**Belot, BJPS 1998** *J.Butterfield* *jb56@cam.ac.uk*

## 1. Introduction

We learn something about our own world when we study the interpretative interaction between two false theories (533).

## 2. Gauge theories and their interpretation

*(i) Hamiltonian systems*: Phase space $\Gamma$; symplectic form $\omega$; dynamical trajectory $\gamma : t \mapsto \gamma(t) \in \Gamma$; determinism. More formally: configuration space $Q$; and the phase space is the cotangent bundle $T^*Q \ni (q, v)$.

*(ii) Gauge systems*: A presymplectic form: whose null space (kernel) has the same dimension at each point. So there is an equivalence relation on phase space, '... is connected by a null curve to...'. The equivalence classes are called gauge orbits; so phase space is partitioned into gauge orbits. There is determinism only with respect to which gauge orbit the state is in. So for a gauge-invariant quantity, i.e a quantity represented by a function $f : \Gamma \to \mathbb{R}$ such that if $[x] = [y]$ then $f(x) = f(y)$, there is determinism of its values.

*Interpretations of gauge systems*

(a): Literal interpretation: indeterminism; though one might hold one can only measure gauge-invariant quantities.

(b): Simply gauge-invariant interpretation: quotient to get a reduced phase space, whose points are the gauge equivalence classes. This is itself a Hamiltonian system. Determinism is restored.

(c): Coarse-grained gauge-invariant interpretation: like (b), except that gauge orbits represent *many-one* the physically possible states.

*Ceteris paribus*, surely (b) is preferable. But the reduced phase space can have a structure that is intractable, and even hard to interpret.

## 3. Interpreting (vacuum) electromagnetism

(1): Physical space is a 3-dimensional Riemannian manifold $S \ni \xi$. A phase space point is a pair of vector fields $A, E$ on $S$, with div $E = 0$.

NB: Here, $A$ is configurational, $E$ is momentum-like.

NB: In field theory, whether classical or quantum, $x$ typically becomes a parameter labelling field coordinates, instead of being a configurational variable.

(2): A presymplectic structure, with $(A, E) \sim (A', E')$ iff (i) $E = E'$ on all of $S$, and (ii) there is a smooth $\Lambda : S \to \mathbb{R}^3$ s.t. $A(\xi) = A'(\xi) + \text{grad}\Lambda$.

(3): The Hamiltonian $H := \int_S \mid E \mid^2 + \mid \text{curl}A \mid^2 \, d\xi$ determines the Hamilton's equations to be Maxwell's equations:

$$\dot{A} = -E \quad ; \quad \dot{E} = \text{curl}(\text{curl A}) \; . \qquad (0.1)$$

(4): Desiderata for interpretation: (i) determinism, (ii) synchronic locality (JNB: cf. Humean supervenience, separability), (iii) diachronic locality, which Belot takes to presuppose (entail) synchronic locality (JNB: cf. signal locality).

### 3.1: Interpretations

(1): *A as a physical field*:

A literal, and so indeterministic, interpretation. Synchronically local. But diachronically non-local: changing $B$ yield instantaneous changes in $A$ throughout space.

(2) *The traditional interpretation*:
Define $B := \text{curl } A$, and take $E, B$ as the physically real fields.
Indeed, curl(grad$\Lambda$) $\equiv 0$ implies $B$ is gauge-invariant. And Maxwell's equations for $E, B$ (viz. Belot's (3.2), which are interdeducible with eq. 0.1, are deterministic for $E, B$. This interpretation is synchronically and diachronically local.

But let us ask: Is it (a) a simply gauge-invariant interpretation, or (b) a coarse-grained gauge-invariant interpretation?

If $S$ is simply connected, the reduced phase space *is* indeed the set of divergence-free fields $E$ and $B$ on $S$. Here, $B$ is configurational, and $E$ is momentum-like. So the answer is: (a). Or in other words: we can adopt a literal interpretation of the reduced phase space.

But: If $S$ is multiply connected, the reduced phase space is as in (3) below: with the upshot that the answer to our question is: (b)—we have a coarse-grained gauge-invariant interpretation. That is: There are points $(A, E)$, $(A', E)$ on different gauge orbits (i.e. $A, A'$ differ by "more than" a gradient), whose $A, A'$ yield the same $B$. This will be the downfall of the traditional interpretation.

(3): *Holonomies*:
We begin by observing that the integral of $A$ around any closed curve $\gamma$ is gauge-invariant. (Reason: recall the elementary Stokes' theorem: if $\gamma$ is the boundary $\partial P$ of a patch $P$, then $\oint_\gamma \mathbf{A}.\mathbf{dr} \equiv \oint_{\partial P} \mathbf{A}.\mathbf{dr} = \int_P \text{curl}\mathbf{A}.\mathbf{dS}$.)

Indeed, the *holonomy* $h_A(\gamma)$ around any closed curve

$$h_A(\gamma) := \exp(\oint_\gamma iA_a(\xi)d\xi^a \ ) \quad \in \quad \mathrm{U}(1)\ . \tag{0.2}$$

is gauge-invariant.

The crucial fact is: the collection of all holonomies encodes the gauge orbit in the sense that: iff $A, A'$ are on the same gauge orbit, thenn for all $\gamma$, $h_A(\gamma) = h_{A'}(\gamma)$.

Indeed, we can use the latter to build a reduced phase space in terms of holonomies, as follows.

Define *loop space*, $\mathcal{L}$, as the set of all closed curves $\gamma$ in $S$.

Define the set $Q$ of all holonomy maps: $Q := \{h : \mathcal{L} \to \mathrm{U}(1)\}$.

$Q$ is a manifold, and it serves as the configuration space, whose cotangent bundle will be our reduced phase space!

So we define $T^*Q \ni (h, E)$, with the canonical symplectic form, and with:

(i) $E$ a divergence-free field on $S$, and

(ii) $h \in Q$ corresponding to a gauge-equivalence class of potentials $A$.

Imposing the correct Hamiltonian as a scalar on $T^*Q$ ("lifted" from the previous $H := \int_S \mid E \mid^2 + \mid B \mid^2 d\xi$ on $S$), we get the reduced phase space formulation of electromagnetism. (JNB: for more details, cf. section 7 of Belot, 'Symmetry and Gauge Freedom', SHPMP 2003, p. 189.)

This formulation supports a literal and deterministic interpretation. But closed curves are the bearers of electromagnetic properties! So obviously, the interpretation is not synchronically local; so it is also not diachronically local.

(4) *Assessment*:

Overall, we would like: (L): locality; and

either (a) a simple gauge-invariant interpretation, or (b) a literal interpretation of the reduced phase space; where both (a) and (b) secure determinism.

But we have seen two conclusions:

(i): For multiply connected $S$, (2) is only a coarse-grained gauge-invariant interpretation.

(ii): (3) gets us (b), but with a flagrant loss of (L).

So it is surely best to prefer (2) and say that its coarse-graining loses no physical information. Classically, this looks good! No classical phenomena discern a difference between gauge-equivalent $A, A'$ that yield the same $B$.

**4: Quantization and the AB effect**

**4.1: Quantization and interpretation**

NB: *Any* Hamiltonian system with a topologically non-trivial phase space $M$ has infinitely many quantizations, parameterized by a cohomology group $H^1(M, U(1))$.

The general philosophical perspective: interpretations, as well as theoretical or empirical considerations, can—and should!—influence which quantizations (or approaches to quantization) are appropriate.

**4.2 Quantizing a charged particle in a (classical) magnetic field**

(1): A classical particle in a static magnetic field. $Q := S$. $\Gamma := T^*S$, but we use a non-canonical $B$-dependent symplectic form, call it $\omega^B$, together with the Hamiltonian of a free particle $H := P^2/2m$.

(2): NB: This has a unique quantization iff $S$ is topologically trivial.

(3): We take $S := \mathbb{R}^3$ - $y$-axis, and we set $E = B = 0$. So the phase space simplifies in that, since $B = 0$, we can take the canonical symplectic form $\omega$.

In fact: $H^1(T^*S, U(1)) = U(1)$.

So: *the infinitely many quantizations are labelled by elements* $e^{i\theta} = z \in U(1)$.

Let us write $Q(z)$ for the quantization labelled by $z$.

The intuitive physical idea: $Q(z)$ predicts that a quantum charged particle going round the $y$-axis gets a phase shift $z$.

(4): What attitude should we take to the various quantizations?

(4 a): Suppose we adopt a simply gauge-invariant or a literal interpretation.

Then the electromagnetic state of the world contains more than just $E$ and $B$, since $B$ corresponds to many gauge orbits.

In fact, in our set-up as described in (3), with $E = B = 0$: it is easy to argue that the extra information, beyond specifying $B$, is encoded in an element $z \in U(1)$, as follows. (Now JNB expands footnote 28, p. 548.)

Collect the following three observations :

(a): Recall the elementary Stokes' theorem applied to a curve $\gamma$ that does not enclose the y-axis: $\oint_\gamma \mathbf{A}.\mathbf{dr} = \int_P \mathrm{curl}\mathbf{A}.\mathbf{dS}$. So if $A_1, A_2$ determine the same $B$, then on any curve *not* enclosing the $y$-axis, $A_1$ and $A_2$ have the same holonomy, $\exp i \oint_\gamma A$. So if two such $A_1, A_2$ are not gauge-equivalent, i.e. not$(A_1 \sim A_2)$, their holonomies can only differ on some loops enclosing the $y$-axis.

(b): Since $B = 0$, any two homotopic loops have the same holonomy.

(c): For $S = \mathbb{R}^3$ - $y$-axis, homotopy-equivalence classes of homotopic loops are labelled by the winding number, a non-negative integer $w \in Z^+$.

Putting these observations together, we get:

The holonomy of any curve with winding number $w$ is given by: $w$ times the holonomy $\exp i \oint_\gamma A$ of *any* curve $\gamma$ that winds just once around the $y$-axis.

To sum up: the extra information in the electromagnetic state, beyond specifying $B$, is in the holonomy of (any element of) the equivalence class with winding number $w = 1$.

So on this view: the fact reported in (3), that there are the infinitely many quantizations labelled by elements $e^{i\theta} = z \in U(1)$, is *no surprise*! Especially when we recall that, since $B = 0$, we could take the canonical symplectic form $\omega$; i.e. the Hamiltonian treatment of the particle encoded nothing about the electromagnetic field.

(4 b): Suppose we adopt the traditional interpretation.
Then we maintain that there are no "hidden variables" which could determine which quantization is physically appropriate. So we surely expect there to be only one physically relevant quantization. Especially when we recall that we use the same classical model for an uncharged free particle; which certainly has a unique correct quantization.

**4.3 The AB effect**

(1): The idea: restrict an electron to a region which can be treated for all practical purposes (FAPP, as John Bell would say!) as multiply connected.

(1 a): We set up a solenoid along the $y$-axis, so that $B = 0$ outside the solenoid, while $B \neq 0$ inside. (In fact, by Stokes' theorem: the magnetic flux through a patch $P$ covering the $y$-axis, $\Phi := \int_P \mathbf{B}.\mathbf{dS}$, is the "holonomy" $\oint_{\partial P} \mathbf{A}.\mathbf{dr}$.)

(1 b): We take the walls of the solenoid to be impenetrable by the electron, and so model the experiment as in (3) of Section 4.2 above.

(2): Experimental description: The interference pattern arising from shooting electrons at the solenoid (so that each electron has two routes round the solenoid) is sensitive to the magnetic flux $\Phi$.

Theoretical description: the quantization that gives the correct predictions is $Q(z_0)$, where $z_0$ is the experiment's actual holonomy around the solenoid.

(3): Physicists tend to agree that:

(i) It is not worth repairing/defending the traditional interpretation, by saying that $B$ acts non-locally.

(ii) The holonomy interpretation (i.e. (3) of Section 3.1) is better than '$A$ as a physical field' interpretation (i.e. (1) of Section 3.1). That is: it is preferable to save determinism, and sacrifice synchronic locality (and so also, diachronic locality); than to save synchronic locality, and sacrifice determinism.

JNB comments:— (1): I think most philosophers would incline the other way: accustomed to indeterminism (from popular accounts of quantum theory?!), they would be more willing to give up determinism than locality.

(2): This failure of synchronic locality is surely more radical than the failure in many-particle quantum theory. There, we describe $N$ particles by a state $\psi \in L^2(\mathbb{R}^{3N})$, which is a function that assigns a complex number $\psi(x_1, ..., z_N) \in \mathbb{C}$ to each $N$-tuple of spatial points, i.e. $3N$-tuple of real numbers $< x_1, ..., z_N >$.

This, I submit, is "in the spirit" of the one-particle case, using $L^2(\mathbb{R}^3)$: not withstanding all the phenomena, and subtlety, that follow from entanglement.

Contrast the holonomy interpretation of classlical electromagnetism, i.e. (3) of Section 3.1. Here, a configuration is $h : \mathcal{L} \rightarrow \mathrm{U}(1)$, with $\mathcal{L}$ both infinite-dimensional and intricate: in a way that $\mathbb{R}^{3N}$ is not.

## 5. Conclusion

Belot gives a wise discussion of how the interpretation of false theories can help tll us about the world. We get especial insight when we assess interpretations of a pair of theories in a domain where they overlap—as in the AB effect.