A. Vector and affine spaces

This appendix reviews basic facts about vector and affine spaces, including the notions of metric and orientation. To a large extent, the treatment follows Malament (2009).

A.1. Matrices

Definition 34. An $m \times n$ *matrix* is a rectangular table of mn real numbers A_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$, arrayed as follows:

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \tag{A.1}$$

♠

The operation of matrix multiplication is defined as follows.

Definition 35. Given an $m \times n$ matrix A and an $n \times p$ matrix B, their *matrix product* AB is the $m \times p$ matrix with entries $(AB)^i{}_k$, where

$$(AB)^i{}_k = A^i{}_j B^j{}_k \tag{A.2}$$

♠

Note that this uses the *Einstein summation convention*: repeated indices are summed over.

Definition 36. Given an $m \times n$ matrix $A^i{}_j$, its *transpose* is an $n \times m$ matrix denoted by $A_i{}^j$, and defined by the condition that for all $1 \leq i \leq m$ and $1 \leq j \leq n$, $A_i{}^j = A^i{}_j$, ♠

An element of $\mathbb{R}^n$ can be identified with an $n \times 1$ matrix; as such, an $m \times n$ matrix A encodes a map $A : \mathbb{R}^m \to \mathbb{R}^n$. Such a map is *linear*, in the sense that for any $x^i, y^i \in \mathbb{R}^m$

and any $a, b \in \mathbb{R}$,

$$A^i{}_j(ax^j + by^j) = a(A^i{}_j x^j) + b(A^i{}_j y^j) \tag{A.3}$$

And, in fact, any linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ corresponds to some matrix.

It follows that a *square* matrix—one that is an $n \times n$ matrix, for some $n \in \mathbb{N}$—can be identified with a linear map $\mathbb{R}^n \to \mathbb{R}^n$.

Definition 37. Given a square $n \times n$ matrix $A^i{}_j$, the *determinant* of $A^i{}_j$ is

$$\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} A^i{}_{\sigma(i)} \right) \tag{A.4}$$

where S_n is the permutation group and $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ (see Appendix B). ♠

Definition 38. Given an $m \times n$ matrix $A^i{}_r$, an $n \times m$ matrix $B^r{}_i$ is its *inverse* if multiplying them together in either order yields an identity matrix: that is, if

$$A^i{}_r B^r{}_j = \delta^i_j \tag{A.5}$$

$$B^r{}_i A^i{}_s = \delta^r_s \tag{A.6}$$

where δ^i_j is the $m \times m$ identity matrix and δ^r_s is the $n \times n$ identity matrix. ♠

Definition 39. A square matrix $A^i{}_j$ is *orthogonal* if its transpose is its inverse: that is, if

$$A^i{}_j A_i{}^k = \delta^k_j \tag{A.7}$$

♠

A.2. Vector spaces

Definition 40. A *(real) vector space* $\mathbb{V}$ consists of a set $|\mathbb{V}|$, equipped with a binary operation $+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ (addition), a unary operation $-$ (additive inversion), an operation $\cdot : \mathbb{R} \times \mathbb{V} \to \mathbb{V}$ (scalar multiplication), and a privileged element $0 \in \mathbb{V}$ (the zero vector), such that the following conditions are obeyed (for any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{V}$ and $a, b \in \mathbb{R}$):

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \tag{A.8}$$

$$\vec{u} + (\vec{v} + w) = (\vec{u} + \vec{v}) + w \tag{A.9}$$

$$\vec{u} + 0 = \vec{u} \tag{A.10}$$

$$(-\vec{u}) + \vec{u} = 0 \tag{A.11}$$

$$a \cdot (b \cdot \vec{u}) = (ab) \cdot \vec{u} \tag{A.12}$$

$$1 \cdot \vec{u} = \vec{u} \tag{A.13}$$

$$a \cdot (\vec{u} + \vec{v}) = a \cdot \vec{u} + a \cdot \vec{v} \tag{A.14}$$

$$(a + b) \cdot \vec{u} = a \cdot \vec{u} + b \cdot \vec{u} \tag{A.15}$$

♠

We will often write $a\vec{u}$ instead of $a \cdot \vec{u}$.

Given a set of vectors $S \subseteq |\mathbb{V}|$, the vectors in S are *linearly dependent* if there exist $\vec{v}_1, \ldots, \vec{v}_k \in S$ and $a_1, \ldots, a_k \in \mathbb{R}$ such that $a_1\vec{u}_1 + \cdots a_k\vec{u}_k = 0$; otherwise, they are *linearly independent*.

Definition 41. A *basis* for $\mathbb{V}$ is a set B of linearly independent vectors such that for every $\vec{v} \in \mathbb{V}$, there exist $\vec{v}_1, \ldots, \vec{v}_k \in B$ and $a_1, \ldots, a_k \in \mathbb{R}$ such that $a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = \vec{v}$. ♠

From now on, we assume that any vector space $\mathbb{V}$ we consider is *finite-dimensional*: that is, that there exists a finite basis B of $\mathbb{V}$. For such a vector space, there is some natural number n such that every basis of $\mathbb{V}$ contain n elements; we say that n is the *dimension* of $\mathbb{V}$, and denote it by $\dim(V)$.

Definition 42. Let $\mathbb{V}$ be an n-dimensional vector space, and let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be a basis for $\mathbb{V}$. Given any $\vec{v} \in \mathbb{V}$, the *components* of $\vec{v}$ are the (unique) numbers $v^1, \ldots, v^n$ such that

$$\vec{v} = v^i\vec{e}_i \tag{A.16}$$

♠

It follows that relative to a choice of basis, an n-dimensional space may be identified with $\mathbb{R}^n$, and so with $n \times 1$ matrices.

Definition 43. Given a vector space $\mathbb{V}$, a *subspace* of $\mathbb{V}$ is a vector space $\mathbb{W}$ such that $|\mathbb{W}| \subseteq |\mathbb{V}|$ and the vector-space structure on $\mathbb{W}$ is the restriction of the vector-space on $\mathbb{V}$ to $\mathbb{W}$. ♠

Definition 44. Let $\mathbb{W}$ be a subspace of $\mathbb{V}$. Two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{V}$ are *equivalent modulo* $\mathbb{W}$ if $(\vec{v}_2 - \vec{v}_1) \in \mathbb{W}$: that is, if there is some $\vec{w} \in \mathbb{W}$ such that $\vec{v}_2 = \vec{v}_1 + \vec{w}$. Let the equivalence class of $\vec{v}$ be denoted $[\vec{v}]$. ♠

Definition 45. Let $\mathbb{W}$ be a subspace of $\mathbb{V}$. The *quotient* of $\mathbb{V}$ by $\mathbb{W}$ is a vector space ${}^{\mathbb{V}}\!/_{\mathbb{W}}$, defined as follows. The underlying set is the partition of $\mathbb{V}$ by equivalence modulo $\mathbb{W}$: i.e. the elements of ${}^{\mathbb{V}}\!/_{\mathbb{W}}$ are equivalence classes $[v]$. Addition and scalar multiplication are defined as follows:

$$[\vec{v}_1] + [\vec{v}_2] = [\vec{v}_1 + \vec{v}_2] \tag{A.17}$$

$$a[\vec{v}] = [a\vec{v}] \tag{A.18}$$

It is straightforward to verify that these definitions do not depend on the choice of representative. ♠

Definition 46. Let V and W be two vector spaces. The *direct sum* of V and W, denoted $V \oplus W$, is the vector space defined as follows: its underlying set is $V \times W$, and addition and scalar multiplication are defined pointwise:

$$(\vec{v}, \vec{w}) + (\vec{v}', \vec{w}') = (\vec{v} + \vec{v}', \vec{w} + \vec{w}') \tag{A.19}$$

$$a(\vec{v}, \vec{w}) = (a\vec{v}, a\vec{w}) \tag{A.20}$$

It is straightforward to show that $V \oplus W$ is a vector space (and that if V and W are finite-dimensional, that $\dim(V \oplus W) = \dim(V) + \dim(W)$). The element $(\vec{v}, \vec{w})$ of $V \oplus W$ will be denoted by $\vec{v} \oplus \vec{w}$. ♠

Proposition 14. For any vector spaces $\mathbb{V}$ and $\mathbb{W}$, $\mathbb{V}$ and $\mathbb{W}$ are both subspaces of $\mathbb{V} \oplus \mathbb{W}$.

Definition 47. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. A *linear map* is a map $f : \mathbb{V} \to \mathbb{W}$ such that for any $\vec{u}, \vec{v} \in \mathbb{V}$ and any $x \in \mathbb{R}$,

$$f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v}) \tag{A.21}$$

$$f(x \cdot \vec{u}) = x \cdot f(\vec{u}) \tag{A.22}$$

♠

Given bases on $\mathbb{V}$ and $\mathbb{W}$, and hence an identification of $\mathbb{V}$ with $\mathbb{R}^m$ and $\mathbb{W}$ with $\mathbb{R}^n$ (where $m = \dim(\mathbb{V})$ and $n = \dim(\mathbb{W})$), a linear map $f : \mathbb{V} \to \mathbb{W}$ may be identified with an $n \times m$ matrix $F^i{}_j$.

Definition 48. A linear map $f : \mathbb{V} \to \mathbb{W}$ is a *linear isomorphism* if it is invertible. ♠

Definition 49. Given an invertible linear map $f : \mathbb{V} \to \mathbb{V}$, the *determinant* of f is the determinant of the matrix $F^i{}_j$ that represents f relative to any basis B of $\mathbb{V}$. ♠

It can be shown that the determinant is independent of the choice of basis, so this definition is well-formed.

Proposition 15. Given a basis B of $\mathbb{V}$, if linear maps $f, g : \mathbb{V} \to \mathbb{W}$ agree on B (i.e. if $f(\vec{v}) = g(\vec{v})$ for all $\vec{v} \in B$), then $f = g$.

Proposition 16. Given two ordered bases $B = \langle \vec{e}_1, \ldots, \vec{e}_n \rangle$ and $B' = \langle \vec{e}'_1, \ldots, \vec{e}'_n \rangle$ of $\mathbb{V}$, there is a unique linear map $f : \mathbb{V} \to \mathbb{V}$ such that $\vec{e}'_i = f(\vec{e}_i)$ for all $1 \leq i \leq n$.

Definition 50. Given a vector space $\mathbb{V}$, an *inner product* on $\mathbb{V}$ is a non-degenerate, bilinear, symmetric map $\langle\, ,\, \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$: that is, a map such that

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \tag{A.23}$$

$$\langle \vec{u}, a\vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle \tag{A.24}$$

$$\langle \vec{u}, \vec{v} + w \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, w \rangle \tag{A.25}$$

$$\text{If } \vec{u} \neq 0\text{, then for some } \vec{v} \in \mathbb{V},\ \langle \vec{u}, \vec{v} \rangle \neq 0 \tag{A.26}$$

♠

A vector space equipped with an inner product will be referred to as an *inner product space*.

Definition 51. Given an inner product space $\mathbb{V}$, two vectors $\vec{u}, \vec{v} \in \mathbb{V}$ are *orthogonal* if $\langle \vec{u}, \vec{v} \rangle = 0$. ♠

Definition 52. Given an inner product space $\mathbb{V}$, a basis B of $\mathbb{V}$ is *orthonormal* if for all $\vec{u}, \vec{v} \in B$,

$$\langle \vec{u}, \vec{v} \rangle^2 = \begin{cases} 0 & \text{if } \vec{u} \neq \vec{v} \\ 1 & \text{if } \vec{u} = \vec{v} \end{cases} \tag{A.27}$$

♠

Given an orthonormal basis B of $\mathbb{V}$, the *signature* of $\mathbb{V}$ is the pair (n^+, n^-), where $n^+, n^- \in \mathbb{N}$, such that there are n^+ elements $u \in B$ such that $\langle u, u \rangle = 1$, and n^- elements $u \in B$ such that $\langle u, u \rangle = -1$. Evidently, $n^+ + n^- = \dim(\mathbb{V})$; moreover, one can show that the signature of $\mathbb{V}$ is independent of what orthonormal basis is chosen.

Definition 53. An inner product on a vector space $\mathbb{V}$ is *positive definite* if $\langle \vec{v}, \vec{v} \rangle \geq 0$ for all $\vec{v} \in \mathbb{V}$; equivalently, if its signature is $(\dim(\mathbb{V}), 0)$. ♠

Definition 54. Given an inner product space $\mathbb{V}$, a linear automorphism $f : \mathbb{V} \to \mathbb{V}$ is an *orthogonal* map if it preserves the inner product: that is, if

$$\langle f(\vec{u}), f(\vec{v}) \rangle = \langle \vec{u}, \vec{v} \rangle \tag{A.28}$$

♠

Definition 55. Let $\mathbb{V}$ be a vector space, and let B and B' be two ordered bases of $\mathbb{V}$. B and B' are *co-oriented* if the linear automorphism of $\mathbb{V}$ taking B into B' (see Proposition 16) has positive determinant. ♠

Proposition 17. Co-orientation is an equivalence relation on the set of ordered bases of $\mathbb{V}$, with exactly two equivalence classes (if $\mathbb{V}$ is non-empty).

Definition 56. An *orientation* on $\mathbb{V}$ is a choice of equivalence class of co-oriented ordered bases on $\mathbb{V}$ as the set of *right-handed* ordered bases; the other equivalence class is referred to as the set of *left-handed* ordered bases. A vector space equipped with an orientation is said to be an *oriented* vector space. ♠

A.3. Affine spaces

Since a vector space is a group, we can form the principal homogeneous space of a vector space. Such a space is known as an *affine space*.

Definition 57. Let $\mathbb{V}$ be a vector space. An *affine space* $\mathcal{V}$ is a set $|\mathcal{V}|$ equipped with a free and transitive action $a \mapsto a + \vec{v}$ of $\mathbb{V}$: that is, for any $a, b \in \mathcal{V}$ there is a unique $\vec{v} \in \mathbb{V}$ such that

$$b = a + \vec{v} \tag{A.29}$$

We will use $(b - a)$ to denote this unique vector. ♠

Proposition 18. If $\mathbb{W}$ is a proper subspace of $\mathbb{V}$, then the action of $\mathbb{W}$ on $\mathcal{V}$ is free but not transitive.

Proposition 19. If $\mathbb{W}$ is a subspace of $\mathbb{V}$, then the quotient $\mathcal{U} = \mathcal{V}/\mathbb{W}$ is an affine space with associated vector space $\mathbb{U} = \mathbb{V}/\mathbb{W}$.

Proof. We define the action of $\mathbb{U}$ on $\mathcal{U}$ as follows. Let $\vec{u} \in \mathbb{U}$ and $x \in \mathcal{U}$: thus $y = [x]$, for some $y \in \mathcal{V}$, and $\vec{u} = [\vec{v}]$, for some $\vec{v} \in \mathbb{V}$. Then define

$$x + \vec{u} := [y + \vec{v}] \tag{A.30}$$

First, we need to check that this is well-defined, i.e. that it is independent of the choice of y and $\vec{v}$. So let y' and $\vec{v}'$ be such that $y' = y + \vec{w}$ and $\vec{v}' = \vec{v} + \vec{w}'$, for $\vec{w}, \vec{w}' \in \mathbb{W}$. Then

$$y' + \vec{v}' = y + \vec{v} + (\vec{w} + \vec{w}') \tag{A.31}$$

and so (since $(\vec{w} + \vec{w}') \in \mathbb{W}$) $[y' + \vec{v}'] = [y + \vec{v}]$, and so our definition is indeed well-defined.

Now suppose that $x_1, x_2 \in \mathcal{U}$, with $x_1 = [y_1]$ and $x_2 = [y_2]$ for $y_1, y_2 \in \mathcal{V}$. Since the action of $\mathbb{V}$ on $\mathcal{V}$ is free and transitive, there is a unique $\vec{v} \in \mathbb{V}$ such that $y_2 = y_1 + \vec{v}$. Let $\vec{u} = [\vec{v}]$. Then:

$$\begin{aligned} x_1 + \vec{u} &= [y_1 + \vec{v}] \\ &= x_2 \end{aligned}$$

So the action of $\mathbb{U}$ on $\mathcal{U}$ is transitive. Furthermore, if $x_2 = x_1 + \vec{u}'$, i.e., $[y_2] = [y_1 + \vec{v}']$ (for some $\vec{v}' \in \mathbb{V}$ such that $[\vec{v}'] = \vec{u}'$), then $y_2 = y_1 + \vec{v}' + \vec{w}$ for some $\vec{w} \in \mathbb{W}$; so $\vec{v} = \vec{v}' + \vec{w}$, and hence $\vec{u}' = [\vec{v}'] = [\vec{v}] = \vec{u}$. So the action of $\mathbb{U}$ on $\mathcal{U}$ is free. □

Definition 58. Let $\mathcal{V}$ and $\mathcal{W}$ be affine spaces, with (respective) underlying vector spaces $\mathbb{V}$ and $\mathbb{W}$. The *product affine space* $\mathcal{V} \times \mathcal{W}$ is the affine space whose underlying set is $|\mathcal{V}| \times |\mathcal{W}|$ and whose associated vector space is $\mathbb{V} \oplus \mathbb{W}$, where the action of $\mathbb{V} \oplus \mathbb{W}$ on $|\mathcal{V}| \times |\mathcal{W}|$ is given by

$$(\vec{v} + \vec{w})(x, y) = (x + \vec{v}, y + \vec{w}) \tag{A.32}$$

♠

Structures on an affine space's associated vector space can be 'transferred' to the affine space, as the following two definitions indicate.

Definition 59. A *metric affine space* is an affine space $\mathcal{V}$ whose associated vector space $\mathbb{V}$ is an inner product space. ♠

A metric affine space carries a notion of distance: given any two points $x, y \in \mathcal{V}$, the distance between them is $|y - x|$.

Definition 60. An *oriented affine space* is an affine space $\mathcal{V}$ whose associated vector space $\mathbb{V}$ has an orientation. ♠

A.4. Vector calculus on Euclidean space

Throughout this section, let $\mathcal{X}$ be an oriented Euclidean space: that is, a three-dimensional affine space whose associated vector space $\mathbb{X}$ is equipped with a positive-definite metric and an orientation.

Definition 61. A *vector field* is a smooth map $\vec{V} : \mathcal{X} \to \mathbb{X}$. ♠

Definition 62. Given a scalar field $\phi : \mathcal{X} \to \mathbb{R}$ and a vector $\vec{v} \in \mathbb{X}$, the *directional derivative* (of λ along $\vec{v}$) is a scalar field whose value at any point $x \in \mathcal{X}$ is given by

$$\nabla_{\vec{v}}\phi = \lim_{\varepsilon \to 0} \frac{\phi(x + \varepsilon\vec{v}) - \phi(x)}{\varepsilon} \tag{A.33}$$

♠

As a notational special case, suppose that we have introduced a right-handed, orthonormal basis $\langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$ on $\mathbb{X}$. Then the directional derivative $\nabla_{\vec{e}_i}\phi$ will be denoted by $\nabla_i\phi$. We use this basis to define the operators grad and div; however, these definitions will pick out the same operators if we use any other right-handed, orthonormal basis.

Given a vector field $\vec{V} : \mathcal{X} \to \mathbb{X}$, the *components* of $\vec{V}$ (relative to this basis) are the three scalar fields V^1, V^2, V^3 such that $\vec{V} = V^i\vec{e}_i\rangle$.

Definition 63. Given a scalar field $\phi : \mathcal{V} \to \mathbb{R}$, the *gradient* of ϕ is a vector field $\mathrm{grad}(\phi)$ whose components are

$$\mathrm{grad}(\phi)^i = \nabla_i\phi \tag{A.34}$$

♠

Geometrically, the gradient is a vector field whose direction is the direction in which ϕ is most strongly changing, and whose magnitude is the rate at which ϕ is changing along that direction.

Definition 64. Given a vector field $\vec{V} : \mathcal{X} \to \mathbb{X}$, the *divergence* of $\vec{V}$ is a scalar field $\mathrm{div}(\vec{V})$ given by

$$\mathrm{div}(\vec{V}) = \sum_i \nabla_i V^i \tag{A.35}$$

♠

Gometrically, the divergence of $\vec{V}$ at a point $x \in \mathcal{X}$ expresses the extent to which x is a source or a sink for $\vec{V}$: if the 'outflow' of $\vec{V}$ around x exceeds the 'inflow', then $\mathrm{div}(\vec{V})$ is positive; if the inflow exceeds the outflow, it is negative; and if inflow is equal to outflow, then it is zero.

Definition 65. Given a vector field $\vec{V} : \mathcal{X} \to \mathbb{X}$, the *curl* of $\vec{V}$ is a vector field $\mathrm{curl}(\vec{V})$ whose components are

$$(\mathrm{curl}(\vec{V}))^1 = \nabla_2 V^3 - \nabla_3 V^2 \tag{A.36}$$

$$(\mathrm{curl}(\vec{V}))^2 = \nabla_3 V^1 - \nabla_1 V^3 \tag{A.37}$$

$$(\mathrm{curl}(\vec{V}))^3 = \nabla_1 V^2 - \nabla_2 V^1 \tag{A.38}$$

♠

Geometrically, the curl of $\vec{V}$ at a point $x \in \mathcal{X}$ expresses the 'rotation' of $\vec{V}$ at x: the direction of $\mathrm{curl}(\vec{V})$ is the axis of rotation (using the right-hand-rule), and its magnitude expresses the amount of rotation.

B. Group theory

B.1. Groups

Definition 66. A *group* consists of a set G, equipped with a binary operation $*$ (of *group multiplication*), a unary operation $^{-1}$ (of *inversion*), and a privileged element e (the *identity*), such that for any $g, h, k \in G$:

$$g * (h * k) = (g * h) * k \tag{B.1}$$

$$g * e = g = e * g \tag{B.2}$$

$$g^{-1} * g = g = g * g^{-1} \tag{B.3}$$

♠

We will frequently abbreviate $g * h$ by gh.

Example 14. Any vector space is a group, with addition as the group operation (and the zero vector as the identity, and additive inverse as group inverse).

Example 15. The real numbers are a group with respect to addition (with 0 as the identity and $-x$ as the inverse of x) and with respect to multiplication (with 1 as the identity and $1/x$ as the inverse of x).

Example 16. Given a set A, a *permutation* of A is a bijection $f : A \to A$. The *symmmetric group* of A is the group $\mathrm{Sym}(A)$ consisting of all permutations of A, with composition as the group operation.

Example 17. If a finite set A has n elements, then $\mathrm{Sym}(A)$ is denoted by S_n. A *transposition* is a permutation $\tau \in S_n$ that just exchanges two elements: i.e. is such that for some $a, b \in A$ where $a \neq b$, $\tau(a) = b$ and $\tau(b) = a$, and for all other $c \in A$, $\tau(c) = c$.

Any $\sigma \in S_n$ can be expressed as a finite composition of transpositions. It can be shown that if σ is expressible as an even number of transpositions, then it is *only* expressible as an even number of transpositions; and similarly in case σ is expressible as an odd number of transpositions. Accordingly, σ is given a *sign* $\mathrm{sgn}(\sigma)$: if it is an even number then $\mathrm{sgn}(\sigma) = +1$, and if it is an odd number then $\mathrm{sgn}(\sigma) = -1$.

Definition 67. Given groups G and H, a *(group) homomorphism* is a map $\phi : G \to H$ such that for any $g, g' \in G$,

$$\phi(g * g') = \phi(g) * \phi(g') \tag{B.4}$$

♠

Definition 68. Given a group G, a subset $H \subseteq G$ is a *subgroup* of G if H is closed under group multiplication and inversion: that is, if for all $g, h \in H$, $g * h \in H$ and $g^{-1} \in H$. ♠

We can also state this as follows: a subset H of G is a subgroup if $e \in H$ and H is a group under the restriction of the operations $*$ and $^{-1}$ to H.

B.2. Group actions

Definition 69. Given a group G and set X, an *action* of G on X assigns every $g \in G$ to some bijection $g\bullet : X \to X$, such that for any $g, h \in G$ and $x \in X$,

$$(gh)x = g(hx) \tag{B.5}$$

♠

Definition 70. Given an action of G on X, two points $x, y \in X$ are *G-related* if the one can be mapped to the other by G: that is, if there is some $g \in G$ such that $y = gx$. (The proof that this is an equivalence relation is left as an exercise.) A *G-orbit* in X is an equivalence class of G-equivalent points of X. ♠

Definition 71. Given an action of a group G on some set Ω, the *quotient* of Ω by G is the set ${}^{\Omega}\!/_{G}$ consisting of G-orbits in Ω. ♠

Definition 72. An action of G on X is *transitive* if for any $x, y \in X$, there is some $g \in G$ such that $y = gx$. ♠

In other words, a transitive group action is one such that any two elements of X are G-related; hence, a transitive group action is one for which there only exists a single orbit.

Definition 73. An action of G on X is *free* if for any $x \in X$ and any $g, h \in G$: if $gx = hx$ then $g = h$ (or, equivalently: if $g \neq h$ then $gx \neq hx$). ♠

Thus, a free action is one where distinct elements of G have distinct effects on *every* element of X.

Definition 74. Let X and Y be G-sets. A map $f : X \to Y$ is *G-equivariant* if for any $g \in G$ and $x \in X$,

$$f(gx) = g(f(x)) \tag{B.6}$$

♠

Actions which are both free and transitive are said to be *regular*, and have the following important feature:

Proposition 20. Suppose that X and Y are G-sets where the action of G is free and transitive. Then there exist G-equivariant bijections $f : X \to Y$ and $f^{-1} : Y \to X$.

Proof. Pick any points $x_0 \in X$ and $y_0 \in Y$, and let $f(x_0) = y_0$. For any other $x \in X$, we know that $x = g_x x_0$ for some unique element g_x of G; now set $f(x) = g_x y_0$. This suffices to determine f; we now show that f is a G-equivariant bijection. First, for any $x \in X$ and any $g \in G$,

$$f(gx) = f(gg_x x_0) = gg_x y_0 = g(f(x))$$

So f is G-equivariant.

Next, consider any $x_1 = g_1 x_0$ and $x_2 = g_2 x_0$ in X. If $f(x_1) = f(x_2)$, i.e. $f(g_1 x_0) = f(g_2 x_0)$, then $g_1 y_0 = g_2 y_0$. But since G acts freely on Y, it follows that $g_1 = g_2$ and so $x_1 = x_2$. So f is injective.

Finally, consider any $y \in Y$, and (again using the fact that G's action on Y is regular), express it in the form $g_y y_0$. Then

$$f(g_y x_0) = g_y(f(x_0)) = g_y y_0 = y$$

So f is surjective, and hence a bijection.

Showing that f^{-1} is G-equivariant is left as an exercise. □

A G-set for which the action of G is regular is said to be a *principal homogeneous space for G*, or alternatively a *G-torsor*. Taking bijective G-equivariant maps as the appropriate notion of isomorphism for G-sets, we see that G has, up to isomorphism, a unique principal homogeneous space.[1]

[1] If that's the case, why don't we speak instead of *the* principal homogeneous space for G, just as we speak of *the* real numbers as the unique (up to isomorphism) complete ordered field? That's a good question; one reason not to do so is that there will typically be multiple isomorphisms between two principal homogeneous spaces for G, so there is no canonical way to identify one principal homogeneous space with another (whereas there is a *unique* isomorphism between two complete ordered fields).

Example 18. Given a vector space $\mathbb{V}$, the principal homogeneous space for $\mathbb{V}$ (regarded as a group) is the affine space $\mathcal{V}$.

C. Differential forms

C.1. Multi-covectors

Definition 75. Let $\mathbb{V}$ be a vector space. A *covector* (over $\mathbb{V}$) is a linear map $\mathbf{p} : \mathbb{V} \to \mathbb{R}$. We refer to the set of covectors over $\mathbb{V}$ as the *dual vector space*, and denote it by $\mathbb{V}^*$. ♠

It is not hard to show that $\mathbb{V}^*$ is also a vector space (by defining addition and scalar multiplication pointwise), of the same dimension as $\mathbb{V}$. The dual vector space to a direct sum of vector spaces is the direct sum of the duals: that is, $(\mathbb{V} \oplus \mathbb{W})^* = \mathbb{V}^* \oplus \mathbb{W}^*$.

Definition 76. Let $\mathbb{V}$ be a vector space. For any $k \in \mathbb{N}$, a *k-covector* is an alternating multilinear map $\mathbf{q} : \mathbb{V}^k \to \mathbb{R}$; that is, a map which is linear in each argument, and which has the property that swapping any two arguments changes the sign of the result. ♠

Thus, a 1-covector is a covector; a 2-covector is an antisymmetric bilinear map $f : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$; and so on. We consider real numbers to be 0-covectors. The set of k-covectors over a vector space $\mathbb{V}$ will be denoted $\Lambda^k(\mathbb{V}^*)$.

If the arguments fed to a k-covector are linearly dependent, then the result will vanish: for example, given a 2-covector $\mathbf{p}$, if we feed it $\vec{u}$ and $a\vec{u}$ (where $a \in \mathbb{R}$),

$$\mathbf{p}(\vec{u}, a\vec{u}) = a\mathbf{p}(\vec{u}, \vec{u}) = 0 \tag{C.1}$$

If $\mathbb{V}$ is n-dimensional, then there can be at most n linearly independent vectors, and so any k-covector for $k > n$ will be trivial; for this reason, we typically treat the n-covectors as the end of the line.

We can form new multi-covectors out of old ones by using the *exterior product*:

Definition 77. Given a k-covector f and an l-covector g, their *exterior product* $f \wedge g$ is the $(k+l)$-covector whose result, for any $\vec{v}_1, \ldots, \vec{v}_{k+l} \in \mathbb{V}$, is given by

$$(f \wedge g)(\vec{v}_1, \ldots, \vec{v}_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) f(\vec{v}_{\sigma(1)}, \ldots, f_{\sigma(k)})\, g(\vec{v}_{\sigma(k+1)}, \ldots, \vec{v}_{\sigma(k+l)}) \tag{C.2}$$

where S_{k+l} is the permutation group for $k+l$ elements, and $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ (see Appendix B). ♠

For example, the exterior product of two covectors $\mathbf{p}$ and $\mathbf{q}$ is a 2-covector $\mathbf{p} \wedge \mathbf{q}$, defined by the condition that for any $\vec{u}, \vec{v} \in \mathbb{V}$,

$$(\mathbf{p} \wedge \mathbf{q})(\vec{u}, \vec{v}) := \mathbf{p}(\vec{u})\mathbf{q}(\vec{v}) - \mathbf{p}(\vec{v})\mathbf{q}(\vec{u}) \tag{C.3}$$

Similarly, the exterior product of a covector $\mathbf{p}$ with a 2-covector $\mathbf{r}$ is a 3-covector $\mathbf{p} \wedge \mathbf{r}$, such that for any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{V}$,

$$(\mathbf{p} \wedge \mathbf{r})(\vec{u}, \vec{v}, \vec{w}) = \mathbf{p}(\vec{u})\mathbf{r}(\vec{v}, \vec{w}) + \mathbf{p}(\vec{v})\mathbf{r}(\vec{w}, \vec{u}) + \mathbf{p}(\vec{w})\mathbf{r}(\vec{u}, \vec{v}) \tag{C.4}$$

C.2. Euclidean multi-covectors

Let $\mathbb{X}$ be oriented Euclidean vector space, i.e. a three-dimensional vector space equipped with a positive-definite inner product and an orientation. The inner product induces a very useful isomorphism between vectors and covectors (i.e. between $\mathbb{X}$ and $\mathbb{X}^*$), known as the *musical isomorphism*. On the one hand, given any vector $\vec{v} \in \mathbb{X}$, its associated covector is the linear map $\vec{v}^\flat : \mathbb{X} \to \mathbb{R}$ such that for any $w \in \mathbb{X}$,

$$\vec{v}^\flat(\vec{w}) = \langle \vec{v}, \vec{w} \rangle \tag{C.5}$$

In the other direction, given a covector $\mathbf{p}$, its associated vector $\mathbf{p}^\sharp$ is defined as the vector such that for any vector $v \in \mathbb{X}$,

$$\langle \mathbf{p}^\sharp, \vec{v} \rangle = \mathbf{p}(\vec{v}) \tag{C.6}$$

In the interests of space, we skip over proving that this condition does pick out a unique vector.

Moreover, since $\mathbb{X}$ carries both an inner product and an orientation, it exhibits *Hodge duality*. That is, for any $1 \leq k \leq 3$, there is an isomorphism between $\Lambda^k(\mathbb{X}^*)$ and $\Lambda^{3-k}(\mathbb{X}^*)$: i.e., between the scalars and the 3-covectors, and between the covectors and the 2-covectors. These isomorphisms are given by the *Hodge star* operator, which we define as follows. Let $\langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$ be an (arbitrarily chosen) right-handed and orthonormal basis of $\mathbb{X}$; and let $\mathbf{e}^i := (\vec{e}_i)^\flat$ (resulting in a basis $\langle \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3 \rangle$ of $\Lambda^1(\mathbb{X})$). Then the isomorphism $\star : \mathbb{R} \to \Lambda^3(\mathbb{X})$ is defined by

$$\star 1 = \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \tag{C.7}$$

and the isomorphism $\star : \Lambda^3(\mathbb{X}) \to \mathbb{R}$ by

$$\star\,(\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3) = 1 \tag{C.8}$$

The isomorphism $\star : \Lambda^1(\mathbb{X}) \to \Lambda^2(\mathbb{X})$ is defined by

$$\star\mathbf{e}^1 = \mathbf{e}^2 \wedge \mathbf{e}^3 \tag{C.9}$$
$$\star\mathbf{e}^2 = \mathbf{e}^3 \wedge \mathbf{e}^1 \tag{C.10}$$
$$\star\mathbf{e}^3 = \mathbf{e}^1 \wedge \mathbf{e}^2 \tag{C.11}$$

and, finally, the isomorphism $\star : \Lambda^2(\mathbb{X}) \to \Lambda^1(\mathbb{X})$ by

$$\star(\mathbf{e}^1 \wedge \mathbf{e}^2) = \mathbf{e}^3 \tag{C.12}$$
$$\star(\mathbf{e}^2 \wedge \mathbf{e}^3) = \mathbf{e}^1 \tag{C.13}$$
$$\star(\mathbf{e}^3 \wedge \mathbf{e}^1) = \mathbf{e}^2 \tag{C.14}$$

Since the Hodge star is required to be linear, these conditions fix its action uniquely. Moreover, it can be shown that the Hodge star (so defined) is independent of which right-handed orthonormal basis of $\mathbb{X}$ is chosen.

We can use the Hodge star and wedge product to express the cross product in a more geometrical fashion: given a pair of vectors $\vec{v}, \vec{w} \in \mathbb{X}$,

$$\vec{v} \times \vec{w} = (\star(\vec{v}^\flat \wedge \vec{w}^\flat))^\sharp \tag{C.15}$$

In other words, we take our vectors, flatten them to a pair of covectors, take their wedge product (a 2-covector), apply the Hodge star to that 2-covector to get a covector back again, and then sharpen that to make a vector. Easy![1]

C.3. Minkowski multi-covectors

Let $\mathbb{M}$ be an oriented Minkowski vector space, i.e. a four-dimensional vector space equipped with a Lorentzian inner product and an orientation. Again, the inner product means that we can establish a musical isomorphism between $\mathbb{M}$ and $\mathbb{M}^*$, again by the

[1] In fact, one can simplify this a bit by defining a wedge product directly on $\mathbb{X}$—thereby constructing an exterior algebra of *multivectors*—and then introducing a Hodge duality between vectors and 2-vectors. With that duality, we can write this expression as $\vec{v} \times \vec{w} = \star(\vec{v} \wedge \vec{w})$. However, such a construction still requires both a metric and an orientation, since those are required to (uniquely) define the Hodge star operator.

conditions that for any $\vec{\xi}, \vec{\eta} \in \mathbb{M}$ and $\mathbf{p} \in \mathbb{M}^*$,

$$\vec{\xi}^\flat(\vec{\eta}) = \langle \vec{\xi}, \vec{\eta} \rangle \tag{C.16}$$

$$\langle \mathbf{p}^\sharp, \vec{\xi} \rangle = \mathbf{p}(\vec{\xi}) \tag{C.17}$$

And again, since $\mathbb{M}$ carries both an inner product and an orientation, it exhibits Hodge duality. In this case, Hodge duality holds between $\Lambda^k(\mathbb{M}^*)$ and $\Lambda^{4-k}(\mathbb{M}^*)$, for each $0 \leq k \leq 4$: that is, between scalars and 4-covectors, covectors and 3-covectors, and between 2-covectors and 2-covectors. Again, take an arbitrary right-handed orthonormal basis $\langle \vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$, set $\mathbf{e}^\mu = (\vec{e}_\mu)^\flat$ to obtain the dual basis $\langle \mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3 \rangle$, and define:

$$\star 1 = \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \tag{C.18}$$

$$\star \mathbf{e}^0 = \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \tag{C.19}$$

$$\star \mathbf{e}^1 = \mathbf{e}^0 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \tag{C.20}$$

$$\star \mathbf{e}^2 = \mathbf{e}^0 \wedge \mathbf{e}^3 \wedge \mathbf{e}^1 \tag{C.21}$$

$$\star \mathbf{e}^3 = \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2 \tag{C.22}$$

$$\star(\mathbf{e}^0 \wedge \mathbf{e}^1) = \mathbf{e}^3 \wedge \mathbf{e}^2 \tag{C.23}$$

$$\star(\mathbf{e}^0 \wedge \mathbf{e}^2) = \mathbf{e}^1 \wedge \mathbf{e}^3 \tag{C.24}$$

$$\star(\mathbf{e}^0 \wedge \mathbf{e}^3) = \mathbf{e}^2 \wedge \mathbf{e}^1 \tag{C.25}$$

$$\star(\mathbf{e}^1 \wedge \mathbf{e}^2) = \mathbf{e}^0 \wedge \mathbf{e}^3 \tag{C.26}$$

$$\star(\mathbf{e}^1 \wedge \mathbf{e}^3) = \mathbf{e}^2 \wedge \mathbf{e}^0 \tag{C.27}$$

$$\star(\mathbf{e}^2 \wedge \mathbf{e}^3) = \mathbf{e}^0 \wedge \mathbf{e}^1 \tag{C.28}$$

$$\star(\mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2) = -\mathbf{e}^3 \tag{C.29}$$

$$\star(\mathbf{e}^0 \wedge \mathbf{e}^3 \wedge \mathbf{e}^1) = -\mathbf{e}^2 \tag{C.30}$$

$$\star(\mathbf{e}^0 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3) = -\mathbf{e}^1 \tag{C.31}$$

$$\star(\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3) = -\mathbf{e}^0 \tag{C.32}$$

$$\star\,(\mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3) = -1 \tag{C.33}$$

C.4. Differential forms

Finally, we introduce differential forms: just as a vector field is a vector-valued field, so a differential form is a multicovector-valued field.

Definition 78. Let $\mathcal{V}$ be an affine space with vector space $\mathbb{V}$. A *k-form on* $\mathcal{V}$ is a smooth map $\mathbf{p} : \mathcal{V} \to \Lambda^k(\mathbb{V}^*)$. ♠

Addition, scalar multiplication, and exterior multiplication of differential forms are defined pointwise. As with multi-covectors, the only non-trivial k-forms on an n-dimensional space are those for $k \leq n$. The set of k-forms on $\mathcal{V}$ is denoted by $\Omega^k(\mathcal{V})$, and the set of all differential forms on $\mathcal{V}$ by $\Omega(\mathcal{V})$.

For oriented Euclidean space and oriented Minkowski spacetime, there is a musical isomorphism between the set of 1-forms and the set of vector fields, and Hodge dualities between the appropriate sets of k-forms; again, these are defined pointwise. Thus, on Euclidean space Hodge duality relates 1-forms to 2-forms, and 3-forms to scalar fields; while on Minkowski spacetime Hodge duality relates 1-forms to 3-forms, 2-forms to 2-forms, and 4-forms to scalar fields.

However, in addition to this, differential forms also exhibit a very natural kind of differential calculus.[2] First, given any scalar field f, we define the *differential* of f to be the 1-form df such that for any vector $\vec{v} \in \mathbb{V}$,

$$df(\vec{v}) = \nabla_{\vec{v}} f \tag{C.34}$$

(where $\nabla_{\vec{v}}$ is the directional derivative with respect to $\vec{v}$; see Appendix A). The extension of this concept to arbitrary differential forms is known as the *exterior derivative*.

Definition 79. Let $\mathcal{V}$ be an n-dimensional affine space. The *exterior derivative* is the unique map $d : \Omega(\mathcal{V}) \to \Omega(\mathcal{V})$ such that for any $k < n$, $d : \Omega^k(\mathcal{V}) \to \Omega^{k+1}(\mathcal{V})$, and which has the following properties:

- for any scalar field (0-form) f,

$$df(V) = \nabla_V f \tag{C.35}$$

- for any k-form $\mathbf{p}$,

$$d(d\mathbf{p}) = 0 \tag{C.36}$$

[2]Although the perspicacious reader might have suspected this would be true, given the name.

- for any $x, y \in \mathbb{R}$ and $\mathbf{p}, \mathbf{q} \in \Omega^k(\mathcal{V})$,

$$d(x\mathbf{p} + y\mathbf{q}) = xd\mathbf{p} + yd\mathbf{q} \tag{C.37}$$

- and for any $\mathbf{p} \in \Omega^k(\mathcal{V})$, $\mathbf{q} \in \Omega(\mathcal{V})$,

$$d(\mathbf{p} \wedge \mathbf{q}) = d\mathbf{p} \wedge \mathbf{q} + (-1)^k \mathbf{p} \wedge d\mathbf{q} \tag{C.38}$$

♠

In fancy lingo, the exterior derivative is a linear and idempotent antiderivation on the exterior algebra of differential forms, which extends the differential on scalar fields. It is non-trivial to show that there exists an operator with these properties, and that it is unique; however, we will just take that fact as given.

C.5. Differential forms and Euclidean vector calculus

We can use differential forms on oriented Euclidean space $\mathcal{X}$ to better understand the vector-calculus operators discussed in Appendix A. First, as discussed above, the differential of a scalar field is a 1-form. The gradient is the vector field obtained from the differential by application of the musical isomorphism, that is:

$$\text{grad}(f) = (df)^\sharp \tag{C.39}$$

Thus, the gradient corresponds to the exterior derivative of a scalar field.

The exterior derivative of a 1-form $\mathbf{P}$ is a 2-form $d\mathbf{P}$, whose components (with respect to some orthonormal basis $\mathbf{e}^i$ on $\mathbb{X}^*$) are

$$(d\mathbf{P})_{ij} = \nabla_i P_j - \nabla_j P_i \tag{C.40}$$

where P_i are the components of $\mathbf{P}$ with respect to that same basis (i.e. $\mathbf{P} = P_i \mathbf{e}^i$), and ∇_i is the directional derivative with respect to the dual basis $\vec{e}_i$. As a result, if we take the Hodge dual then we obtain a 1-form

$$(\star d\mathbf{P})_1 = \nabla_2 P_3 - \nabla_3 P_2 \tag{C.41}$$

$$(\star d\mathbf{P})_2 = \nabla_3 P_1 - \nabla_1 P_3 \tag{C.42}$$

$$(\star d\mathbf{P})_3 = \nabla_1 P_2 - \nabla_2 P_1 \tag{C.43}$$

which we recognise as the same pattern of components as the curl; in more intrinsic geometric language, for any vector field $\vec{V}$,

$$\mathrm{curl}(\vec{V})^\flat = \star d(\vec{V}^\flat) \tag{C.44}$$

So we can take the curl of a vector field by flattening it (to a 1-form), taking its exterior derivative (2-form), applying the Hodge dual (1-form) and sharpening it (vector field). Hence, the curl operator corresponds to the exterior derivative of a 1-form.

Finally, the exterior derivative of a 2-form $\mathbf{T}$ is a 3-form $d\mathbf{T}$, which we can express as

$$d\mathbf{T} = (\nabla_1 T_{23} + \nabla_2 T_{31} + \nabla_3 T_{12})\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \tag{C.45}$$

Thus, when we apply the Hodge star we obtain a scalar field

$$\star\, d\mathbf{T} = \nabla_1 T_{23} + \nabla_2 T_{31} + \nabla_3 T_{12} \tag{C.46}$$

It follows that given a vector field $\vec{V}$, if we first flatten it to a 1-form, then turn into a 2-form (via the Hodge star), then take the exterior derivative (to get a 3-form) and finally convert it into a scalar field (by the Hodge star again), we have obtained the divergence; that is,

$$\mathrm{div}(\vec{V}) = \star d \star (\vec{V}^\flat) \tag{C.47}$$

So the divergence operator corresponds to taking the exterior derivative of a 2-form.

Bibliography

Aharonov, Y. and Bohm, D. (1959). Significance of Electromagnetic Potentials in the Quantum Theory. *Physical Review*, 115(3):485–491.

Albert, D. Z. (2000). *Time and Chance*. Harvard University Press.

Alexander, H. G., editor (1956). *The Leibniz-Clarke Correspondence*. Manchester University Press, Manchester.

Andreas, H. (2017). Theoretical Terms in Science. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, fall 2017 edition.

Arntzenius, F. and Greaves, H. (2009). Time Reversal in Classical Electromagnetism. *The British Journal for the Philosophy of Science*, 60(3):557–584.

Awodey, S. (2010). *Category Theory*. Oxford University Press, Oxford; New York.

Baez, J. C. and Muniain, J. P. (1994). *Gauge Fields, Knots, and Gravity*. World Scientific, Singapore.

Baez, J. C. and Shulman, M. (2010). Lectures on *n*-categories and cohomology. In Baez, J. C. and May, J. P., editors, *Towards Higher Categories*, number v. 152 in The IMA Volumes in Mathematics and Its Applications, pages 1–68. Springer, New York.

Barbour, J. B. (1989). *Absolute or Relative Motion: A Study from a Machian Point of View of the Discovery and the Structure of Dynamical Theories*. Cambridge University Press, Cambridge.

Barrett, T. W. (n.d.). Structure and Equivalence. *Philosophy of Science*.

Barrett, T. W. and Halvorson, H. (2016a). Glymour and Quine on Theoretical Equivalence. *Journal of Philosophical Logic*, 45(5):467–483.

Barrett, T. W. and Halvorson, H. (2016b). Morita Equivalence. *The Review of Symbolic Logic*, 9(3):556–582.

Belot, G. (1998). Understanding Electromagnetism. *The British Journal for the Philosophy of Science*, 49(4):531–555.

Belot, G. (2013). Symmetry and equivalence. In Batterman, R. W., editor, *The Oxford Handbook of Philosophy of Physics*. Oxford University Press, New York.

Brading, K. and Brown, H. R. (2004). Are Gauge Symmetry Transformations Observable? *The British Journal for the Philosophy of Science*, 55(4):645–665.

Brown, H. R. (2005). *Physical Relativity: Space-Time Structure from a Dynamical Perspective*. Oxford University Press, Oxford.

Carnap, V. R. (1958). Beobachtungssprache Und Theoretische Sprache. *Dialectica*, 12(3-4):236–248.

Dasgupta, S. (2016). Symmetry as an Epistemic Notion (Twice Over). *The British Journal for the Philosophy of Science*, 67(3):837–878.

Dewar, N. (2019a). Ramsey Equivalence. *Erkenntnis*, 84(1):77–99.

Dewar, N. (2019b). Sophistication about Symmetries. *The British Journal for the Philosophy of Science*, 70(2):485–521.

Dewar, N. (n.d.). On recovering internal structure from categorical structure.

Earman, J. (1986). *A Primer on Determinism*. The Western Ontario Series in Philosophy of Science. Springer Netherlands.

Earman, J. and Norton, J. (1987). What Price Spacetime Substantivalism? The Hole Story. *The British Journal for the Philosophy of Science*, 38(4):515–525.

Feynman, R. P., Leighton, R. B., and Sands, M. (2011). *The Feynman Lectures on Physics, Vol. II: The New Millennium Edition: Mainly Electromagnetism and Matter*. Basic Books.

Fletcher, S. C. (2012). What counts as a Newtonian system? The view from Norton's dome. *European Journal for Philosophy of Science*, 2(3):275–297.

Gomes, H. and Butterfield, J. (n.d.). On the empirical significance of symmetries. page 48.

Greaves, H. and Wallace, D. (2014). Empirical Consequences of Symmetries. *The British Journal for the Philosophy of Science*, 65(1):59–89.

Halvorson, H. (2012). What Scientific Theories Could Not Be. *Philosophy of Science*, 79(2):183–206.

Halvorson, H. (2019). *The Logic in Philosophy of Science*. Cambridge University Press, Cambridge.

Healey, R. (1997). Nonlocality and the Aharonov-Bohm Effect. *Philosophy of Science*, 64(1):18–41.

Healey, R. (2007). *Gauging What's Real*. Oxford University Press, Oxford.

Healey, R. (2009). Perfect Symmetries. *The British Journal for the Philosophy of Science*, 60(4):697–720.

Hodges, W. (1993). *Model Theory*. Number 42 in Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge.

Hodges, W. (1997). *A Shorter Model Theory*. Cambridge University Press, Cambridge.

Hudetz, L. (2019). Definable Categorical Equivalence. *Philosophy of Science*, 86(1):47–75.

Ismael, J. and van Fraassen, B. C. (2003). Symmetry as a guide to superfluous theoretical structure. In Brading, K. and Castellani, E., editors, *Symmetries in Physics: Philosophical Reflections*, pages 371–392. Cambridge University Press, Cambridge.

Ketland, J. (2004). Empirical Adequacy and Ramsification. *The British Journal for the Philosophy of Science*, 55(2):287–300.

Kosso, P. (2000). The empirical status of symmetries in physics. *The British Journal for the Philosophy of Science*, 51(1):81–98.

Leeds, S. (1999). Gauges: Aharonov, Bohm, Yang, Healey. *Philosophy of Science*, 66(4):606–627.

Leeds, S. (2006). Discussion: Malament on Time Reversal. *Philosophy of Science*, 73(4):448–458.

Lutz, S. (2015). What Was the Syntax-Semantics Debate in the Philosophy of Science About? *Philosophy and Phenomenological Research*, 91(3).

Malament, D. B. (2004). On the time reversal invariance of classical electromagnetic theory. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 35(2):295–315.

Malament, D. B. (2008). Norton's slippery slope. *Philosophy of Science*, 75(5):799–816.

Malament, D. B. (2009). Notes on Geometry and Spacetime.

Malament, D. B. (2012). *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*. University of Chicago Press, Chicago, IL.

Manzano, M. (1996). *Extensions of First Order Logic*. Cambridge University Press, Cambridge.

Marquis, J.-P. (2020). Category Theory. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, fall 2020 edition.

Maudlin, T. (1998). Healey on the Aharonov-Bohm Effect. *Philosophy of Science*, 65(2):361–368.

Maudlin, T. (2012). *Philosophy of Physics: Space and Time*. Princeton University Press, Princeton, NJ.

Maudlin, T. (2018). Ontological Clarity via Canonical Presentation: Electromagnetism and the Aharonov–Bohm Effect. *Entropy*, 20(6):465.

Møller-Nielsen, T. (2017). Invariance, Interpretation, and Motivation. *Philosophy of Science*, 84(5):1253–1264.

Newman, M. H. A. (1928). Mr. Russell's "Causal Theory of Perception". *Mind*, 37(146):137–148.

Norton, J. D. (2008). The Dome: An Unexpectedly Simple Failure of Determinism. *Philosophy of Science*, 75(5):786–798.

Nounou, A. M. (2003). A fourth way to the Aharanov-Bohm effect. In Brading, K. and Castellani, E., editors, *Symmetries in Physics: Philosophical Reflections*, pages 174–199. Cambridge University Press, Cambridge.

Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*. Springer-Verlag, New York, NY.

Pooley, O. (2006). Points, particles, and structural realism. In Rickles, D., French, S., and Saatsi, J., editors, *The Structural Foundations of Quantum Gravity*, pages 83–120. Oxford University Press, Oxford, UK.

Psillos, S. (2000). Carnap, the Ramsey-Sentence and Realistic Empiricism. *Erkenntnis*, 52(2):253–279.

Ramsey, F. P. (1931). Theories (1929). In Braithwaite, R. B., editor, *The Foundations of Mathematics and Other Logical Essays*. Routledge and Kegan Paul, London.

Read, J. and Møller-Nielsen, T. (2018). Motivating dualities. *Synthese*.

Roberts, J. T. (2008). A Puzzle about Laws, Symmetries and Measurability. *The British Journal for the Philosophy of Science*, 59(2):143–168.

Russell, B. (1927). *The Analysis of Matter*. Kegan Paul, London.

Saunders, S. (2003). Physics and Leibniz's principles. In Brading, K. and Castellani, E., editors, *Symmetries in Physics: Philosophical Reflections*, pages 289–308. Cambridge University Press, Cambridge, UK.

Saunders, S. (2013). Rethinking Newton's Principia. *Philosophy of Science*, 80(1):22–48.

Shapiro, S. (1991). *Foundations without Foundationalism: A Case for Second-Order Logic*. Oxford University Press, Oxford.

Sklar, L. (1982). Saving the noumena. *Philosophical Topics*.

Stein, H. (1967). Newtonian space-time. *Texas Quarterly*, 10:174–200.

Teh, N. J. (2016). Galileo's Gauge: Understanding the Empirical Significance of Gauge Symmetry. *Philosophy of Science*, 83(1):93–118.

Wallace, D. (2003). Time-dependent symmetries: The link between gauge symmetries and indeterminism. In Brading, K. and Castellani, E., editors, *Symmetries in Physics: Philosophical Reflections*, pages 163–173. Cambridge University Press, Cambridge.

Wallace, D. (2014). Deflating the Aharonov-Bohm Effect. *arXiv:1407.5073 [quant-ph]*.

Wallace, D. (n.d.a). Isolated Systems and their Symmetries, Part I: General Framework and Particle-Mechanics Examples. http://philsci-archive.pitt.edu/16623/.

Wallace, D. (n.d.b). Isolated systems and their symmetries, part II: Local and global symmetries of field theories. http://philsci-archive.pitt.edu/16624/.

Wallace, D. (n.d.c). Observability, redundancy and modality for dynamical symmetry transformations.

Weatherall, J. O. (2016). Are Newtonian Gravitation and Geometrized Newtonian Gravitation Theoretically Equivalent? *Erkenntnis*, 81(5):1073–1091.

Wheeler, J. T. (2007). Gauging Newton's law. *Canadian Journal of Physics*, 85(4):307–344.

Wilson, M. (2009). Determinism and the Mystery of the Missing Physics. *The British Journal for the Philosophy of Science*, 60(1):173–193.

Winnie, J. A. (1986). Invariants and objectivity: A theory with applications to relativity and geometry. In Colodny, R. G., editor, *From Quarks to Quasars, University of Pittsburgh Press, Pittsburgh*, pages 71–180. University of Pittsburgh Press, Pittsburgh.