j-1,t+j}) \). Figures 4 and 5 plot the distributions at the end of 2017.

The one-year mode of inflation is 2%.\(^{25}\) All of the distributions in figure 3 are bell-shaped and their medians and modes are all between 1.75% and 2.25%; the means all lie between 2.0% and 2.3%. All of the distributions have fat tails and are significantly non-normal. Depending on the maturity, excess kurtosis is between 3.5 and 16.6. Kitsul and Wright (2013) interpret the tails as reflecting investors’ perception that both very high and very low inflation are the costly states of the world. The patterns for the one-year forward distributions are similar, except that the means at longer maturities are a little higher, rising from 2.0% (1-year) to 2.6% (10-year).

An interesting feature of the cumulative distributions is that, as the horizon increases, the variance is close to constant, and does not fall, something we would expect if the variance in one-year forward inflation does not rise significantly with the horizon and shocks across years are imperfectly correlated. The standard deviation for cumulative inflation rises slightly from 0.78% at maturity 5 to 0.94% for maturity 15. This could be either because extreme events far in the future are perceived as more costly, or because there is more uncertainty about inflation. For year-on-year inflation, the standard deviations range between 0.8% and 1.9%, but standard deviations from 4 to 10 years all lie between 1.7% and 1.9%.

Overall, the distributions show that the probability of average annual inflation exceeding 4% is below 1% across all horizons up to 10 years, and never exceeds 1.6%; the probability of one-year forward inflation above 4% lies above 3% only at the 3 and 4-year horizons and lies below 3% for all other years. Sustained high risk-adjusted inflation is perceived as being an extremely remote possibility.

### 3.3 The joint distribution of inflation

From these marginal distributions we next construct joint risk-adjusted distributions of inflation across multiple years. Why do we need the joint distribution and how does the data that we have allow us to construct it? In order to construct a distribution of future inflation paths, one needs to know the joint distribution of inflation in year one (over the next year) and year two (the following year). Only then can one figure out the conditional distribution in year two based on a given realization in year one. Data on cumulative and one-year forward distributions provides information on co-movement across years. For example, if

\(^{25}\)Because of the 0.5% granularity of our data, we report the probability of lying between 1% and 1.5% as a probability mass at 1.25%. The probabilities plotted at 1.75% and 2.25% are almost exactly the same.
Figure 4: Marginal risk-adjusted distributions for cumulative annualized inflation

Figure 5: Marginal risk-adjusted distributions for year-on-year inflation
the correlation across years is high, then the two-year cumulative distribution will have a relatively higher dispersion compared to the case where the co-movement across years is low. This section turns this simple observation into a method to estimate the joint distribution of inflation over time.

3.3.1 A method of moments copula-based estimator

Understanding how the realizations of a random variable are related over time is, of course, the classic problem in time-series modeling. Our particular data on inflation contracts provides a novel way to approach this problem that has the intriguing feature of only using data at one date in time. We do so by exploiting agents’ expectations in financial markets to recover their beliefs about the time series of risk-adjusted inflation.

Consider the problem of obtaining the risk-adjusted joint density between annual inflation over the next two years: \( f(\ln \pi_{t,t+1}, \ln \pi_{t+1,t+2}) \). Sklar (1959) shows that there exists a copula function \( C(.): [0, 1]^2 \rightarrow [0, 1] \) such that:

\[
f(\ln \pi_{t,t+1}, \ln \pi_{t+1,t+2}) = C(f(\ln \pi_{t,t+1}), f(\ln \pi_{t+1,t+2})).
\] (11)

This function captures the co-dependence between the two random variables, so that we can obtain the joint density given information on the marginals. We use a parametric version of the copula function \( \hat{C}(f(\ln \pi_{t,t+1}), f(\ln \pi_{t+1,t+2}), \rho) \), so the unknown copula function is fully characterized by a vector of parameters \( \rho \) in the known function \( \hat{C}(.) \), where \( \rho \) is of dimension \( M \). The typical approach in the literature that estimates copulas would be to use the time series for past inflation to estimate both the marginal densities and the parameters in \( \rho \). Our unusual data allows us to approach the problem differently.

To start, we already have estimates of the marginal densities of forward inflation implied by year-on-year inflation contracts. Moreover, from the zero-coupon options, we also have another marginal distribution: \( f(\ln \pi_{t,t+2}) = f(\ln \pi_{t,t+1} + \ln \pi_{t+1,t+2}) \). From the definition of the distribution:

\[
f(\ln \pi_{t,t+2}) = \int_{\ln \pi_{t+1,t+2} = \ln \pi_{t,t+2}} \hat{C}(f(\ln \pi_{t,t+1}), f(\ln \pi_{t+1,t+2}), \rho) d\pi_{t,t+1} d\pi_{t+1,t+2}. \quad (12)
\]

Since we have \( N \) bins for the marginal distributions, this expression gives \( N \) moment conditions with which to estimate the \( M \) unknown parameters in \( \rho \).

The appendix extends this logic to show that:
Proposition 2. Given data for the marginal distributions of spot (cumulative) inflation \( f(\ln \pi_{t,t+j}) \) and forward (year-on-year) inflation \( f(\ln \pi_{t+1,t+1}, \ln \pi_{t+2,t+2}, \ldots, \ln \pi_{t+J,t+J}) \) for \( j = 1, \ldots, J \), one can estimate the joint distribution \( f(\ln \pi_{t,t+1}, \ln \pi_{t+1,t+2}, \ldots, \ln \pi_{t+J-1,t+J}) \) by estimating the \( M \) parameters in the \( \rho \) vector that satisfy the \((N-1)(J-1) \geq M\) conditions:

\[
f(\ln \pi_{t,t+j}) = \int_{\Pi} \hat{C}(f(\ln \pi_{t,t+1}), \ldots, f(\ln \pi_{t+J-1,t+J}), \rho) d\ln \pi_{t,t+1} \ldots d\ln \pi_{t+J-1,t+J}.
\] (13)

The integration set \( \Pi \) is such that: \( \ln \pi_{t,t+1} + \ldots + \ln \pi_{t+J-1,t+J} = \ln \pi_{t,t+j} \), for \( j = 1, \ldots, J \).

We can use these moments to estimate \( \rho \), akin to GMM, although these are not moments of the distribution of the random variable, as is usual, but rather the distributions themselves.\(^{26}\)

3.3.2 The estimates

Proposition 2 allows one to estimate a very general process for inflation, since there are 180 moments to use with our inflation distributions covering 10 years. We find that a simple process with a few parameters fits the data quite well.\(^{27}\) First, we use a multivariate Gaussian copula, which has a single parameter to model the co-dependence of inflation over two horizons. This does not impose normality for inflation, it simply assumes that the joint dependence of inflation over time resembles a normal distribution in the sense that if the marginals were normal, then the multivariate would be normal too. Second, we assume that inflation dynamics have a stationary and a non-stationary component, and that the non-stationary component is a random walk. This is a generalization of the common Beveridge and Nelson (1981) model for inflation that is frequently used. This parsimonious model has only 10 parameters, all of which are easy to interpret: the relative variance of the non-stationary and stationary components of inflation, and the 9 autocorrelations of the stationary part. We search for the parameters that minimize the equally-weighted squared deviations of the moments in proposition 2. The model fits the data quite well, in the sense that the discrepancies between the two sides of the moment conditions in proposition 2 are small.\(^{28}\)

\(^{26}\)Our data on options contracts only goes to 10 years, but the debt maturity goes all the way to 30 years. For inflation beyond 10 years, we extrapolate by assuming that the joint distribution is a stationary Markov process of order 9 with parameters given by the distribution from 1 to 10 years. The appendix discusses the details.

\(^{27}\)Choosing a simple special case of the general process aids interpretation. In addition, as discussed below, an out-of-sample test supports our choice.

\(^{28}\)See the appendix for additional details.
Our first finding is that, according to the options data, risk-adjusted inflation is generally close to being a stationary process. We estimate the variance of the innovations to the random walk component of risk-adjusted inflation to be small. Its ratio to the variance of the innovations to the transitory shocks is 0.09 in 2017 and lies above 0.25 only in 2010 and 2011. This suggests that in risk-adjusted terms, long-run inflation expectations are well anchored. To understand what drives this result in the options data, note that if risk-adjusted inflation were non-stationary then the variance of year-on-year inflation should tend to increase with the horizon. Yet, figure 5 shows that after 5 years the distributions do not spread out in a significant way.\textsuperscript{29}

The second interesting result is that the correlation coefficients do not noticeably fall with the horizon. Table 1 shows the estimated correlation parameters across maturities. Even as far as eight years into the future, the autocorrelation is still 0.42. In spite of the evidence for stationarity, risk-adjusted inflation is a process with a long memory, where shocks persist for many years. Intuitively, figure 4 shows that the distribution of cumulative average inflation does not shrink with the horizon, so risk-adjusted inflation must be expected to revert slowly to its long-run mean.\textsuperscript{30} The figure shows that the distribution of inflation only gradually shifts to the right with the horizon, so the expected reversion to the mean is quite slow.

The last column of table 1 relaxes the restriction of the Beveridge-Nelson model, estimating instead $M = J(J - 1)/2 = 45$ parameters corresponding to the number of cross-horizon correlations for inflation. While the estimates move around more across horizons, the two general conclusions are similar. Interestingly, we find that the in-sample fit is only slightly better than the restricted model. There is an out-of-sample test of our estimation procedure that uses only the maturities 1 to 10, which consists of comparing the copula model’s prediction for cumulative inflation in years 12 and 15, for which we have data. We cannot use these data for estimation purposes since the maturity of the year-on-year contracts does not extend that far but we can use the data for this test. We find that the restricted model performs either noticeably better or similar for all of the years.

\textsuperscript{29}The standard deviation is equal to 2.3\% for 5 years and 2.6\% for 10 years.

\textsuperscript{30}Curiously, this is consistent with the findings of Gürkaynak, Sack and Swanson (2005) who also use market expectations of the future in forward contracts to find that 10-year forward interest rates are more correlated with 1-year interest rates than what usual mean-reverting models of short term rates would suggest.
Table 1: Estimated correlation coefficients of year-on-year inflation, joint distribution

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Restricted model</th>
<th>Unrestricted (average)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.11</td>
<td>0.24</td>
</tr>
<tr>
<td>2</td>
<td>-0.45</td>
<td>0.09</td>
</tr>
<tr>
<td>3</td>
<td>-0.38</td>
<td>0.11</td>
</tr>
<tr>
<td>4</td>
<td>0.04</td>
<td>0.15</td>
</tr>
<tr>
<td>5</td>
<td>0.30</td>
<td>0.14</td>
</tr>
<tr>
<td>6</td>
<td>0.07</td>
<td>0.05</td>
</tr>
<tr>
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</tr>
<tr>
<td>8</td>
<td>0.42</td>
<td>0.30</td>
</tr>
<tr>
<td>9</td>
<td>0.08</td>
<td>0.54</td>
</tr>
</tbody>
</table>

Notes: Estimated correlation coefficients for year-on-year inflation between 2017 and 2017+j, where j is the maturity in each row.

4 Results: debt debasement distribution and stress tests

We now combine the joint distribution for risk-adjusted inflation with the data on holders of nominal government debt. Different scenarios have probability weights disciplined by the data giving rise to a distribution of debt debasement that we now characterize.

4.1 Value-at-Risk measures of debasement

Proposition 1 provides a way to measure the probabilities of debt debasement of various sizes. Using our estimated risk-adjusted joint distribution for inflation, we draw a large number of paths and, ordering them by their impact on the real value of debt, we estimate probabilities that debt will fall by more than some threshold. This is akin to measures of risk-adjusted Value-at-Risk (VaR).\textsuperscript{31} These measures take into account all possible future

\textsuperscript{31}Veronesi and Zingales (2010) and Kocherlakota (2012) point out that risk-adjusted probabilities are appropriate when evaluating the effects of policies on welfare; see also Ait-Sahalia and Lo (2000) and Jackwerth and Rubinstein (1996). As we discuss later in section 4.4.1, constructing value-at-risk measures using physical probabilities instead would require knowledge of the joint distribution of inflation and the stochastic discount factor. Standard value-at-risk measures are calculated for specific time horizons. Proposition 1 shows that we are calculating potential losses over the entire life of the outstanding payments and then
scenarios for inflation as perceived today by market participants, including changes in the policy of the central bank, shocks to inflation beyond the control of monetary policy, and current as well as possible changes to future risk-attitudes towards inflation.\textsuperscript{32}

Figure 6 shows the probability that the fiscal burden will fall by more than a few percentage points of GDP, according to the risk-adjusted distribution for inflation. Strikingly, the numbers are all quite small. The probability that debt falls by more than 4% of GDP is 0.3%. Having the real value of the debt fall by at least 1% of GDP due to inflation variation is likely, with a probability of 47% but anything more than a mere 3% of GDP has the small probability of 5.5%.

Table 2 presents percentiles of the distribution of debasement separately for each investor. Most of the gains are at the expense of foreigners. This happens not just because they hold more debt than domestics, but mostly because they hold longer maturity debt. Therefore, extreme situations where a succession of high realizations of annual inflation lead to large cumulative inflation affect foreigners more than domestics.\textsuperscript{33}

The last column in the table shows the effect on a non-private holder of debt, the central discounting back to today.

\textsuperscript{32}Because they include perceptions of different policy regimes, these measures are robust to the Lucas critique, as pointed out by Sims (1982). Moreover, because Proposition 1 calls for risk-adjusted probabilities, these are the ones used here. We discuss at the end of this section how to interpret these, and how our results extend to using physical probabilities.

\textsuperscript{33}From the perspective of foreigners, how much they ultimately lose also depends on how much the exchange rate devalues with the increase in inflation and on whether there are taxes on capital flows. We are not measuring the foreigners’ losses. Rather, we are measuring how much the fiscal authority would gain at their expense. Our measure does not depend on whether the holder is domestic or foreign, or what happens to the exchange rate, even if these factors would be important for foreign debtholders.
bank. The Federal Reserve would potentially suffer large losses as a share of its portfolio. Even though it holds about the same debt as domestic private investors, the central bank loses significantly more than they do. At the end of 2017, the central bank held mostly long-term bonds, which are more exposed to inflation risk.

4.2 Inflation counterfactuals

The magnitude of debt debasement in figure 6 and table 2 is surprisingly small. To understand why this is the case, we consider a series of counterfactuals that shift the inflation distributions. In the next subsection, we consider instead the effect of changing the distribution of debt.

For each counterfactual, we propose a new distribution \( \hat{f}(\cdot) \), recalculate the real value of the debt using equation (8), and subtract it from the market value of debt to obtain our estimate of the fall in real debt.\(^{34}\) This provides the stochastic equivalent of asking what would happen if inflation was \( x\% \) higher but for a set of inflation paths. We discipline the experiments using the data by pinning down the shift \( x \) to be consistent with the plausible set of scenarios in our original distribution \( f(\cdot) \). Importantly, these are shifts in the risk-adjusted

\(^{34}\)The appendix provides more details.
distribution for inflation, not changes in inflation per se. They are useful to understand how different properties of inflation affect the debt debasement effect, but they do not cleanly correspond to actual policy changes.\footnote{Unlike the Value at Risk results in the previous section, but like any counterfactual experiment, these stress tests are potentially exposed to the Lucas critique. However, because these are shifts to risk-adjusted distributions they do incorporate agents’ shifting expectations about inflation and risk. It is not the shifts per se that are subject to criticism, but only their interpretation as policy changes.}

The first experiment shifts horizontally the marginal distributions for annual inflation at every maturity so that the new median is at the old 90th percentile. One can think of this experiment as capturing an announcement that the inflation target of the Fed is now expected to be higher, as suggested by Blanchard, Dell’Ariccia and Mauro (2010).\footnote{Again to be clear, we are shifting the risk-neutral distribution of inflation, not the actual inflation target of the central bank. The link between the two depends both on the effectiveness of central bank policy as well as on changes in private assessments of risk. In the extreme case where the central bank controls the distribution of inflation, and where inflation is “pure” in the sense of Reis and Watson (2010), so that changes in inflation are independent of changes in relative prices, the two are the same.} The second experiment instead sets the density below the 90th percentile to zero. Probabilities are then scaled up proportionately to integrate to one. This corresponds to a commitment that inflation will be higher for sure in the future. Only inflation realizations at the right tail of the current distribution become possible. At the same time, because there is no shift to the right as in the first case, very high levels of inflation are also not that likely.

The three next stress tests consider the effect of uncertainty. The third experiment again shifts the marginal density so the new median is the old 90th percentile, but now this is accomplished by scaling inflation proportionately at every maturity. It is often said that higher average inflation comes with more variable inflation, and this experiment tries to capture this possibility. In the other direction, in the fourth counterfactual, we assume that year-on-year inflation is exactly equal to average inflation in the first stress test. This shows what would happen if inflation became deterministic, so we can understand better the effect of volatility. The fifth counterfactual studies a shift in inflation that is partially expected. We assume that after an initial unexpected jump of inflation upwards, the distribution of inflation looking forward is equal to the conditional distribution that we have estimated. Therefore, whereas in the previous experiments all of the changes at all maturities were unexpected, now only the change in the first year catches agents by surprise, but they adjust their expectations right after.

So far, the experiments assumed that the risk-adjusted inflation distribution would change immediately and permanently. The sixth stress test considers instead a temporary increase in inflation, with the distribution for year-on-year inflation shifting rightwards so the new
median is at the 90th percentile the next year, but only at the 80th percentile the year after, and so on, so that for maturities above 5 years there is no change. The seventh case considers a gradual increase, with the one-year inflation distribution unchanged, while the two-year shifts horizontally so the new median is at the old 60th percentile, and so on until the fifth year, after which we have the same permanent shift as in the first case.

Table 3 shows the risk-adjusted harmonic mean of inflation for both the baseline and each of the counterfactuals at different maturities. Table 4 reports the effect of each of the counterfactuals on the real value of the debt. All of the experiments have a relatively small impact on the real value of the debt. Ensuring larger inflation, as in experiment 2, lowers the fiscal burden by 3.9%, the highest number in the table. More uncertainty lowers the effectiveness of inflation at debasing the debt because the real value of future nominal payments are convex in inflation, so uncertainty raises their value and lowers the benefits of raising inflation. If agents adjust their expectations after one year of surprise inflation, as in case 5, the estimates are significantly smaller. Moreover, the last two rows in table 4 show that more reasonable time paths for inflation than a sudden and permanent change also cut significantly the effect of inflation on debt. Finally, confirming our previous results, again foreigners absorb a large share of the losses, and the Federal Reserve has likewise large losses relative to the size of its portfolio.

The reason for these small effects is due to the interaction of two features of the inflation data. First, large increases in inflation are unlikely. Even in the most extreme case in table 3, expected inflation in one year never increases by more than 1.6% relative to the baseline. From the perspective of actual market-based distributions, anything larger than this seems unreasonable. But from the perspective of debt valuation, such modest increases in inflation are incapable of generating large debt debasement. In order to raise the debt debasement effect to 10%, it would take shifting all year-on-year distributions of inflation in experiment 1 by 6.5% to the right. For the first year, the new median would be 8.5%, and the shift is equal to five and a half standard deviations of the average volatility for the first three years. Therefore, in that case, the new and old distributions for inflation would have close to zero overlap, making this scenario literally incredible.

Second, as table 1 shows, the persistence of inflation is quite low in the first few years, closer to the temporary increase in our sixth counterfactual. While shocks then persist for many years, the increase in the horizon implies that these effects are partially anticipated

---

37We take the harmonic, instead of an arithmetic mean, since proposition 1 shows that it is the expectation of the inverse of inflation that matters for debt valuation.
Table 3: Expected adjusted average annual inflation for different counterfactuals

<table>
<thead>
<tr>
<th>Distribution for inflation</th>
<th>Horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1-year</td>
</tr>
<tr>
<td><strong>Baseline</strong></td>
<td>2.0%</td>
</tr>
<tr>
<td><strong>Counterfactuals</strong></td>
<td></td>
</tr>
<tr>
<td>1. Permanently higher</td>
<td>2.9%</td>
</tr>
<tr>
<td>2. Right tail only</td>
<td>3.6%</td>
</tr>
<tr>
<td>3. Higher and more variable</td>
<td>2.9%</td>
</tr>
<tr>
<td>4. Higher for sure</td>
<td>2.9%</td>
</tr>
<tr>
<td>5. Partially anticipated</td>
<td>3.0%</td>
</tr>
<tr>
<td>6. Temporary increase</td>
<td>2.9%</td>
</tr>
<tr>
<td>7. Gradual increase</td>
<td>2.0%</td>
</tr>
</tbody>
</table>

Notes: Each cell reports $1/E(n/\pi_0,j)$ the harmonic mean of inflation until horizon j.

Table 4: Counterfactual impact of higher inflation on the real present value of debt

<table>
<thead>
<tr>
<th>Inflation counterfactual</th>
<th>Holders of the debt</th>
<th>Privately held (49.6%)</th>
<th>Domestic (17.2%)</th>
<th>Foreign (32.4%)</th>
<th>Central Bank (12.6%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Permanently higher</td>
<td></td>
<td>2.3%</td>
<td>0.9%</td>
<td>1.4%</td>
<td>0.8%</td>
</tr>
<tr>
<td>2. Right tail only</td>
<td></td>
<td>3.9%</td>
<td>1.8%</td>
<td>2.0%</td>
<td>1.5%</td>
</tr>
<tr>
<td>3. Higher and more variable</td>
<td></td>
<td>1.6%</td>
<td>0.5%</td>
<td>1.1%</td>
<td>0.5%</td>
</tr>
<tr>
<td>4. Higher for sure</td>
<td></td>
<td>2.8%</td>
<td>1.2%</td>
<td>1.6%</td>
<td>0.9%</td>
</tr>
<tr>
<td>5. Partially anticipated</td>
<td></td>
<td>0.4%</td>
<td>0.2%</td>
<td>0.2%</td>
<td>0.1%</td>
</tr>
<tr>
<td>6. Temporary increase</td>
<td></td>
<td>0.6%</td>
<td>0.2%</td>
<td>0.4%</td>
<td>0.2%</td>
</tr>
<tr>
<td>7. Gradual increase</td>
<td></td>
<td>1.5%</td>
<td>0.7%</td>
<td>0.8%</td>
<td>0.6%</td>
</tr>
</tbody>
</table>

Notes: Each cell shows the fall in the real present value of debt as a ratio of GDP.
as in our fifth counterfactual. As the two counterfactuals combined show, this reduces the effect on the real debt. This is due to the interaction of inflation with the debt maturity, which we discuss next.

4.3 The drivers of debasement

In this section, we examine variation in debasement probabilities over time and identify its key determinants: debt maturity, inflation volatility, and inflation persistence.

4.3.1 The role of debt maturity

Starting with debt maturity, over several years, an additional percentage point of inflation can accumulate to a large effect on debt debasement. If all privately-held debt was at long maturities, inflation might significantly reduce its real value. Table 5 investigates the effect of maturity on our estimates by considering the effect of higher inflation only on debt with maturity below 1 year, or only below 5 years. The numbers are significantly lower than when all the debt is included. Moreover, even though in 2017 almost three quarters of the market value of debt has a maturity below 5 years, the debt debasement effect is well below 0.75 of our estimates when all the debt is included. Combining the fact that the maturity of the debt is low, and that the serial correlation of inflation in the first few years is low, gives a low effect on debt debasement. Inflation persists into the more distant future, but by then there is not much debt to debase. In the table, the temporary increase (counterfactual 6) has a larger effect than the gradual (counterfactual 7) on debt with maturity below 5 years, since the former has a greater impact on the short end of the maturity structure. Thus, ceteris paribus, higher debt levels and a longer maturity will result in more debasement.

We repeat the work of section 3 building the distribution of privately-held debt across maturities, but now extending it back to 2009. Figure 7 shows the distributions cumulatively over time. As is well-known, the distribution has shifted up as the U.S. government issued more debt. Less appreciated, privately-held debt has become more concentrated at short maturities over time.

Figure 8 plots debt debasement value at risk levels from 2009 to 2017. There is substantial variation. In 2009 debasement of more than 2% was very unlikely; in 2011, only all the way to a debasement of more than 6% was now unlikely. Importantly, debasement does not increase primarily with the debt burden. If it did, we would see a monotonic increase in the value-at-risk probability of debasement, reflecting the monotonic increase in the debt
burden. Instead, we see that using only debt maturity or variables calculated from debt maturity provides an incomplete picture.

4.3.2 A simple, but misleading, rule of thumb

From proposition 1, we know that empirically evaluating inflationary debasement requires the simulation of future inflation paths and knowledge of the debt maturity distribution. A related measure is the *debasement sensitivity*, which uses only information on debt maturity and discount rates. This quantifies how much the public debt changes in response to a marginal change in inflation. A common rule of thumb used to evaluate the effect of \( \varepsilon \)% more inflation for sure and forever on the public debt multiplies the debt-to-GDP ratio by the average maturity of the debt.\(^{38}\) With the headline debt-to-GDP ratio, 103.6% in 2017, and an average maturity of 5.8 years, this common rule of thumb would suggest a sensitivity of 6.0%. Our estimates instead suggest that the probability of such a large sensitivity to an inflation shock today is virtually zero.

The rule of thumb is misleading and incomplete for a few reasons. First, because a

\(^{38}\)This rule of thumb comes from making the very strong assumptions that: (i) all debt is nominal, (ii) the maturity distribution is exponential with expected maturity \(m\), (iii) inflation is a constant \(\pi\), (iv) the nominal interest rate is constant and approximately zero, (v) the sensitivity is with respect to \(\varepsilon\) permanently higher inflation. Then, the fiscal burden is \(\int_0^{\infty} (B_t/P_t)e^{-rt}dt = \int_0^{\infty} (B_0(e^{-t/m}))/P_0 e^{\pi t}e^{-rt}dt = (B_0/P_0)(1/m - r - \pi)^{-1}\). A first order Taylor approximation for an \(\varepsilon\) increase in \(\pi\) at a point where \(r + \pi = 0\) then gives a debasement sensitivity of \((B_0/P_0)m\varepsilon\).
Figure 7: Debt maturity distribution 2009-2017

Figure 8: Probability that the fall in the value of debt exceeds a threshold, 2009-2017
sensitivity is not a distribution. At a minimum, the volatility of inflation must also play a role; without such an input it is not possible to move from sensitivity to distribution or value at risk. Simply put, if inflation volatility is zero, debasement sensitivity plays no role. Debasement will always be equal to zero. A back-of-the-envelope calculation might assume that inflation is normally distributed and has a standard deviation of 1% (its historical average in our sample). Then $\varepsilon = 2.33\%$ for the 1% VaR. We end up with a predicted 1% VaR of 14.1%, much higher than the actual 3.6%.

Second, the estimate above does not use the correct number for debt, which is the privately held one: 49.6% of GDP. By itself, this cuts the estimate in half.

Third, average maturity is not the correct measure to capture the sensitivity of real debt to the inflation term structure. A more accurate, but still highly imperfect, rule of thumb measures the response to a $\varepsilon$ parallel shift of the inflation distributions at all horizons, using the privately-held debt, and using information on future inflation distributions. Doing a first-order Taylor approximation on the size of the shift gives a debasement sensitivity equal to: debt value times $\varepsilon$ times the Fisher and Weil (1971) duration of the bond portfolio. While this formula is simple, note that calculating the Fisher-Weil duration requires knowing the entire distribution of privately-held bond holdings $\{B_t^d\}_t$ and associated prices. Therefore, calculating it requires the same debt maturity structure data that is needed for the accurate debasement distributions that we calculate.

Fourth, and even after all the adjustment above, a sensitivity measure is not forward looking. It is not the historical distribution of inflation, but rather future inflation that is relevant for debasement. Using the detailed knowledge of future inflation distributions we have constructed, we can calculate the mean and the 99th percentile of the inflation distribution in each of the next three years. This calculation requires almost the same data that we use for our exact calculations. The average difference is a good measure of the parallel shift needed. Using this approach, we get a 1% VaR of 7.0%, based on an inflation shift of 2.7%, a Fisher Weil duration of 5.1% and the debt-to-GDP ratio of 49.6%. The number is still quite a bit higher than our accurate estimate of 3.6%.

That is because of a fifth and final problem. This measure neglects the joint distribution of inflation over horizons. It assumes a permanent shift of inflation, that is, a parallel shift of inflation at all maturities. It thus ignores the stochastic properties of inflation, especially its persistence, simply assuming an annual serial correlation of inflation equal to one. In the data, this is 0.11 or 0.24 (restricted or unrestricted model), making parallel shifts extremely unlikely. In our 2009-17 data on inflation distributions, we never see such parallel shifts.
Assuming perfect correlation, as a calculation based on duration does, will result in an overestimate of debasement.

### 4.3.3 A more accurate rule of thumb

Figure 8 shows value-at-risk distributions. Another way to present the estimates shows instead the variation in value-at-risk levels over time at one level, say 1%. Figure 9 shows these: the debasement cutoffs, over time such that, with risk-adjusted probability of 1%, debasement will exceed this level. In 2009, it is equal to 2.1%, it then increases to 6% in 2011, and it stabilizes a little below 4% from 2015 to 2017.

The picture also shows a measure based on the Fisher-Weil duration, which assumes a 1% parallel shift of the entire inflation term structure. This measure, by contrast, increases over the sample period, missing the initial strong rise, subsequent fall, and following stabilization of the debasement VaR. The $R^2$ of a regression of actual VaR on Fisher-Weil predicted VaR is a modest 14%. That rule of thumb is highly inaccurate not just in the level, but also in the change, of the debasement distribution.

To bring persistence into the picture, we use the average one-year correlation from our
restricted model estimates. There is substantial variation in this measure over the years, which sometimes is close to zero and even negative. In the figure, we add one to the average correlation and divide by 100 to easily plot the numbers in the same graph; for example, in 2017 the average \((t, t + 1), (t + 1, t + 2), \ldots\) one-year correlation is 31%, which in the figure is shown as 1.3%. We then multiply our measure of persistence by the Fisher-Weil duration. This new rule of thumb, that combines information on the debt distribution with information on the perceived persistence of inflation, now explains 46% of the actual measure, according to the \(R^2\), more than three times what duration itself could explain. Looking at individual years, from 2013 to 2017 duration steadily increases but persistence first declines and then stabilizes; as a result, the debasement VaR is relatively stable.

Finally, we add the volatility of inflation shocks. We calculate the average annual volatility of year-on-year inflation for years one to three. Multiplying volatility with persistence and duration completes the picture. The predicted VaR, plotted in the figure, now leads to an \(R^2\) of 89%, another substantial increase. In the sample, there has been a large decline in inflation volatility, from 2.8% in 2009 to 1.2% in 2017. This further explains why, in spite of rising debt levels, debt debasement via inflation has remained unlikely through the sample.

In short, our new rule of thumb, the product of a duration-based measure, persistence, and volatility of inflation captures the variation in debasement probability well. It shows that the three necessary conditions for the presence of significant debasement probabilities are: (i) a long maturity structure, (ii) a reasonable probability of higher-than-expected inflation, and (iii) reasonable levels of inflation persistence.

### 4.3.4 Variation in GDP

The goal of this paper is to measure inflation-driven debt debasement. Debt-to-GDP may fall for other reasons, notably if GDP rises. Of clear interest is whether states of high inflation are also states of high growth, further lowering the real value of debt compared to GDP. While this does not answer the question of inflation-induced debt debasement directly posed in the introduction, it complements the analysis.

To accurately measure this perceived risk-adjusted correlation between inflation and GDP would require having traded options on the joint outcomes for inflation and GDP. These do not exist. We proceed by providing estimates based on a sample of historical data, and the (strong) assumption that the past empirical correlation of inflation and output provides a rough estimate of the expected risk-adjusted correlation.

To calculate it, first we estimate a structural vector autoregression on inflation, GDP
growth, and nominal interests rate using annual data and one lag (results are robust to more lags). At any horizon, this gives the response of inflation as the weighted sum of the three structural shocks. Then, taking a draw from the risk-adjusted inflation distribution, just as we do when constructing our VaR levels, we can back out the underlying vector of shocks that exactly match this draw at three different horizons (we do so at horizons 1, 3 and 5, but the results are robust to other choices). Given this vector of shocks, the vector autoregression then provides the matching impulse response for the corresponding draw of output, that has embedded in it the correlation between inflation and output at all horizons. Combining these output draws with the original inflation draws, we calculate the distribution of debasement of the debt. In order for GDP growth to materialize, we need to examine the distribution of debt-to-GDP levels in the future, not the present, and so we calculate maturity specific burdens of debt that is currently outstanding relative to future GDP levels.

To take away the effect of the correlation between output and inflation, we next identify one structural shock in the vector autoregression that has the properties that it has no contemporaneous effect on either interest rates or inflation. This is then a pure output shock, accounting for output volatility but not its correlation with inflation. Again, for each inflation draw, this identification restriction allows us to back out the pure output shock and thus the corresponding output draw.

A third and final exercise considers shutting off both output’s correlation and volatility, and simply adds a 2% growth trend to output paths for each inflation draw.

Figure 10 shows the maturity structure of 5% VaR debt-to-GDP reductions for these three cases. Note that adding up the whole term structure is similar, but not the same, as calculating the 5% VaR directly.\(^{39}\)

When the GDP growth path is fixed at 2%, so that only inflation is stochastic, the maturity structure of inflation-induced debasement reflects the drivers of our main results. At short maturities, because there is a lot of debt, and at long maturities because inflation may be more extreme at longer horizons. Allowing for GDP growth volatility, but no correlation with inflation, evaluates by how much extreme growth spurts can reduce the debt-to-GDP level for each maturity. Inflation is still allowed to vary and so large debt to GDP reductions can be caused by either inflation, growth, or both. The effects are unsurprisingly much larger, since volatility of GDP through the vector autoregression means that there are extreme cases on the tails where there are significant growth spurts in the future. Because there is a lot of

\(^{39}\)Our main calculation compares the discounted debt value relative to GDP today, whereas in this exercise we do not discount and instead compare today’s debt to future GDP. Our main calculation also finds the overall 95th percentile, which is not the same as the sum of the period by period 95th percentile levels.

37
debate at short maturities, debase from extreme shocks is large. The third case allows for correlation between shocks to inflation and GDP. Now, the extreme growth spurts can be times of high inflation, given the positive correlation of GDP growth and inflation shocks, especially at short horizons. As a result, the increased extreme debase relative to assuming no correlation is larger at shorter horizons. Once the horizon rises, the Phillips correlation implicit in the vector autoregression is smaller, and there is less debt, so the difference becomes negligible.

Altogether, the results highlight the important role that GDP growth can play in the evolution of the debt-to-GDP ratio. Hall and Sargent (2011) find that GDP growth was an important determinant of the historical evolution of the U.S. debt to GDP ratio since the Second World War. Our results complement their findings, from a forward-looking perspective.
4.3.5 Isolating the sources of debt to GDP changes over one year

The debt-to-GDP ratio may decline because of inflation and growth, as well as if the real interest rate used to discount future debt payments changes. Hall and Sargent (2011) find that this was also a significant determinant of the historical evolution of the debt-to-GDP ratio. This section offers a related exercise, but one that takes a forward-looking approach to decompose surprise changes in the debt-to-GDP ratio into surprises on the real return of the debt and the output growth rate every year over our sample period.

Adapting the approach of Campbell and Shiller (1988) and Campbell (1991), start with the unexpected real return of government debt. For expositional purposes, assume that there is a single zero-coupon bond with maturity \( T \). Over time, the cumulative nominal return of the bond is equal to the total discount at which it is currently trading. Breaking the return up into one-period returns and noting that the final nominal payoff of the bond is fixed, the surprise one-period real return is:

\[
\begin{align*}
    r_{t+1} - \mathbb{E}_t(r_{t+1}) &= -(\pi_{t+1} - \mathbb{E}_t \pi_{t+1}) - (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{s=t+2}^{T} (r_s + \pi_s).
\end{align*}
\]  

(14)

Lower case letters denote logs: \( r_{t+1} \) is the real return on the debt, \( \pi_{t+1} \) is the realized inflation rate, \( i_s \) is the one-period nominal interest rate, and by assumption \( \sum_{s=t+2}^{T} (r_s + \pi_s) = \sum_{s=t+2}^{T} i_s \). The real return on the debt is high if inflation is low or if real rates or expected future inflation have declined. The surprise percent change in the debt to GDP ratio is then equal to:

\[
(\mathbb{E}_{t+1} - \mathbb{E}_t)(w_{t+1} - y_{t+1}) = (\mathbb{E}_{t+1} - \mathbb{E}_t)(r_{t+1} - g_{t+1}).
\]

(15)

To get the unexpected change of the debt-to-GDP ratio from one period to the next, we scale this expression by the current debt-to-GDP. All combined, note that this is the change in the real value of currently outstanding debt, without including next period’s deficit.\(^{40}\)

Table 6 reports the one-year surprise changes for our sample period. We also report one-period surprise inflation and GDP growth, which are the direct inputs into the calculation. Changes in the term structure of future inflation and real rates are the remaining components; for these we report changes at the 10-year point. From the table we can see that changes in expected inflation and real rates are the main drivers of debt-to-GDP changes. The main

\(^{40}\)Doing so would require a forecast of next period’s deficit since we are interested in the surprise change of the debt to GDP ratio. Since the purpose of the current exercise is to understand the effects of inflation, GDP growth, and revaluation we choose to not model the deficit.
Table 6: Decomposing one-year surprise changes in the debt-to-GDP ratio

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Surprise debt/GDP change</td>
<td>0.83%</td>
<td>1.89%</td>
<td>0.39%</td>
<td>-0.99%</td>
<td>1.43%</td>
<td>-0.62%</td>
<td>0.05%</td>
<td>0.31%</td>
</tr>
<tr>
<td>Inflation surprise</td>
<td>0.30%</td>
<td>1.43%</td>
<td>0.08%</td>
<td>0.13%</td>
<td>-0.91%</td>
<td>0.96%</td>
<td>0.91%</td>
<td>-0.04%</td>
</tr>
<tr>
<td>GDP growth surprise</td>
<td>0.66%</td>
<td>-0.32%</td>
<td>0.27%</td>
<td>-0.12%</td>
<td>0.56%</td>
<td>0.95%</td>
<td>-0.32%</td>
<td>0.47%</td>
</tr>
<tr>
<td>Change in expected inflation</td>
<td>-0.05%</td>
<td>-0.29%</td>
<td>0.39%</td>
<td>-0.18%</td>
<td>-0.57%</td>
<td>0.10%</td>
<td>0.33%</td>
<td>-0.03%</td>
</tr>
<tr>
<td>Change in real rates</td>
<td>-0.60%</td>
<td>-1.27%</td>
<td>-0.55%</td>
<td>1.56%</td>
<td>-0.40%</td>
<td>0.03%</td>
<td>-0.16%</td>
<td>-0.08%</td>
</tr>
</tbody>
</table>

Notes: Surprise change in debt/GDP is the one-year surprise real return of current debt adjusted for real GDP growth and scaled by current debt/GDP; inflation surprise is the one-year surprise inflation, change in expected inflation and change in real rates are the one-year changes in the 10-year expected inflation and real interest rates.

reason is that the one-year horizon, at which inflation and GDP growth are measured, is short relative to the average maturity of the debt.\textsuperscript{41} For example, in 2011, even though there was a large positive inflation surprise, debt to GDP went up, driven by the reduction in expected future inflation and real rates. The correlation of the overall change is equal to 47% with future inflation, and 85% with real interest rates. Some other times, one-year shocks are important. For instance, in 2015, when changes in rates were small, debt-to-GDP was reduced by positive inflation and growth surprises.

4.4 Interpreting the probabilities

The probabilities we use are all in risk-adjusted space. This is exactly what proposition 1 calls for. From a policy perspective, Veronesi and Zingales (2010) and Kocherlakota (2012) make the case that governments should weight different inflation policies according to their effect on welfare. Because they adjust for the growth of marginal utility, risk-adjusted probabilities are appropriate when considering policy. Moreover, the debasement question is a valuation question, so it is appropriate to use valuation weights.

In order to better interpret our results, it is useful to understand the relationship with physical probabilities. First, all probabilities are conditional on information available at the time of measurement, for example 2017. While we look at paths for inflation into the future, there is no new information revealed by the joint distribution of 2018 and 2019 that affects the joint distributions between 2021 and 2022; both are measured given information in 2017.

\textsuperscript{41}One reason for choosing to calculate debasement over the life of the debt, as we do in our main exercise, is that it naturally abstracts from when the changes in real rates and changes in inflation expectations happen over the life of the bonds.
Second, the relevant multi-period risk-adjusted expectation of inflation depends on the product of actual inflation and state-dependent risk compensation, which includes anything that changes marginal utility for different realizations of inflation. Our estimates have no way of distinguishing whether the risk-adjusted correlations between inflation at different horizons are due to serial correlation of actual inflation or of the stochastic discount factor. They measure the debt debasement distribution according to the marginal distribution of risk-adjusted inflation, which already includes the correlation of inflation with other factors that affect marginal utility.

A concrete example makes these two points clearer. When we evaluate the expectation of inflation between dates 0 and 2 using the estimated risk-neutral densities, we are computing:

\[
\mathbb{E}
\left[
\frac{m_{0,2}}{\pi_{0,2}}
\right]
= \mathbb{E}
\left[
\left(
\frac{m_{0,1}}{\pi_{0,1}}
\right)
\left(
\frac{m_{1,2}}{\pi_{1,2}}
\right)
\left(
\frac{m_{0,2}}{m_{0,1}m_{1,2}}
\right)
\right]
= \mathbb{E}
\left[
\frac{m_{0,1}}{\pi_{0,1}}
\right]
\mathbb{E}
\left[
\frac{m_{1,2}}{\pi_{1,2}}
\right]
+ \text{cov}
\left[
\frac{m_{0,1}}{\pi_{0,1}}, \frac{m_{1,2}}{\pi_{1,2}}
\right].
\]

Our first point is that all expectations are taken conditional on information at time 0. Therefore, the last term in brackets in the first line is exactly equal to 1, no matter what states are realized at dates 1 and 2. Risk-adjustments (distributions of the stochastic discount factor across future states) or real interest rates (conditional means of the stochastic discount factor) may change over time, but that is taken into account by the date 0 expectation. Our second point is that the covariance term can depend on the covariance between \(\pi_{0,1}\) and \(\pi_{1,2}\), as well as that between \(m_{0,1}\) and \(m_{1,2}\), and between the two pairs. It takes a model for \(m_{0,t}\) or for \(\pi_{0,t}\) to separate them.

4.4.1 Physical probabilities

Even though proposition 1 calls for the use of risk-adjusted probabilities, one might wonder how to translate them to physical probabilities. In that case, in order to aggregate the effect of different inflation realizations, one would require the joint distribution of future inflation paths and future realizations of the stochastic discount factor. In general, this requires a full general-equilibrium model linking real payoffs and inflation. In the appendix, we formally show that our risk-adjusted probabilities likely are either very similar to or provide an upper bound for the physical probabilities, by considering three broad classes of models, which we now verbally describe.

One class assumes that the classical dichotomy holds. This property states that inflation
realizations are uncorrelated with risk preferences and discount rates. In this case, drawing paths only from the marginal distribution of inflation, like we did, delivers the right answer as the physical and risk-neutral distributions of debt debasement coincide.

A second class of models has the property that inflation has a risk premium, and this may vary over time or with shocks, but the risk premium is additive; in this case, to move from one distribution to the other, one needs to adjust the drift of the stochastic process for inflation. Under this approach, the physical probabilities of debasement are still exactly the same as the ones reported in table 2 and figure 6. Since the risk-adjustment is additive, the relative ordering of simulated paths for inflation does not change. In other words, the value at risk depends on the probability of inflation tail events, and it does not change under this approach.\footnote{In addition, the inflation risk premium estimated with this approach tends to be small: Fleckenstein, Longstaff and Lustig (2017) estimate it to be only between 10 and 20 bps. The two distributions are thus close to each other.}

A third class of models has a price of inflation risk, but one that must fit the empirical evidence at the tail of the distribution. Kitsul and Wright (2013) estimate the price of risk in the past using a process for actual inflation fit to the historical data and a flexible parametric pricing kernel, and assume the same model applies to the future. Relevant for our calculations are their estimate at the right tail of the distribution. They find that states of the world with high realizations of inflation have higher risk prices, or risk-adjusted probabilities above the physical ones. In this case, our VaR estimates provide an upper bound for debasement under the physical measure, since they use the higher risk-neutral mass at the tails. Extreme debasement based on physical probabilities is even less likely than what we report.

5 Applications: Financial repression and links to theoretical models

In this section, we modify our estimates to account for financial repression, discuss their relationship to rules of thumb that measure debasement sensitivity, and describe how our estimates can be used to inform models of monetary and fiscal policy.

5.1 Financial repression

It may seem surprising that our calculations so far could be silent on the hotly disputed topic of whether inflation has real effects. One might think that if inflation lowers real interest
rates, then because the government needs to pay less to roll over its debt, the fiscal burden will be smaller. Yet, if the interest rate is lower, this also means that investors discount the future debt by less. By our assumption of no arbitrage opportunities, the real interest paid on the government bonds and the real interest that private agents use to discount the future are the same. Therefore, in the present value of the fiscal burden, these two effects exactly offset, whatever is the impact of inflation on the real interest rate. To be clear, this does not mean that changes in ex post realized real interest rates do not affect the value of the debt at intermediate dates prior to maturity (of course they do), but rather that for a forward-looking measure of the market value of the debt, changes in ex-ante expected real rates are exactly offset by changes in the real discounting factors for these payoffs.

Financial repression is a way to drive a wedge between these two interest rates. This wedge works like a tax on the returns of government debt and as such provides a source of revenue that reduces the fiscal burden. The literature on financial repression, which dates back at least to McKinnon (1973), offers many examples of how this tax is collected and enforced, through channels like caps on interest rates, direct lending to the government by captive domestic savers, or financial regulation, among others. In theory, this would show up as a factor \((1 - \tau_t)\) in each of the terms in our formula in proposition 1. But, at this general level, we cannot say more empirically about its size or how it varies with inflation. We therefore next consider a specific form of financial repression.

### 5.1.1 Repression as financial regulation

Reinhart and Sbrancia (2015) discuss how many developed countries, including the United States, used a combination of caps on the interest rates on government bonds and inflation between 1945 and 1980 to pay for the World War II debt. One particular way in which this is achieved is by forcing the holders of outstanding debt to roll it over for “special” debt that sells for a higher price (or pays a lower return) than the market price for identical private securities. This is achieved for instance by forcing banks to accept this special debt and hold it under the guise of financial regulation and stability. An extreme case of this hidden financial repression is to require banks to hold zero-interest reserves at the central bank. Effectively, one type of government liability that pays market interest is replaced by another type that pays no interest.

To model this formally, assume for simplicity that all debt is nominal and has maturity of one period, and that the holders of maturing bonds are forced to take special bonds as payment that promise to pay \(\tilde{B}_{t+1}\) next period and sell today for price \(\tilde{H}_t\). The price of
the bond is higher than the market price for nominal bonds, \( H_t \), capturing repression, and below one, capturing the zero lower bound on interest rates. When it is equal to one, we have extreme financial repression with the special bonds being zero-interest reserves. The law of motion at any date after 0 for debt now becomes:

\[
W_t = s_t + \left( \frac{\tilde{H}_t - H_t}{P_t} \right) \tilde{B}_{t+1} + W_{t+1} \frac{H_t P_{t+1}}{P_t}.
\]  

(17)

Because \( \tilde{H}_t > H_t \), this expression makes clear that financial repression works like a source of tax revenue. Because \( B_t^0 = \tilde{H}_t \tilde{B}_{t+1} \), this revenue subtracts from the real value of outstanding debt just like a tax on its holders would. Similar algebra as the one that led to proposition 1 shows that the debt burden now is:

\[
\mathbb{E}_0 \left[ \sum_{t=0}^{\infty} m_{0,t} \left( \frac{H_t}{\tilde{H}_t} \right) \left( \frac{B_t^0}{P_t} \right) \right] \leq \sum_{t=0}^{\infty} \frac{B_t^0}{P_0} \mathbb{E}_0 \left[ m_{0,t+1} \sigma_{0,t+1} \right].
\]  

(18)

Since \( \tilde{H}_t > H_t \), the debt burden is lower the higher is financial repression. The inequality becomes an equality in the case of extreme financial repression (\( \tilde{H}_t = 1 \)), where the government rolls over its past debt through zero-interest required reserves. This expression shows that the effect of extreme financial repression is essentially equivalent to delaying all payments on the debt for one year at a zero interest rate, or shifting the maturity structure by one year.

Generalizing the previous argument to have financial repression for \( N \) periods, we obtain a new version of proposition 1:

**Proposition 3.** With financial repression for \( N \) periods, the debt burden is still equal to \( \sum_{t=0}^{\infty} \omega_t^r B_t^0 \) but now the weights are:

\[
\omega_t^r = R_t^{-1} \int \left( \frac{f(\pi_{0,t})}{\pi_{0,t}} \right) \left( \frac{H_t^N}{\tilde{H}_t^N} \right) d\pi_{0,t} \leq \omega_{t+N}.
\]  

(19)

With extreme financial repression (\( \tilde{H}_t^N = 1 \)), the inequality becomes an equality.

The weights in the formula for the debt burden are lower, as long as nominal interest rates are positive. Higher inflation is now more effective at lowering the real value of the debt. Inflation not only debases the debt but also lowers the real return on the zero-interest reserves.
Table 7: The effect of inflation with financial repression

<table>
<thead>
<tr>
<th>Duration of repression</th>
<th>Repression</th>
<th>Higher inflation</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>1.4%</td>
<td>2.7%</td>
<td>4.1%</td>
</tr>
<tr>
<td>5 year</td>
<td>7.5%</td>
<td>4.0%</td>
<td>11.5%</td>
</tr>
<tr>
<td>10 year</td>
<td>13.7%</td>
<td>4.9%</td>
<td>18.7%</td>
</tr>
</tbody>
</table>

Notes: Each cell shows the fall in the real value of debt as a ratio of GDP as a result first of repression, and then of higher inflation under experiment 1. The last column is the sum of the two previous ones.

5.1.2 Estimating the joint effectiveness of inflation and repression

Table 7 shows the effect of extreme financial repression. To be clear, these are counterfactual measures, not debt debasement distributions. If agents had anticipated repression coming, this would have affected the debt outstanding in the first place. We display them, not because they substitute for or qualify our debt debasement distribution, but because they help to demonstrate how a government may try to inflate the debt.

The second column shows the direct effects of financial repression alone, which can be calculated by just using the information in the yield curve. Even with the low nominal interest rates in 2017, repression for 10 years would wipe out more than one quarter of the privately held debt.

Column four is our main interest since it shows how repression interacts with inflation. It conducts our experiment 1 within the financially repressed economy, so it measures the effect solely of the inflation distribution shifting to the right. The effect of inflation is much higher than before. This confirms our conclusion of the previous section that a longer maturity of current debt makes inflation effective at lowering the real value of debt. The last column shows the joint effect, at date 0, of both imposing financial repression and shifting the distribution of inflation to the right. The reduction of the real value of debt is substantial.

5.2 Using our estimates in different classes of theoretical models

We have calculated estimates of debt debasement motivated by the high and growing debt levels in many countries and the question of what role inflation can play in reducing debt levels. In addition, our estimates can also be used to measure magnitudes of variables in several classes of theoretical models, including models of monetary policy, fiscal theory of the price level, and fiscal policy. This section expands on these uses.
5.2.1 The fiscal backing of inflation-targeting central banks

The intertemporal budget constraint (omitting indexed debt because it plays no role) is:

$$W_0 = \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} s_t \right].$$

(20)

This equation, which we have already seen as equation (5) only assumes no arbitrage and the absence of Ponzi schemes. It therefore holds in most models of monetary and fiscal policy.

In traditional new Keynesian models, an independent, inflation-targeting, central bank sets a policy tool (like the interest rate) to achieve a desired level of inflation. That desired level is the sum of an inflation target and a term proportional to the output gap, a policy known as flexible inflation targeting (e.g., Woodford, 2010). Actual inflation is then the inflation target set in the policy regime plus a function of shocks that cause a tradeoff between stabilizing inflation and real activity. Completing the equilibrium is the assumption that the fiscal authorities follow a passive, or Ricardian, policy of adjusting fiscal surpluses to satisfy equation (20).43

Our distribution of debt debasement is an estimate of the distribution of how taxes and spending rise and fall to support independent monetary policy. Given shocks to the implementation of monetary policy (like shocks to the Taylor rule), shocks that move the output gap, and shocks to the inflation target in the mandate, there will be a distribution of inflation outcomes. This creates a distribution of $W_0$, which, given the fiscal support of independent central banking, implies a distribution for the present value of fiscal surpluses. In short, an independent central bank that targets inflation leaves a fiscal footprint, and our estimated distribution of debasement measures part of this surplus. Our distribution provides a set of estimates under different realizations of shocks for the fiscal index proposed by Cochrane (2017) and the extent of fiscal backing discussed by Del Negro and Sims (2015).44

5.2.2 Fiscal-driven inflation

If fiscal policy is non-Ricardian, or active, it does not provide fiscal backing to the central bank. In simple models, this is modeled as an exogenous present value of fiscal surpluses, the right-hand side of equation (20). When it falls, then $W_0$ must fall. Since $W_0$ depends on

---

43The equations behind the model in this paragraph can be found in any survey on the new Keynesian model, like Woodford (2010).

44There may of course be extra fiscal effects from the operation of automatic stabilizers, or changes in the discounting of the future, but in the baseline new Keynesian model both of these are assumed away.
which is predetermined, and the price level, then a negative fiscal shock raises inflation. This is the fiscal theory of the price level. The equilibrium of this class of models is a joint distribution of inflation and fiscal surpluses, \((\pi_{0,t}, W_0)\), e.g., see Sims (2013). Our estimates of the distribution of debt debasement map into the marginal distribution for \(W_0\) in this large class of models.

Recently, Bianchi and Melosi (2017) assume that fiscal policy alternates between active and passive regimes, and estimate these regimes using data on U.S. inflation. They simplify the government budget constraint by assuming that all debt is in nominal terms, one-period, privately-held bonds, which requires them to assume an exogenous “term-premium” shock that turns out to have a significant posterior variance. Our estimates can map, with some adjustments, to the joint posterior distribution of these shocks and inflation, providing a validation test of their estimates.

Both this example and the previous one make clear that our distribution of debasement includes both shocks to fundamentals and shocks to the policy regime, in terms of the inflation target and the fiscal regime.\(^{45}\) Rational agents anticipate these policy changes, and incorporate them probabilistically into the risk-adjusted distributions of inflation that we measure. To answer “what-if” questions, in the sense of the effect on the fiscal burden of inflation policy changes that surprise agents, one must use our estimates of counterfactuals in section 4. The distribution of debasement gives an assessment of the consequences of policy changes when markets may get it wrong but are not completely surprised by policy in the sense of having put zero probability on it ex ante.\(^{46}\)

\subsection{5.2.3 The maturity of debt and inflation, tax smoothing, and sovereign default}

Missale and Blanchard (1994) write a model where the government can either inflate away the debt or raise taxes to pay for it, and bondholders form expectations of inflation before the policymaker sets actual inflation. The government faces a temptation to generate unexpected inflation to reduce socially costly taxes, and this causes a time inconsistency problem in the tradition of Barro and Gordon (1983). Missale and Blanchard (1994) solve for a reputation equilibrium where the government keeps expected inflation low by choosing the maturity of its debt ex ante. A shorter maturity reduces the ex post ability to inflate the debt away, and thus controls the temptation to do so. The model has predictions for inflation, debt

\(^{45}\)Schmitt-Grohe and Uribe (2010) discuss many factors that affect the optimal inflation target. These can change over time.

\(^{46}\)Sims (1982) is an early argument for using distributions like we do in our VaR estimates in order to deal with the Lucas critique of counterfactuals.
debasement, and the optimal maturity of debt.

Their model assumes a reduced-form sensitivity of the real value of debt to inflation, which is decreasing in the maturity of privately-held debt. Our estimates in the previous section accurately measure this sensitivity. Adding shocks to inflation in their model would produce a distribution of debasement that corresponds to our estimates. Moreover, section 4 links our estimates of the distribution of debasement to the maturity distribution of privately-held debt in a numerically accurate way. Finally, while Missale and Blanchard (1994) use the simple average maturity of public debt to link their theory to the data, we show in section 5 that this rule of thumb is inaccurate and provide more accurate alternatives.

Persson, Persson and Svensson (2006) implement the optimal fiscal and monetary policy under commitment by choosing a maturity structure for debt that exactly balances the marginal benefit of a surprise inflation with its marginal cost. They show that a crucial determinant of the marginal benefit is the debt debasement distribution at the optimal Ramsey outcome. This is precisely the object that we measure. Future research can accurately quantify and take these models to the data (e.g., extending Debortoli, Nunes and Yared, 2017, to consider inflation).

Finally, Aguiar et al. (2013) study the incentives of governments to default on sovereign debt or to inflate it away, when there may be self-fulfilling crises. They assume a grace period for the repayment of debt in order to proxy for the maturity of the debt, and this parameter plays an important role in their theoretical results. Taking their model to the data requires calibrating this parameter. This can be done by matching it to our estimates of debt debasement.

6 Conclusion

This paper presents and implements a new method to evaluate the effect of higher-than-expected future inflation on the real value of debt. We identify four main drivers of debasement: who holds which debt, its maturity structure, and the stochastic properties of future inflation, including its volatility and persistence. For the United States in 2017: (i) the value of nominal debt held by the private sector is only about half the size of the total outstanding government debt. (ii) The maturity structure, especially of debt held domestically, is skewed towards the short end. (iii) The likelihood of near-term inflation rising significantly is quite low. And (iv) inflation takes a few years to build up, so its persistence in the short horizon is small. Taken together, these four features imply that inflationary debt debasement by
more than a few percentage points of GDP is unlikely. Confirming our results, we find that financial repression, an active policy tool that extends debt maturity, substantially increases the likelihood of large inflationary debasement, and that rules of thumb, like duration, that ignore the stochastic properties of inflation, provide inaccurate estimates. We propose and validate a new rule of thumb for inflationary debasement that takes into account duration, inflation persistence and volatility. Finally, we point out that our estimates of both the sensitivity of the fiscal burden to inflation and their joint distribution provide numerical targets for theoretical models of fiscal and monetary policy to hit in their calibrations.

In the process, we assemble new data and develop new methods that are of independent interest. We compile data measuring the maturity structure of debt held by different sets of investors for the U.S. from 2009 to 2017. We show that the debt maturity structure has changed significantly over time, and that its distribution is poorly approximated by a simple exponential formula. We use new data on derivatives contracts for both cumulative and forward inflation. We show that combining both sets of distributions allows us to extract information about the time-series behavior of inflation. Finally, we propose a new estimator for the risk-adjusted joint distribution of inflation using copulas. We find that inflation expectations are well-anchored. The market expects risk-adjusted inflation to be stationary but also believes that the process has long memory. We find that inflation volatility has declined significantly from 2009 to 2017 and that there is important variation in inflation persistence. We also explore the additional effect of changes in GDP or in real interest rates, finding that they can contribute to significantly more reductions in the debt to GDP ratio, and have done so in the past.

Increasing the debasement of debt by inflation would require that one of the drivers of our low estimates changes. One way would be to have significantly higher inflation that completely surprises the market even though informed, profit-maximizing agents view such outcomes as being virtually impossible. At least in our sample period, this has not happened. Indeed, as noted, expected inflation volatility has declined. Another way would be to have the private sector hold more long-term public debt. Yet, perhaps it is precisely by holding short-term debt that private agents are reducing the incentive of the government to inflate away the debt, so that low maturity holdings and low inflation expectations are mutually consistent. If so, issuing more long-term debt might result in lower debt prices. Substantial debasement would arise if inflation were combined with financial repression, which could be implemented by forcing an ex-post extension of maturities. Of course, such a policy would likely impose significant costs on the economy by impairing financial intermediation.
None of these possibilities appear likely and market prices today put the probability of the United States defaulting at close to zero. Therefore, the markets seem to be expecting either economic growth, a favorable evolution of real interest rates, or perhaps primary budget surpluses, so that debt to GDP stabilizes. What is sure and inescapable is that, one way or another, the budget constraint of the government will have to hold.
References


Appendix to “Inflating Away the Public Debt? An Empirical Assessment”

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A  Proof of proposition 1

While the bulk of the proof is already in section 2, here we fill in some missing steps. First, adding and subtracting $Q_t^1 W_{t+1}$ on the right-hand side of equation (2), and using equation (1) to replace out $W_{t+1}$, a few steps of algebra deliver the law of motion for the market value of government debt:

$$W_t = Q_t^1 W_{t+1} + s_t + x_{t+1}$$  \hspace{1cm} (A1)

where the revaluation term $x_{t+1}$ is equal to:

$$\sum_{j=0}^{\infty} (Q_{t+1}^j - Q^j_{t+1}) K_{t+1}^j + \sum_{j=0}^{\infty} (H_{t+1}^j - H_t^j H_{t+1}^j) \frac{B_{t+1}^j}{P_t} + \sum_{j=0}^{\infty} \frac{H_{t+1}^j B_{t+1}^j}{P_{t+1}} - Q_t^1.$$

(A2)

Iterating this equation forward, from date 0 to date $t+1$, delivers equation (3) in the text. Dividing both sides of the law of motion for $W_t$ by $Q^1_t$, multiplying by $m_{t,t+1}$ and taking expectations gives:

$$W_t = \mathbb{E}_t(m_{t,t+1} W_{t+1}) + s_t + \mathbb{E}_t \left( \frac{m_{t,t+1} x_{t+1}}{Q^1_t} \right).$$

(A3)

For now, assume that the last term on the right-hand side is zero. We will show it shortly. Multiply both sides of (A3) by $m_{0,t}$ and take expectations as of date 0, so that using the law
of iterated expectations you get the recursion:

$$E_0(m_{0,t}W_t) = E_0(m_{0,t+1}W_{t+1}) + E_0(m_{0,t}s_t).$$

(A4)

Iterate this forward from date 0 to date $T$, and take the limit as $T$ goes to infinity. With the no-Ponzi scheme condition $\lim_{T \to \infty} E_0(m_{0,T}W_T) = 0$, you get the result in expression (5):

$$W_0 = E_0 \left[ \sum_{t=0}^{\infty} m_{0,t}s_t \right].$$

(A5)

Finally, for the first equality in expression (5), replace out the bond prices from equation (1) using equation (4).

The missing step was to show that $E_t(m_{t,t+1}x_{t+1}/Q_t^1) = 0$ for all $t$. Consider the first element of $x_{t+1}$ and take the following steps:

$$E_t \left[ m_{t,t+1} \sum_{j=0}^{\infty} \left( \frac{Q_t^{j+1}}{Q_t^1} - Q_t^{j+1} \right) K_{t+1}^j \right] =$$

$$\sum_{j=0}^{\infty} K_{t+1}^j \left[ Q_t^{j+1} E_t \left( \frac{m_{t,t+1}}{Q_t^1} \right) - E_t \left( m_{t,t+1}Q_t^{j+1} \right) \right] =$$

$$\sum_{j=0}^{\infty} K_{t+1}^j \left[ Q_t^{j+1} - E_t \left( m_{t,t+1} \frac{E_t (m_{t,t+1}m_{t+1,t+1+j})}{E_t (m_{t,t+1}x_{t+1}/Q_t^1)} \right) \right] =$$

$$\sum_{j=0}^{\infty} K_{t+1}^j \left[ Q_t^{j+1} - E_t \left( m_{t,t+1}x_{t+1+j} \right) \right] = 0.$$  

(A6)

The first equality comes from replacing the expectation of the sum by the sum of the expectations and from taking the prices known at date $t$ out of the expectation; the second from using the result in equation (4) twice; the third from using the law of iterated expectations; and the fourth from using equation (4). Identical steps show that the other terms in $E_t(m_{t,t+1}x_{t+1}/Q_t^1)$ are also equal to zero.

The final part of the proof to clarify are expressions (6) and (7). First note that in principle, $m_{0,t}$ can depend on many random variables. However, all we need to evaluate is $E(m_{0,t}/\pi_{0,t})$. Therefore, only the dependence of $m_{0,t}$ on $\pi_{0,t}$ will lead to a non-zero term once the expectation of the product of the discount factor and inverse inflation is evaluated. Therefore, we can write $E(m(\pi_{0,t})/\pi_{0,t})$.

Second, let $\hat{f}(.)$ be the probability density function for inflation. Then, we can re-write
the expression as: \( f(\pi_{0,t})m(\pi_{0,t})/\pi_{0,t} \). Define the inverse of the risk-free rate, from the perspective of date 0, as \( R_t^{-1} = \int f(\pi_{0,t})m(\pi_{0,t})d\pi_{0,t} \). From the definition of the stochastic discount factor, this is the price of a bond that would pay one dollar for sure in \( t \) periods, regardless of the realization of inflation. Then, define the risk-adjusted density for inflation as: \( f(\pi_{0,t}) = f(\pi_{0,t})m(\pi_{0,t})R_t \). Using these results it follows that:

\[
\mathbb{E}(m(\pi_{0,t})/\pi_{0,t}) = \int \left( \frac{f(\pi_{0,t})m(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t} = R_t^{-1} \int \left( \frac{f(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t}.
\]  

(A7)

This confirms expression (7).

## B Debt holdings

We construct monthly maturity structures, that is \( B_t^i \), for four groups of investors: (1) private total, which is publicly held debt, excluding the Federal Reserve and state and local government holdings, (2) foreign, (3) domestic, which is private minus foreign, and (4) the central bank. The data sources are the CRSP U.S. Treasury database, the Treasury Bulletin, the “Foreign Portfolio Holdings of U.S. Securities” report available from the U.S. Treasury, and the System Open Market Account (SOMA) holdings available from the FRBNY. The numbers for face value of total outstanding debt for different categories in section 3.1 come from the Monthly Statement of Public Debt, the Treasury Bulletin (OFS-2, FD-3), and the SOMA. Data are all for end of December 2009 to 2017 for the United States.

We construct holdings of notes and bonds as follows. For (1), the data is available from CRSP, and we subtract out state and local government holdings assuming they have the same maturity structure. Detailed information for (2) is available for June 2015 and we assume proportionate growth of foreign holdings to construct December 2015 numbers. (3) is the difference of (1) and (2). For (4) we use the security level data available from the FRBNY. We assume that all coupon and principal payments mature in the middle of each month.

CRSP does not have data on Treasury bills. We use the issues of the Treasury bulletin to obtain information on bills and follow the same steps as we did above for notes and bonds. In particular, we construct holding of T-Bills as follows. For (1) we subtract Fed holdings from Treasury Bulletin Table FD-2 T-Bill holdings and then subtract out the proportionate amount of state and local government holdings; for (2) and (3) we do the same as for notes and bonds; for (4) we use the SOMA holdings.
Aside from calculating debt holdings for more categories of investors, our method for constructing the maturity structure of (1) has the following differences relative to Hall and Sargent (2011). First, we construct a monthly term structure and assume that promised payments in a month are paid in the middle of the month (instead of using an annual frequency). Second, we exclude state and local government holdings. Third, we base the T-Bill holdings on Table FD-2 of the Treasury bulletin and Fed holdings (rather than Table FD-5 of the Treasury bulletin).

Real interest rates are constructed according to equation (8) in the paper and the spot curve (nominal zero coupon yield curve) provided by Gürkaynak, Sack and Wright (2007). The harmonic means of inflation are calculated using simulated 30-year distributions based on our fitted restricted model. As a check, we also separately calculate the market value of the debt using the real term structure from Gürkaynak, Sack and Wright (2010) for real interest rates. These only extend up to 20 years. To construct real spot rates for longer maturities we assume that forward rates for years 21 to 30 are equal to the average forward rate for years 18 to 20. Our results are robust to this alternative.

C Estimating the marginal distributions for inflation

We estimate inflation distributions from data on zero coupon and year-on-year inflation caps and floors, which we collect from Bloomberg, as do Fleckenstein, Longstaff and Lustig (2017). Kitsul and Wright (2013) use data provided by an interdealer organization, whereas we use the raw reported numbers.

C.1 Zero-coupon inflation options

The basic methodology for construction of the distributions from zero coupon inflation options is fairly standard and similar to Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2017). Still, we take several steps in cleaning the data that this section of the appendix clarifies.

The zero coupon inflation call options are traded with strike prices between 1% and 6%, in 0.5% increments, and expiration dates ranging from 1 to 10 years as well as 12 and 15 years, although the data for the 2 and 9 year maturities are of generally lower quality. The zero coupon put options are available for strike prices between -2% and 3% and identical maturities as the caps. For the overlapping range of strike prices we use both option prices to reduce measurement error.
A zero coupon inflation cap is the most traded contract among inflation derivatives. It pays, at expiry, the maximum between zero and the difference between the cumulate inflation during the period and the strike price so its payoff at maturity is max[0, (1 + \pi_{0,t}) - (1 + k)^t]. Adapting the argument in Breeden and Litzenberger (1978) we can non-parametrically construct risk neutral density functions using these option prices.

In particular, for our options, the asset pricing formula is:

\[ a_0 = R_t^{-1} \mathbb{E}_t f(\max\{(P_t - k)/P_t, 0\}). \]

where \( a_0 \) is the price of the bond, \( R_t \) is the real safe return, \( f(.) \) is the real risk-neutral measure, \( k \) is the strike, while \( P_t \) is the price index, with \( P_0 = 1 \) so \( \pi_{0,t} = P_t \). Taking the derivative of the pricing equation with respect to \( k \) gives:

\[ R_t \frac{\partial a_0}{\partial k} = -\int_k^\infty \frac{1}{P_t} f(P_t) dP_t. \]

Then take a second derivative

\[ R_t \frac{\partial^2 a_0}{\partial k^2} = \frac{f(k)}{k}. \]

Therefore, to get the real risk neutral measure of inflation, one needs to calculate:

\[ f(\pi_{0,t}) = \pi_{0,t} R_t \frac{\partial^2 a_0}{\partial \pi_{0,t}^2} \]

We can extract the risk-neutral density by observing how the price of the option varies with changes in the strike price.

Breeden and Litzenberger (1978) suggest using a butterfly trading strategy to construct Arrow-Debreu securities, claims that pay one unit of currency if at some specific time in the future the underlying asset price is equal to a specific value and zero otherwise. While this method provides a good first approximation to risk neutral probabilities, it does not adjust for irregular options prices, due to, for example, non-synchronous trading (Bahra, 1997). Therefore, while we check that all such prices are positive, we must smooth the data otherwise the results are very inaccurate.

To overcome the drawbacks of the unadjusted butterfly strategy, we do the following. First, we drop data that represent simple arbitrage opportunities (discussed in the text). We next calculate Black-Scholes implied volatilities, and, following Shimko (1993) and Campa, Chang and Reider (1998), for each set of options at any expiry date, we fit a four factor
stochastic volatility model, the SABR model, which was developed by Hagan et al. (2002). This calibration minimizes the norm of the difference between the observed data and the candidate SABR function, resulting in a vector of optimal parameters for the SABR model which is used to create a volatility curve. We constrain the estimated implied volatility function to ensure that the smoothing does not re-introduce arbitrage opportunities. This method reduces the weight of irregular data, while preserving its overall form. We convert back to option prices and construct risk neutral distributions.

To be clear, this method does not assume that we can use the Black-Scholes formula for pricing. Instead, it is simply used as a nonlinear transformation on which smoothing is performed.

C.2 Year-on-year inflation options

The construction of distributions from year-on-year options requires a bootstrapping method where cap and floor contracts, which are portfolios of annual caplets and floorlets, are unbundled to recover prices of the underlying options. First, we use a bootstrapping procedure to extract the caplet and the floorlet prices from the cap and floor prices respectively. Second, when calculating the option’s implied volatility we use the Rubinstein (1991) transformation which enables us to price the option as a plain vanilla option with a time to maturity equal to the option tenor between inflation reset times, discounting back using the real interest rate, calculated assuming that put-call parity holds for these options (Birru and Figlewski, 2012).

For each horizon, for the smallest (largest) bin we report the risk-adjusted probability of inflation lying in or below (above) that bin. If there is considerable mass in the last bin for year-on-year inflation, we use our smoothed estimates to project additional bins.

C.3 Comparing option prices at different dates

For each year, we inspect option prices at the end of December and start of January (our data of interest) and select out of the last five trading days of the year and the first five trading days of the new year, the day that yields the maximum number of option prices that do not violate the threshold criteria, described above and in the text. For end of 2017 data, this was January 9, 2018. Each year, we also look at distributions for several days before and after. All of them look almost identical to the ones we then end up using.
D Estimating the joint distributions for inflation

The marginal distributions for inflation are enough to evaluate the formula in proposition 1. Yet, to calculate inflation paths or counterfactuals, we need joint distributions.

D.1 Proof of proposition 2

Using \( F(.) \) to denote the cumulative density function, we have data for one-year inflation \( F(\ln \pi_{t+j-1,t+j}) \) for \( j = 1 \ldots J \) where \( J = 10 \) years, and for cumulative inflation \( F(\ln \pi_{t,t+j}) \). The data comes in \( N \) bins expressed as ranges for inflation.

Sklar’s theorem states that there exists a function \( c : [0, 1]^J \to [0, 1] \) such that:

\[
F(\ln \pi_{t,t+1}, \ldots, \ln \pi_{t+J-1,t+J}) = c(F(\ln \pi_{t,t+1}), F(\ln \pi_{t+1,t+2}), \ldots, F(\ln \pi_{t+J-1,t+J})) . \tag{A8}
\]

In turn, it follows from the link between marginal and joint distributions and the definition of cumulative inflation that:

\[
F(\ln \pi_{t,t+j}) = \int_{\Pi} F(\ln \pi_{t,t+1}, \ln \pi_{t+1,t+2}, \ldots, \ln \pi_{t+J-1,t+J}) \prod_{i=1}^{j} \ln \pi_{t+i-1,t+i} \, d\ln \pi_{t,t+1} \ldots d\ln \pi_{t+J-1,t+J} \tag{A9}
\]

where the set \( \Pi \) is defined as: \( \left\{ \ln \pi_{t+t+j} : \ln \pi_{t,t+n} \prod_{i=1}^{j} \ln \pi_{t+i-1,t+i} \right\} \).

Combining these two results delivers the proposition in terms of cumulative distributions. We maintain the assumption throughout that all distributions are continuous. Therefore, Sklar’s theorem also applies to the marginal density functions, with \( C(.) \) replacing \( c(.) \). For numerical purposes, it was better to work with densities rather than with cumulative distribution functions.

A final point to note is that there are \( N \) bins and so \( N \) equalities in the distribution, of which one is redundant since probabilities must add up to one. There are \( J \) maturities, but for maturity one the equality is trivial. Therefore, in total there are \((N - 1)(J - 1)\) conditions.
D.2 Parametric copula and method of moments

We use the parametric normal copula, whose formula is:

\[
\hat{C}(f(\ln \pi_{t,t+1}), \ldots f(\ln \pi_{t+J-1,t+J})) = \\
\left(\frac{1}{\det R}\right) \exp \left(-\frac{1}{2} \begin{pmatrix}
\Phi^{-1}(f(\ln \pi_{t,t+1})) \\
\vdots \\
\Phi^{-1}(f(\ln \pi_{t+J-1,t+J}))
\end{pmatrix} \left(\rho^{-1} - I_J\right) \begin{pmatrix}
\Phi^{-1}(f(\ln \pi_{t,t+1})) \\
\vdots \\
\Phi^{-1}(f(\ln \pi_{t+J-1,t+J}))
\end{pmatrix}\right) 
\]

where \(\Phi(.)^{-1}\) is the inverse of the standard normal cdf, and \(\rho\) is a correlation matrix of dimension \(J\).

The matrix \(\rho\) would only exactly equal the correlation matrix of the variables in the joint distribution if the marginal distributions happened to be normal. Yet, by drawing from the joint distribution using the formula above, and calculating correlation coefficients across many draws, we found that the difference between the actual correlation matrix and \(\rho\) was almost always less than 0.01.

To find the unrestricted estimates, we minimize over the \(J(J - 1)/2 = 45\) independent components of the correlation matrix \(\rho\) that lie between \(-1\) and \(1\). The objective function is the equally weighted average of the \((N - 1)(J - 1) = 20 \times 9 = 180\) squared deviations from the moments in proposition 2. This is a difficult global minimization over a large parameter space, which we handle through a combination of global and local minimization algorithms and many repeated searches from Sobol-sampled starting points.

D.3 Inflation after 10 years

In drawing paths for inflation, for the first 10 years we use the joint distribution given by the multivariate copula. After that, we assume that inflation is a 9th order Markov process. Therefore, the distribution for inflation in year 11, conditional on inflation in years 2 to 10 is the same as the distribution for inflation in year 10, conditional on years 1 to 9. Since we have the joint distribution for inflation from year 1 to 10, it is easy to derive the conditional distribution for inflation in year 10, conditional on years 1 to 9. Thus, we have the conditional distribution for year 11, conditional on the draws so far. The same applies to year 12, and so on, all the way to 30. Note that, since we assume that the joint distribution of maturities one to ten follows a Gaussian copula, then this procedure implicitly assumes that the joint distribution in maturities one to thirty is likewise a Gaussian copula. The key restriction is
that the correlation matrix of $30 \times 29/2 = 435$ elements for the 1-30 copula only have 45 independent separate elements that we estimated for the 1-10 distribution.

### D.4 Restricted distribution

We can represent inflation between two successive dates (or maturities) as:

$$\ln \pi_{t,t+1} = \mathbb{E}_t(\ln \pi_{t,t+1}) + p_{t+1} + s_{t+1},$$

(A11)

where $p_{t+1}$ is a non-stationary part, and $s_{t+1}$ a stationary one, with the two being independent and zero mean. From Wold’s theorem, the stationary process is fully characterized by its covariance function $v_j = \mathbb{E}_t(s_{t+n}s_{t+n+j})$ for any arbitrary $j$. The definition of stationary is that this is true for any positive $n$.

The restriction that we impose is that the non-stationary process is a random walk, so that $\mathbb{E}_t(p_{t+n}p_{t+n+j}) = \sigma n$. This constrains the way in which non-stationarity affects the correlation matrix over time. In particular, the correlation between inflation at date $t + n$ and date $t + j$, or the $(n, j)$ element of the matrix $\rho$ is given by the expression:

$$\rho_{n,n+j} = \frac{v_j + n\sigma}{\sqrt{(v_0 + n\sigma)(v_0 + (n + j)\sigma)}}$$

(A12)

While the unrestricted $\rho$ matrix has 45 parameters, with the random-walk constraint, there are only 10 parameters. Nine are the correlations of the stationary part \{\(v_1/v_0, \ldots, v_9/v_0\}\}, and one more is the ratio of the relative variances of the permanent and transitory components $\sigma/v_0$. We minimize the same objective function but over this smaller parameter space. Note that in this case, the $\rho$ matrix has the easily identified form, $\rho_{n,n+j} = v_j/v_0$, which is the same whatever is $n$.

Table 1 reports the restricted model parameters. Importantly, note that these are not correlations;\(^1\) instead, the numbers are parameters that enter the restricted copula model. The variance parameter in the bottom row is the ratio of the variance of the shock to the random walk component to the variance of the shock to the stationary component. We can see that, for most years, the process is close to stationary. We can translate these parameters to correlations, which are reported in 2.

---

\(^1\)Correlations cannot lie above one, of course. However, within the copula model it is possible for the rho parameters to lie above one. Note that when we report corresponding correlations, the numbers are all below one.
### Table 1: Estimates of restricted model parameters, 2009 to 2017

<table>
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<tr>
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<td>0.11</td>
<td>0.00</td>
<td>0.16</td>
<td>0.09</td>
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</tbody>
</table>

Notes: Estimated parameters from the restricted model, composed of nine autocorrelations parameters of the stationary part and the variance of the random walk process.

### Table 2: Restricted model implied correlations, 2009 to 2017

<table>
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<tr>
<th></th>
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<th></th>
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<td>-0.04</td>
<td>0.42</td>
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<td>0.79</td>
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</table>

Notes: Estimated correlation coefficients from the restricted models for year-on-year inflation between a given year and that year+j, where j is the maturity in each row.
D.5 Goodness of fit and restricted versus unrestricted distribution

Table 3 shows the estimated $\rho$ for the unrestricted model in 2017. Because the estimates vary so much, they are a little hard to interpret. Therefore, in Table 4 we calculate the average (off-diagonal) correlation coefficients between short term (1 to 3 years), medium term (4 to 7 years) and long term maturities (8 to 10 years). The two key features that are in common with the restricted estimates are the ones emphasized in the text: the autocorrelation coefficients are not very high, and they do not fall with maturity. But the estimates are quite noisy in that, across contiguous maturities, they jump up and down.

We assess goodness of fit in multiple ways. First, figure 1 shows the data for the risk-neutral density of cumulative inflation at maturities 2 to 10 against the predicted densities, according to the restricted and unrestricted models. Our method of moments, following proposition 2, consisted of picking parameters $\rho$ to minimize the difference between the data in these 9 plots, and the models. Visually the fit is quite good, and the restriction seems to have almost no effect on the ability of the copula model to fit the data. The patterns the other years, not reporter here, are similar.$^2$

Second, we compare the models’ prediction for inflation at horizons 12 and 15 with the data for those maturities. Note that this tests not only the normal copula, but also our 9th order Markov assumption to simulate beyond 10 years. Figure 2 shows the model-implied and data distributions. The restricted model fits better.

Third, we compare the model’s predicted standard deviations of risk-adjusted inflation, with those in the marginal distributions. Figure 3 shows the two models. The restricted model again does a slightly better job when applied to the horizon 12 and 15 distributions, although again it is clear that, in spite of the slightly different estimates of $\rho$, the two models have similar fits.

E Estimating debt debasement

We draw 500,000 samples for 40 years of inflation using our joint distribution. We convert these into 480 month histories by assuming continuous compounding and a constant inflation rate within each year. We calculate the real value of the nominal payments for each of these draws, and order them, calculating their percentiles in table 2. We repeat the calculations using the $B_t^0$ for each group of investors instead.

$^2$Graphs are available upon request.
Table 3: Estimated correlation coefficients of year-on-year inflation in the joint distribution

<table>
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Notes: Estimated correlation coefficients for year-on-year inflation between date 2017+j and 2017+l, in column j, row l.

Table 4: Average correlation coefficients of year-on-year inflation in the joint distribution

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<th>Short term (1-3 years)</th>
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<td>Medium term (4-7 years)</td>
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Figure 1: Marginal distribution for risk-adjusted cumulative inflation: data and models

Figure 2: Distribution for $\ln \pi_{t,t+11}$ and $\ln \pi_{t,t+15}$, model and data
For the counterfactuals, we use alternative distributions for inflation from which to take draws, recalculate the value of the debt using the formula in proposition 1, and subtract it from the original number. The alternative distributions are:

1. Permanently higher: Shift all the year-on-year distributions by the difference between the 90th and the 50th percentile at each maturity.

2. Right tail only: We draw from the baseline distribution but discard all histories in which average inflation is below its 90th percentile in any one of the years in the simulation.

3. Higher and more variable: For each maturity we multiply baseline inflation levels by a scaling factor so that the new mean is equal to the mean in case 1. This results in more variable inflation.

4. Higher for sure: This is the same as case 1 but we now assign all the weight to the mean at each maturity.

5. Partially expected: We set inflation equal to 3% in year 1. For the following histories we use marginal distributions conditional on the year 1 realization. After 10 years we assume that year-on-year inflation is 9th order Markov.
6. Temporary increase: We shift the year-on-year distributions in the same way as case 1 but we now shift them to the 90th percentile in year 1, 80th in year 2, 70th in year 3, and 60th in year 4. There is no change in the year-on-year distributions for maturities equal to and above five years. Note that this is different from the unexpected shock in the previous case since distributions in years 2 to 4 are shifted directly instead of changing only due to the new distribution in year 1.

7. Gradual increase: The one year distribution is unchanged, the 2 year median shifts to the previous 60th percentile, 70th for year 3, 80th in year 4, and 90th for 5 years and above.

F Risk neutral and physical probabilities

To simplify the discussion, assume that (i) there is a single bond outstanding that pays no coupon but just a principal in $T$ periods ($B_{0,t} = 0, t \neq T$), (ii) the principal is 1 ($B_{0,T} = 1$), (iii) there is a countable set of states of the world, $s$ drawn from a set $S$ with physical probability distribution $p(s)$ so that $p(s) \geq 0$ for all $s$ and $\sum_{s \in S} p(s) = 1$, (iv) the stochastic discount factor in $T$ periods is stochastic, $m_{0,T} = m(s)$, (v) inflation is likewise random $\pi_{0,T} = \pi(s)$. The sole goal of these assumptions is to focus on scalar realizations of inflation at a fixed horizon, instead of sequences of inflation over time. They imply that the burden of the debt is:

$$W_0 = \sum_{s \in S} \frac{p(s)m(s)}{\pi(s)}.$$

With this setup, building the distribution of debasement due to inflation works as follows. First, because the goal is to capture inflation variation, not all variation that affects debt, one needs a change of measure. The (marginal) distribution of inflation is given by the standard formula:

$$p(\pi) = \sum_{s : \pi(s) = \pi} p(s).$$

which is calculated over the set of all possible values of inflation $\Pi$. The cardinality of $\Pi$ may be lower than that of $S$ because there may be some states $s'$ and $s''$ such that $\pi(s') = \pi(s'')$. To obtain the distribution of the SDF as a function of inflation $m(\pi)$ is not as straightforward. The reason is that for states $s'$ and $s''$ such that $\pi(s') = \pi(s'')$, it may be that $m(s') \neq m(s'')$. To be able to continue using it as a valid SDF that is positive and that has an expectation that is equal to the observed inverse of the safe rate, requires building it by averaging over
such states:

\[ m(\pi) = \frac{\sum_{s:\pi(s)=\pi} p(s)m(s)}{\sum_{s:\pi(s)=\pi} p(s)} \]

This way, \( \sum_{\pi \in \Pi} p(\pi)m(\pi) = \sum_{s \in S} p(s)m(s) \). This is the relevant \( p(.) \) measure for the paper, not the one over states.

Second, for every possible outcome for inflation, consider a second stochastic variable:

\[ \omega(\pi) = W_0 - \frac{\mathbb{E}^p(m)}{\pi} \]

measuring by how much does debt fall for that draw of inflation. To be completely clear, the expectation is with regards to the distribution of inflation: \( \mathbb{E}^p(m) = \sum_{\pi \in \Pi} p(\pi)m(\pi) \). This is still stochastic in that it may be different for different realizations of \( s \), as in the example of \( s' \) and \( s'' \) above. But it is risk free when it comes to variation in inflation.

We define the risk-adjusted probabilities as: \( f(\pi) = m(\pi)p(\pi)/\mathbb{E}^p(m) \). Our distribution is then given by

\[ \Phi(\bar{w}) = \text{Prob}^f(\omega(\pi) > \bar{w}) \]

that is: the risk-adjusted probability that debt debasement due to inflation variance exceeds a certain amount \( \bar{w} \) in dollars or as a share of GDP. Define the set of possible realizations of inflation for which debasement exceeds this threshold to be: \( \Pi^f = \{ \pi : \omega(\pi) > \bar{w} \} \). Then, we calculate our measure using the simple formula:

\[ \Phi(\bar{w}) = \sum_{\pi \in \Pi^f} f(\pi) \]

These are our estimates, which rely on the \( f(\pi) \) distribution that we extract from the options. With multiple maturities, one needs instead a risk-adjusted distribution over paths of inflation over time, and a distribution of the maturity of debt coming due held in private hands, as we do in the main paper.

The alternative object:

\[ \text{Prob}^p \left( W_0 - \frac{m(s)}{\pi(s)} > \bar{w} \right) = \sum_{s \in S^p} p(s) \]

where \( S^p = \{ s : W_0 - \frac{m(s)}{\pi(s)} > \bar{w} \} \) does not answer the question posed in the paper. It would give the probability of debasement resulting from all variation in the data, whether that comes from inflation or not. That is the first major difference between our estimates and
the physical probability ones: we focus on inflation risk only. The inflation options provide
the right distribution for our question, since they are over $\Pi$ not $S$.

Second, consider the natural alternative object under the physical probabilities:

$$\phi(\tilde{\omega}) = \text{Prob}^p \left( W_0 - \frac{m(\pi)}{\pi} > \tilde{\omega} \right) = \sum_{\pi \in \Pi^p} p(\pi)$$

where the set is defined as $\Pi^f = \left\{ \pi : W_0 - \frac{m(\pi)}{\pi} > \tilde{\omega} \right\}$. Taking into account risk adjustment, that is comparing $\Phi(.)$ with $\phi(.)$ then requires not just comparing $f(\pi)$ and $p(\pi)$ but also
$\Pi^f$ and $\Pi^p$.

To make this comparison, we consider three cases in the text, each corresponding to
different assumptions about $m(\pi)$, which we now state formally:

**Case 1: classical dichotomy.** In this case, there is no correlation between the SDF and inflation, so there is no inflation risk premium. That is, $m(\pi) = m$ is a constant. Given the
way we defined it above, this constant is given by the expectation of the SDF over all the possible states:

$$m = \sum_{s \in S} p(s)m(s)$$

There can still be plenty of variability in $m(s)$ and so lots of other risk in the economy, but
the relevant $m$ with respect to inflation risk is just this constant. It then follow immediately that:

$$f(\pi) = m(\pi)p(\pi)/\mathbb{E}^p(m) = p(\pi)$$

$$\Pi^f = \left\{ \pi : W_0 - \frac{\mathbb{E}^p(m)}{\pi} > \tilde{\omega} \right\} = \left\{ \pi : W_0 - \frac{m(\pi)}{\pi} > \tilde{\omega} \right\} = \Pi^p$$

Therefore physical and risk-adjusted debasement probabilities coincide: $\Phi(\tilde{\omega}) = \phi(\tilde{\omega})$.

**Case 2: Additive risk premium in Gaussian models.** These are models in which
$m(\pi) = k(s)$. This $k(s)$ can vary over time, it is stochastic, but it is constant with respect to inflation. But since the definition of the risk-adjusted distribution $f(\pi)$ is only taking into
account inflation risk, it follows that $\mathbb{E}^p(m) = k(s)$. In other words, our risk-free rate is only risk-free w.r.t. inflation risk. Then, by the same steps as in the previous case, it follows that
physical and risk-adjusted probabilities of debasement coincide.

**Case 3: High inflation commands higher risk premium.** Our reading of the estimates
of Kitsul and Wright is the following: for \( \pi > \bar{\pi} \) we have \( m(\pi) > \mathbb{E}^p(m) \). That is, in the right tail of the distributions, it is always the case that \( f(\pi) > p(\pi) \). Loosely, the risk-adjusted probability distribution had a fatter right tail than the physical distribution.

Now, we are always looking at high percentiles in the debt debasement distribution. That is, we always consider large \( \bar{w} \). Therefore, we are always in the tail of the distribution above this \( \bar{\pi} \) where the risk-adjustment is a positive factor over the physical distribution. Now, this immediately delivers that, if one were to keep fixed the same set of inflation paths, the risk-adjusted probability is going to be higher. However to show that

\[
\Phi(\bar{w}) = \sum_{\pi \in \Pi^f} f(\pi) > \sum_{\pi \in \Pi^p} p(\pi) = \phi(\bar{w})
\]

also requires that \( \Pi^f \) is not too “small” in the sense of this expression relative to \( \Pi^p \). From the definitions above of these two sets, a sufficient (but not necessary) condition for this to be the case is that:

\[
\Pi^p \subset \Pi^f \quad \equiv \quad \left\{ \pi : \pi > \frac{m(\pi)}{\bar{w} - W_0} \right\} \subset \left\{ \pi : \pi > \frac{\mathbb{E}^p(m)}{\bar{w} - W_0} \right\}
\]

Since we already know that in this range \( m(\pi) > \mathbb{E}^p(m) \), only if \( \pi/m(\pi) \) is strongly non-monotonic would this not hold. But note that this is not necessary, and it seems plausible that it holds (see Kitsul and Wright (2013)). Therefore, our (very small) percentiles in the VAR are upper bounds on the physical probabilities.

Finally, to conclude, the assumptions we introduced in the first paragraph were just to make the maths reduce to simple operations. Considering a rich maturity of the debt would be the same but now the sets would be defined as sets of sequences of draws of inflation over time, and the assumption in case 3 would be with respect to cumulative inflation at several horizons (which still matches what Kitsul and Wright (2013) report). The assumption of countable sets makes it easier to explain what we do draw by draw instead of using measures and integrating, which requires introducing more elements of real analysis with no change in the underlying logic.
G Proof of proposition 3

Financial repression consists of paying nominal bonds that are due at date $t$ with new $N$-period special debt that sells for the price $\tilde{H}_t^N$:

$$B_t^0 = \tilde{H}_t^N \tilde{B}_{t+1}^{N-1}. \quad (A13)$$

The value of outstanding debt at date $t$ now is:

$$W_t = \sum_{j=0}^{\infty} \frac{H_t^j B_t^j}{P_t} + \sum_{j=0}^{\infty} Q_t^j K_t^j + \frac{\tilde{B}_{t-N+1}^{N-1}}{P_t}, \quad (A14)$$

while the government budget constraint now is:

$$W_t = s_t + \sum_{j=0}^{\infty} \frac{H_t^{j+1} B_t^{j+1}}{P_t} + \sum_{j=0}^{\infty} Q_t^{j+1} K_{t+1}^j + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1}}{P_t}. \quad (A15)$$

The text covered the special case where $N = 1$, there is no real debt, and all nominal debt had one period maturity. This appendix proves the general case.

Combining these two equations just as we did in the proof of proposition 1, we end up with a law of motion for debt:

$$W_t = Q_t^1 W_{t+1} + s_t + x_{t+1} + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1}}{P_t} - Q_t^1 \frac{\tilde{B}_{t-N+2}^{N-1}}{P_{t+1}}. \quad (A16)$$

By precisely the same steps as in the proof of proposition 1, it then follows that:

$$W_0 = \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( \frac{B_0^t}{P_t} \right) \right] + \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} K_0^t \right] = \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( s_t + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1} - Q_t^1 \tilde{B}_{t-N+2}^{N-1} P/P_{t+1}}{P_t} \right) \right]. \quad (A17)$$

Now, replacing the new debt with old debt using equation (A13), we get that the nominal debt burden then becomes:

$$\mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( B_0^t \right) \right] - \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( B_0^t \right) \right] + \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( \frac{Q_1 B_0^{t-N+1}}{H_t^N P_{t+1}} \right) \right]. \quad (A18)$$

Canceling terms and relabeling the limits of the sums (since financial repression started at
date 0) we get the nominal debt burden:

\[
\mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t+N-1} \frac{Q_{t+N-1}^1 B_t^0}{H_t^N P_{t+N}} \right].
\]  

(A19)

Finally, recall that \( Q_{t+N-1}^1 = \mathbb{E}_{t+N-1}(m_{t+N-1,t+N}) \). Using the law of iterated expectations we end up with:

\[
\mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t+N} \left( \frac{1}{\pi_{t,t+N} H_t^N} \right) \left( \frac{B_t^0}{P_t} \right) \right].
\]  

(A20)

To get the equality in the proposition, simply use the upper bound \( H_t^N = 1 \). To get the equality written in terms of the price of nominal bonds, simply use the law of iterated expectations and the arbitrage condition: \( H_t^N = \mathbb{E}(m_{t,t+N}/\pi_{t+N}) \).

References


