When we do (and do not) have a classical arrow of time

Bryan W. Roberts

School of Philosophy
University of Southern California
3709 Trousdale Parkway
Los Angeles, CA 90089-0451
bryan.roberts@usc.edu

Abstract. I point out that some common folk wisdom about time reversal invariance in classical mechanics is strictly incorrect, by showing some explicit examples in which classical time reversal invariance fails, even among conservative systems. I then show that there is nevertheless a broad class of familiar classical systems that are time reversal invariant.

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1. Introduction

A physical law is time reversal invariant if whenever a motion is allowed by the law, there is a nemesis "time-reversed" motion that is also allowed by the law, corresponding roughly to what one would see if a film of the original motion were played in reverse. A time reversal invariant law lacks the structure to distinguish future-directed motion from past-directed motion. In this sense it does not admit an "arrow of time." The laws of many familiar classical systems do have this interesting property. But despite what is often said, time reversal invariance is not a general property of the laws of classical mechanics – not without further qualification.

1.1. The meaning of time reversal.

In the Newtonian "force" formulation of classical mechanics, time reversal is simply the reversal of the order of events in a trajectory $x(t)$. That is, if $x(t)$ is the curve describing the position x of a particle in space at every moment in time t, then the time-reversed motion is given by the curve $x(-t)$, which describes the same positions occurring in the reverse order.

In the Hamiltonian formulation of classical mechanics, reversing the order of events in a trajectory is not enough. Since time reversal transforms a rightward-moving body into a leftward-moving body, the momentum of an instantaneous state is reversed, $T(q,p) = (q,-p)$.

The operator T appearing in the first part of the transformation is referred to as the *time reversal operator*. One can speak quite generally about time reversal operators, in a way that applies to both the Hamiltonian and the Newtonian formulations. In general, the time reversal operator is a bijection on a theory's space of states, whatever that space may be. In the Hamiltonian description of a particle on a string, it is an operator on phase space, $T : \mathbb{R}^2 \to \mathbb{R}^2$. In the Newtonian description, it is a transformation of *physical* space, $T : \mathbb{R} \to \mathbb{R}$. The latter is easy to miss, because it is simply the identity transformation $T(x) = x$. The former is more conspicuous, because it is in general not the identity. When we say *time reversal*, we mean both the application of the time reversal operator on states, together with the reversal of the order of states in trajectories.

1.2. Time reversal invariance.

Invariance of a law under a transformation means that if one motion is possible according to the law, then the transformed motion is possible as well. More precisely:

Definition 1 (time reversal invariance)**.** Let $\gamma(t) : \mathbb{R} \to \mathcal{M}$ be a curve through some manifold of states $\mathcal{M}$, which characterizes a dynamical trajectory. Let $T : \mathcal{M} \to \mathcal{M}$ be the time reversal operator with respect to $\mathcal{M}$. A theory of curves on $\mathcal{M}$ is called *T-reversal invariant* (or simply *time reversal invariant*) if, whenever $\gamma(t)$ is a possible trajectory according to the theory, then so is $T\gamma(-t)$.

We say "the time reversal operator with respect to $\mathcal{M}$" because, at this level of generality, one cannot say much more than that about T. Its meaning depends on physical facts about the degrees of freedom that the space of states $\mathcal{M}$ represents. If $\mathcal{M} = \mathbb{R}^3$ represents the location of a particle in space, T is the identity operator. On the other hand, if $\mathcal{M} = \mathbb{R}^6$ represents the position and linear momentum of a particle, T is not the identity.

1.3. **Two useful facts.** Let me conclude this section by stating two useful (though not novel) facts, which will facilitate the identification of time reversal invariance in the remainder of our discussion.

Lemma 1. *The statement that* $h(q,p) = h(q,-p) + k$ *(for some* $k \in \mathbb{R}$*) is equivalent to the statement that* $(q(-t), -p(-t))$ *satisfies Hamilton's equations whenever* $(q(t), p(t))$ *does.*

Proof. ($\Rightarrow$): Suppose $h(q,p) = h(q,-p) + k$. Let $(q(t), p(t))$ satisfy Hamilton's equations with $h(q,p)$. Since these equations hold for all t, we can substitute $t \mapsto -t$ to get $\frac{dq(-t)}{d(-t)} = \frac{\partial h(q,p)}{\partial p}$ and $\frac{dp(-t)}{d(-t)} = -\frac{\partial h(q,p)}{\partial q}$. The former implies $\frac{dq(-t)}{dt} = \frac{\partial h(q,-p)}{\partial(-p)}$, and the latter implies $\frac{d(-p(-t))}{dt} = -\frac{\partial h(q,-p)}{\partial q}$, by simply pushing negative signs around and by our hypothesis that $h(q,p) = h(q,-p) + k$. But this just says that the time-reversed trajectory $(q(-t), -p(-t))$ satisfies Hamilton's equations.

($\Leftarrow$): Suppose that (a) $(q(t), p(t))$ and (b) $q(-t), -p(-t)$ are both solutions. Substituting $t \mapsto -t$ into Hamilton's equations with (a) gives $\frac{d}{d(-t)}q(-t) = \frac{\partial h(q,p)}{\partial p}$ and $\frac{d}{d(-t)}p(-t) = -\frac{\partial h(q,p)}{\partial q}$. Hamilton's equations with (b) give $\frac{d}{dt}q(-t) = \frac{\partial h(q,-p)}{\partial(-p)}$ and $\frac{d}{dt}(-p(-t)) = -\frac{\partial h(q,-p)}{\partial q}$. Combining, we find that

$$\frac{\partial h(q,p)}{\partial p} = \frac{\partial h(q,-p)}{\partial p}, \qquad \frac{\partial h(q,p)}{\partial q} = \frac{\partial h(q,-p)}{\partial q}.$$

This implies that $h(q,p) = h(q,-p) + f(q,p)$, for some function f such that $\partial f/\partial q = \partial f/\partial p = 0$. But the only such function is a constant function, so $h(q,p) = h(q,-p) + k$ (for some $k \in \mathbb{R}$). $\square$

Lemma 2. *Let* $F(x, v, t)$ *be a force that might depend on position, time, or velocity* $v = dx/dt$*. The statement that* $F(x, -v, -t) = F(x, v, t)$ *is equivalent to the statement that Newton's equation is time reversal invariant, i.e. that* $x(t)$ *satisfies Newton's equation only if* $x(-t)$ *does.*

The proof of this statement is very similar to that of Lemma 1, and so I omit it.

2. What does *not* underpin classical TRI

Overzealous textbook authors have been known to make the following sweeping claim.

Claim 1. *Classical mechanics is time reversal invariant.*

Philosophers have sometimes fallen for this ruse as well. For example, Frigg (2008) writes that time reversal invariance (TRI) cannot fail in the Hamiltonian formulation of classical mechanics (which he calls HM).

> HM is TRI in this sense. This can be seen by time-reversing the Hamiltonian equations: carry out the transformations $t \to \tau$ [where $\tau = -t$] and $(q,p) \to R(q,p)$ and after some elementary algbraic manipulations you find $dq_i/d\tau = \partial H/\partial p_i$ and $dp_i/d\tau = -\partial H/\partial p_i, i = 1, \ldots, m$. Hence the equations have the same form in either direction of time. (Frigg 2008, p.181)

Frigg's conclusion, like Claim 1, is strictly incorrect. A simple counterexample is a classical system with a so-called "dissipative" force[1]. For example, Newton's laws (and Hamilton's equations) allow trajectories in which a block slides along a smooth surface, subject to the force of friction, until eventually coming to a stop. However, the time-reversed trajectory of a block that spontaneously begins accelerating from rest is not a possible solution. These systems are described by Hamiltonians for which $h(q,p) \neq h(q,-p) + k$. As we observed in Lemma 1, this is sufficient for the failure of time reversal invariance. The significance of such examples for time reversal has been emphasized by Hutchison (1993).

More charitably, Frigg and other authors sympathetic to Claim 1 must make a tacit assumption about the scope of classical mechanics. For example, we can avoid dissipative forces in the description of elementary classical systems by requiring (for example) that $dH/dt = 0$ in the Hamiltonian formulation. One might hope to conclude time reversal invariance on the basis of such an assumption. Indeed, Callender (1995) responds to Hutchison by arguing that systems with dissipative forces like friction are not "interesting" examples of classical systems, at least from a foundational perspective. The apparent "force" of friction only arises out of an incomplete description of the block on the surface. If the more elementary interactions between the block and the surface were accounted for, then the force describing the system would take a very different form. Time reversal invariance would stand a chance of being regained.

However, we are not out of the woods yet. These authors are suggesting the following.

Claim 2. *Classical mechanical systems that are "conservative" are time reversal invariant.*

For example, Callender (1995, p.334) writes that, on the assumption that there are no non-conservative forces, "it is easy to verify that classical mechanics is TRI." The correctness of that claim, however, hinges on the precise definition of the term 'conservative.' There are two points that I would like to make about this. First, on the usual definition of a conservative system, as one that "conserves work" or "conserves energy," Claim 2 is simply false. Even the assumption that $dH/dt = 0$ is not enough for Frigg to conclude that the Hamiltonian formulation is time reversal invariant. I provide counterexamples in Sections 2.1 and 2.2 below. Second, there is a stronger definition of "conservative" that requires the force $F(x,t)$ (or the Hamiltonian $h(q,p)$) to take a certain functional form, which depends only on position. This *is* sufficient for time reversal invariance. However, the physical motivation for that stronger requirement has not yet been made clear. I will discuss this point in Section 2.3.

2.1. **Conservative but not TRI, part I.** Velocity-dependent forces can be conservative in a natural sense, but still fail to be time reversal invariant. To see this, we'll make use of a textbook definition of "conservative" in the Newtonian formulation. Consider a segment of a smooth curve $x(t)$ through configuration space, with

[1] To put an even finer point on the problem with Frigg's statement: there are many Hamiltonians with the property that $h(q,p) \neq h(q,-p)$. As we noted at the end of the last section, this is sufficient for the failure of time reversal invariance.

an initial point $x(t_1)$ and a final point $x(t_2)$. The work required to transport the system from $x(t_1)$ to $x(t_2)$ is given by the work integral,

$$W\left(x(t_1), x(t_2)\right) = \int_{t_1}^{t_2} (F \cdot v)\, dt,$$

where $v = dx/dt$. We say[2] that a system is *conservative* if, whenever the curve $x(t)$ is a closed loop with $x(t_1) = x(t_2)$, it follows that the work done around the loop is zero. For example, if a particle sets out along a path through space that ends where it started, then no work is accrued during the round-trip journey. Work is "conserved" along such a loop, in the sense that nothing is added or lost.

Although many conservative forces depend only on position, this definition of a conservative system makes perfect sense when a force depends on velocity as well. For example, consider a particle in a force field $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, which exerts a force $F(x, v)$ on the particle when it has position and velocity (x, v). Defining the work integral as before, we can again take a conservative system to be one in which work is conserved on a closed loop $x(t)$ through configuration space $\mathbb{R}^3$. We must only take care that some other well-known definitions of a conservative system, such as one involving path-independence, may not be mathematically well-defined in this context[3].

Here is a simple example of a system that is conservative in this sense but not time reversal invariant[4]. Take a particle in three spatial dimensions, with position $x = (x_1, x_2, x_3)$. As a shorthand, we will write $v = dx/dt$, and thus denote the particle's velocity by $v = (v_1, v_2, v_3)$. Suppose the particle is subject to a force field defined by the cross-product,

$$F = x \times v.$$

This describes a force on a particle that is orthogonal to both its position and velocity vectors, which is strange but easy to model[5].

This system is "conservative" on the definition above. The reason is that the cross product $(x \times v)$ is orthogonal to v. So, the integral characterizing work W along a section of any curve $x(t)$ vanishes:

$$W(x(t_1), x(t_2)) = \int_{t_1}^{t_2} (F \cdot v)\, dt = \int_{t_1}^{t_2} ((x \times v) \cdot v) dt = 0.$$

[2](c.f. Goldstein et al. 2002, p.3).

[3]It is often said that a system is conservative if work is *path independent*, in that the work integral depends only on the initial position $x(t_1)$ and the final position $x(t_2)$. This is equivalent to our definition when the force F depends only on position; however, it is not necessarily well-defined in the presence of velocity-dependent forces. This is because there can be many velocity-dependent force vectors $F(x, v)$ at the same position x, corresponding to the different values of v. A work integral depending only on initial and final position is not well-defined in such cases, since a position x is not enough to determine the value of $F(x, v)$. I thank David Malament for pointing this out to me.

[4]I thank Wayne Myrvold for pointing out this example.

[5]There is an animation of the motion for a typical initial state available at `http://www.youtube.com/watch?v=-3hv3-YVA-E`. I thank Peter Distelzweig for showing me how to simulate this motion using vPython.

No work is ever done along any path whatsoever, and the system is trivially conservative. There are also less trivial conservative forces of this kind, such as,

$$F = x + (x \times v).$$

This is a less trivial force, in the work done is not always zero. However, it is still a conservative force, because it is the sum of conservative forces.

Nevertheless, both systems fail to be time reversal invariant. To verify this formally, we simply observe that $F(x, -v, -t) = x \times (-v) = -F(x, v, t)$. So, $F(x, -v, -t) \neq F(x, v, t)$, and time reversal invariance fails by Lemma 2.

2.2. **Conservative but not TRI, part II.** There is another natural definition of a "conservative" system in the context of Hamiltonian mechanics. Namely, since we interpret the Hamiltonian h to represent a system's total energy, "conservative" can naturally be taken to mean that h is a conserved quantity, $dh/dt = 0$.

However, there are many conservative systems of this kind that violate time reversal invariance. A simple example is the system for which $h(q, p) = p^3$ at every point $(q, p) \in \mathbb{R}^2$. Another is $h(q, p) = \frac{1}{2}(p - q)^2$. Since each satisfies $dh/dt = \partial h/\partial t = 0$, each system is conservative in the required sense[6]. However, since $h(q, -p) \neq h(q, p) + k$, Lemma 1 implies that each system violates time reversal invariance.

2.3. **'Strong' Conservative implies TRI.** In the context of Newtonian force mechanics, Arnold (1989, p.22) defines a conservative system to be one in which all forces have a particular functional form:

$$F(x, t) = \nabla V(x),$$

for some scalar field $V(x)$, which (crucially) depends only on position. We might refer to this as "strong" conservativeness. On this definition, Newton's equation is manifestly time reversal invariant, because the force has no time or velocity dependence. Thus $F(x, v, t) = F(x, -v, -t)$, and we have time reversal invariance by Lemma 2.

This is one way to guarantee time reversal invariance. But it would be nice to determine *physical* conditions that guarantee time reversal invariance, in a way that is more informative than just demanding forces have a certain functional form. It is certainly the case the many familiar forces appear to have a form that is time reversal invariant. But is it possible to explain why this is the case?

The question has an analogue in the Hamiltonian formulation. If the Hamiltonian h has its "common" form $h = (m/2)p^2 + v(q)$, then $h(q, p) = h(q, -p)$, and we are guaranteed time reversal invariance. But what reason do we have to think that the Hamiltonian must have this functional form? One would like to go beyond the superficial fact that many classical systems happen to be that way, and ask *why* familiar classical Hamiltonians (or classical force fields) tend to have a form that is time reversal invariant. In the next section, I will point out two ways to answer this question.

[6] Here I make use of the fact that $dh/dt = \partial h/\partial t$.

3. What *does* underpin classical TRI

We have seen some common claims that do not guarantee time reversal invariance. Now let us turn to some examples of claims that do. We noticed before that many "familiar" classical systems are time reversal invariant. This is, I claim, because some properties that make a classical system "familiar" are also sufficient to guarantee time reversal invariance. In the first place, a classical system is familiar if it satisfies $p = mv$, the property of having momentum proportional to velocity. In the second place, familiar classical systems often satisfy certain symmetries, such as invariance under translations and Galilei boosts. I will show that each guarantees time reversal invariance.

3.1. **Velocity-momentum proportionality and TRI.** Suppose a classical system is described by a $2n$-dimensional phase space $\mathcal{P}$ and a Hamiltonian function $h : \mathcal{P} \to \mathbb{R}$. Such a system is only time reversal invariant if $h(q, p) = h(q, -p)$, according to Lemma 1. Nevertheless, time reversal invariance *is* guaranteed in the presence of velocity-momentum proportionality.

Claim 3. *If the momentum of a system is proportional to its velocity, $p = mv$, then the system is time reversal invariant.*

For example, we saw in Section 1.1 that $p = mv$ is an essential part of what it means to be a classical bob on a spring. Indeed, part of what makes many classical systems familiar is that they have their momentum pointed in the same direction as their velocity. Such systems, it turns out, are always time reversal invariant. This can be shown by the following simple calculation.

Proposition 1. *Suppose $p = mv$ on every trajectory $(q(t), p(t))$ of a system with Hamiltonian $h(q, p)$, where m is a constant and $v = dq/dt$. Then $h(q, p) = \frac{1}{2}mv^2 + f(q)$ for some function f, and the system is time reversal invariant.*

Proof. Taking partial derivatives of $p = mv$ with respect to v gives $\partial p = m\partial v$. So, we may substitute $m\partial v$ in for the ∂p appearing in hamilton's equations,

$$v = \frac{d}{dt}q = \frac{\partial h(q,p)}{\partial p} = \frac{\partial h(q,p)}{m\partial v}.$$

Now multiply by $m\partial v$ on both sides, so that we can integrate for h to get

$$h(q,p) = \int mv\partial v = \tfrac{1}{2}mv^2 + f(q)$$

for some function $f(q)$ of q alone. This Hamiltonian obviously satisfies $h(q, p) = h(q, -p)$, and so we have time reversal invariance by Lemma 1. $\square$

This provides a first step toward understanding the extent to which classical mechanics is time reversal invariant. It may be summarized as follows. Classical mechanics does allow a variety of "anomalous" systems that are not time reversal invariant, even among those systems that conserve energy. But, if the momentum of a particle is proportional to its velocity, then none of these anomalous systems are allowed. Time reversal invariance is guaranteed.

3.2. **Galilei invariance and TRI.** Another aspect of familiar classical systems is that they have certain symmetries. For example, one may be used to the idea of obtaining the same results when the same experiment is set up in two different locations in space. This is known as invariance under *spatial translations*. Similarly, one might think that if an experiment were done in the cabin of a boat moving with constant velocity across the sea, then we would find the same result as if the experiment were set up on shore. This is known as invariance under *Galilei boosts*. There is a precise sense in which systems that satisfy these familiar properties are guaranteed to be time reversal invariant as well. A rough statement of this fact may be summarized as follows, although it is no substitute for the more precise expression given in Proposition 2 below.

Claim 4. *If a classical Hamiltonian system is invariant under spatial translations and Galilei boosts, then it is also time reversal invariant.*

Settling this statement requires a precise language in which to talk about symmetries in the Hamiltonian formulation. Unfortunately, talk of symmetries in classical mechanics is often reduced to imprecise language involving "preserving the form" of an equation of motion. In order to avoid risking the pitfalls of that language and of coordinate-based approaches in general, I have chosen in what follows to adopt a more precise geometric formalism, in which Claim 4 can be clearly stated and settled. The price is that a fair amount of mathematical machinery must be introduced along the way. I proceed hoping that the precise result might be worth the cost.

3.2.1. *Notation.* The geometric formalism for Hamiltonian mechanics[7] eliminates the assumption that classical phase space has the topology of $\mathbb{R}^{2n}$, and eliminates any preferred coordinate system (q, p). Instead, we focus on coordinate-invariant structures built on a smooth $2n$-dimensional manifold $\mathcal{P}$ (for "$\mathcal{P}$hase space"), which may or may not be $\mathbb{R}^{2n}$. Each point $x \in \mathcal{P}$ is still interpreted as a "possible state" of a classical system, and we can still think of an observable quantity (like energy or spatial position) as a function $f : \mathcal{P} \to \mathbb{R}$. I adopt Penrose's "abstract index" notation to talk about invariant structures on phase space, denoting a vector v^a on $\mathcal{P}$ with an index upstairs, and a covector w_a with an index downstairs. The operation of contraction (sometimes called "interior multiplication" or "index summation") between tensors will be indicated by a common index in both upper and lower positions, such as $w^a v_a$. The unique exterior derivative on k-forms of a manifold will be denoted d_a. I denote[8] the symplectic form Ω_{ab}, its inverse by Ω^{ab}, and the Poisson bracket by $\{\cdot, \cdot\}$.

To describe how a classical system changes over time, we equip our symplectic manifold $(\mathcal{P}, \Omega_{ab})$ with a distinguished smooth function $h : \mathcal{P} \to \mathbb{R}$, which we call the "Hamiltonian." This h represents the energy at each point in phase space, and also generates the possible trajectories of the system, given by the integral curves that thread the associated vector field $H^a := \Omega^{ba} d_b h$. We denote the group

[7]For a reference on this approach, try (Geroch 1974) or (Marsden and Ratiu 2010).

[8]A *symplectic form* Ω_{ab} is a 2-form on $\mathcal{P}$, which is closed ($d_a \Omega_{bc} = \mathbf{0}$) and non-degenerate ($\Omega_{ab} v^a = \mathbf{0} \Rightarrow v^a = \mathbf{0}$. The *Poisson bracket* on a pair of functions $f, h : \mathcal{P} \to \mathbb{R}$ is defined by $\{f, h\} := \Omega^{ab}(d_a h)(d_b f)$. In coordinate form, it is usually written in terms of partial derivatives, $\{f, h\} = \frac{\partial f}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial h}{\partial q}$.

of transformations describing how a system evolves along those possible trajectories by $\varphi^h_t : \mathcal{P} \to \mathcal{P}$, which maps an initial state $x \in \mathcal{P}$ to the state $x(t)$ that occurs a duration of time t later. Each φ^h_t has the property that $\frac{d}{dt}\left(f \circ \varphi^h_t\right)\big|_{t=t_0} = H^a d_a f$, for all functions $f : \mathcal{P} \to \mathbb{R}$.

3.2.2. *Defining invariance under translations and boosts.* We noted that the presence of certain symmetries, such as invariance under spatial translations and Galilei boosts, may make a classical system $(\mathcal{P}, \Omega_{ab}, h)$ look "familiar." In this section, we will motivate and give precise expression to those symmetries. The beautiful generality of the Hamiltonian formulation is that a state in phase space can represent anything at all, from the position of a particle to the angle of a swinging pendulum. But we must now talk about spatial translations and Galilei boosts. This requires introducing additional structure to guarantee we're talking about *position*.

Following Woodhouse (1991, §4.5), we will talk about position in terms of what is called a "maximal orthogonal set" (or sometimes a "real polarization") on $\mathcal{P}$.

Definition 2. A *maximal orthogonal set* for a $2n$-dimensional manifold $\mathcal{P}$ is a set $\{\overset{1}{q}, \overset{2}{q}, \ldots, \overset{n}{q}\}$ of n smooth functions $\overset{i}{q} : \mathcal{P} \to \mathbb{R}$ such that (i) $\{\overset{i}{q}, \overset{j}{q}\} = 0$ for each $i, j = 1, \ldots, n$, and (ii) if f is another smooth function satisfying $\{f, \overset{i}{q}\} = 0$ for all i, then $f = f(q) = f(\overset{1}{q}, \ldots, \overset{n}{q})$ is a function of the $\overset{i}{q}$.

For example, if think of positions as points in $\mathbb{R}^n$, and represent phase space by the cotangent bundle $\mathcal{P} = T^*\mathbb{R}^n$, then the analogues of those positions in $\mathcal{P}$ form a maximal orthogonal set[9]. This particular maximal orthogonal set allows us to talk about position on the manifold $\mathcal{P} = T^*\mathbb{R}^n$. A maximal orthogonal set in the abstract allows us to talk about position in classical mechanics for any arbitrary manifold $\mathcal{P}$.

Now that we have a structure we can refer to as "position," we can define "velocity" or instantaneous change in position over time. Since change over time is given by the phase flow φ^h_t generated by h, the velocity of a function q is given by $\dot{q}(t) := \frac{d}{dt}(q \circ \varphi^h_t)$. In what follows, we will make use in particular of the *initial velocity* $\dot{q}$ of a classical system, given by

$$\dot{q} := \dot{q}(0) = \frac{d}{dt}(q \circ \varphi^h_t)\bigg|_{t=0} = H^b d_b f = \{q, h\}.$$

With a definition of position and velocity in hand, we may now define what spatial translations and Galilei boosts mean.

Definition 3 (Translations and Boosts)**.** We take a *translation and boost group* for a classical system $(\mathcal{P}, \Omega_{ab}, h)$ to be a $2n$-parameter family of diffeomorphisms $\Phi(\sigma, \rho) : \mathcal{P} \to \mathcal{P}$, which forms a representation of $\mathbb{R}^{2n}$, and such that

(1) $q \circ \Phi(\sigma, \rho) = q + \sigma$
(2) $\dot{q} \circ \Phi(\sigma, \rho) = \dot{q} + \rho$

[9]That is, given a Cartesian coordinate chart $\{\overset{1}{q}, \overset{2}{q}, \ldots, \overset{n}{q}\}$ on $\mathbb{R}^n$, the set $\{\overset{1}{q} \circ \pi, \overset{2}{q} \circ \pi, \ldots, \overset{n}{q} \circ \pi\}$ is a maximal orthogonal set for $\mathcal{P}$, where π is the canonical projection, $\pi : (q, p) \mapsto q$ (Woodhouse 1991, §4.5).

where $q = \{\overset{1}{q}, \ldots, \overset{n}{q}\}$ is a maximal set of orthogonal functions, and $\dot{q}$ is the corresponding initial velocity. We define two associated diffeomorphism groups $\varphi^s_\sigma := \Phi(\sigma, 0)$ and $\varphi^r_\rho := \Phi(0, \rho)$, and refer to them as the *translation group* and the *boost group*, respectively. When these groups have a generator, we denote those generators by $s : \mathcal{P} \to \mathbb{R}$ and $r : \mathcal{P} \to \mathbb{R}$ respectively, and ask that they satisfy $\{\overset{i}{s}, \overset{j}{s}\} = \{\overset{i}{r}, \overset{j}{r}\} = 0$ for all i, j, to guarantee the translations and boosts are defined in n independent directions.

The notion of "invariance" under translations and boosts will be similar to what we have discussed so far: if a trajectory is possible, then so is the transformed trajectory. We may state this precisely as follows. Let H^a be a vector field for a given Hamiltonian h, which represents a set of possible trajectories. Let $\Phi : \mathcal{P} \to \mathcal{P}$ some transformation, and let us use a starred Φ^* and Φ_* to denote its pullback and pushforward. For a classical system to be *invariant* under Φ means that transformed trajectories $\tilde{H}^a := \Phi_* H^a$ are also a set possible dynamical trajectories with respect to some Hamiltonian function $\tilde{h}$; that is, $\tilde{H}^a = \Omega^{ba} d_b \tilde{h}$ for some smooth $\tilde{h} : \mathcal{P} \to \mathbb{R}$.

One can show that $\tilde{H}^a := \Phi_* H^a$ if and only if Φ is *symplectic*, meaning that it preserves the symplectic form: $\Phi^* \Omega_{ab} = \Omega_{ab}$ (Marsden and Ratiu 2010, Proposition 2.6.1). So, requiring classical systems to be invariant under translations and boosts can be expressed by the requirement that translations and boosts be symplectic. This motivates the following simple definition.

Definition 4 (Translation and Boost Covariance)**.** A classical system $(\mathcal{P}, \Omega_{ab}, h)$ is *invariant under translations and boosts* if there exists a translation and boost group $\Phi(\sigma, \rho)$ on $\mathcal{P}$ such that each element of the group is symplectic, in that $\Phi^*(\sigma, \rho)\Omega_{ab} = \Omega_{ab}$ for all σ, ρ.

3.2.3. *Establishing time reversal invariance.* With these definitions in hand, we may now formulate our main result. We take the *time reversal operator* to be a transformation $\tau : \mathcal{P} \to \mathcal{P}$ such that $\tau^* q = q$ and $\tau^* \dot{q} = -\dot{q}$. The time reverse of a classical system $(\mathcal{P}, \Omega_{ab}, h)$ with Hamiltonian vector field H^a is then the transformation that takes each integral curve $c(t)$ of H^a to $\tau \circ c(-t)$.

Proposition 2. *If a classical system $(\mathcal{P}, \Omega_{ab}, h)$ is invariant under translations and boosts, then it is time reversal invariant, in that if $c(t)$ is an integral curve of the Hamiltonian vector field generated by h, then so is $\tau \circ c(-t)$.*

The proof of this proposition makes use of a lemma, which shows that translation and boost invariance places strong constraints on the form of the Hamiltonian and of q and $\dot{q}$ (for a proof, see Roberts 2013).

Lemma 3. *If $(\mathcal{P}, \Omega_{ab}, h)$ is translation and Galilei boost invariant with respect to a maximal orthogonal set $\{\overset{1}{q}, \ldots, \overset{n}{q}\}$, then q and $\mu\dot{q}$ form a local orthonormal coordinate chart $\{q, \mu\dot{q}\} = 1$ for some (non-zero) $\mu \in \mathbb{R}$, and $h = (\mu/2)\dot{q}^2 + v(q)$ for some function v of q alone.*

From this lemma our result follows straightforwardly.

Proof of Proposition 2. The lemma shows that q and $\mu\dot{q}$ form a local orthonormal coordinate chart. This implies that the symplectic form Ω_{ab} can be expressed as the product $\Omega_{ab} = (d_a q)(d_b \mu \dot{q})$. Let $\tau : \mathcal{P} \to \mathcal{P}$ be the mapping such that $\tau^* \dot{q} = -\dot{q}$ and $\tau^* q = q$. Then,

$$\tau^* \Omega_{ab} = \tau^*(d_a q)(d_b \mu \dot{q}) = (d_a \tau^* q)(d_b \tau^* \mu \dot{q}) = -(d_a q)(d_b \mu \dot{q}) = -\Omega_{ab}.$$

Moreover, since the lemma guarantees that $h = (\mu/2)\dot{q}^2 + v(q)$, we have $\tau^* h = (\mu/2)(-\dot{q})^2 + v(q) = h$. But if $\tau^* \Omega_{ab} = -\Omega_{ab}$ and $\tau^* h = h$, then it follows from Proposition 4.3.13 of Abraham and Marsden (1978, p.308) that $(\mathcal{P}, \Omega_{ab}, h)$ is time reversal invariant in the sense we have stated. □

4. Conclusion

The common dogma that classical mechanics is time reversal invariant requires careful qualification. There are various ways that it can go awry, even for conservative systems, in the presence of uncommon interactions such as velocity-dependent forces. So, what makes the "familiar" classical systems time reversal invariant? I have argued that two familiar ways of characterizing classical mechanical systems are enough. Time reversal invariance is guaranteed when momentum is proportional to velocity, and it is guaranteed when we have invariance under spatial translations and Galilei boosts. Although these are not the only kinds of time reversal invariant systems, I hope that they may provide a start toward understanding what kinds of classical systems are.

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