Chapter 7

Dimensioned Algebra and Geometry

As a motivation for the mathematical definitions that will be introduced in this chapter, let us begin by briefly discussing the use of physical quantities in classical thermodynamics. It can be determined empirically that there are four parameters that uniquely characterise the state of a gas, its pressure P, volume V, amount of matter N and temperature T, and that they can all be measured independently by comparison to some reference unit, for example an atmosphere atm, a litre L, a mol and a Kelvin K, respectively. The gas under study will then have a state specified by $(P = p\,\text{atm}, V = v\,\text{L}, N = n\,\text{mol}, T = t\,\text{K})$ with p, v, n, t $\mathbb{R}$-valued variables. By assuming, for instance, the ideal gas law to model the equilibrium behaviour of the gas, one can derive other physical quantities such as the energy $U = PVN^{-1}$, carrying units atm·L/mol, or the entropy $E = PVN^{-1}T^{-1}$, carrying units atm·L/mol/K. In general, any measurable physical quantity defined for the state of the gas will be of the form $Q = q\,P^aV^bN^cT^d$ for some $q \in \mathbb{R}$ and $a, b, c, d \in \mathbb{Z}$. The product of two physical quantities Q and Q' is then given by

$$Q \cdot Q' = qq'\,P^{a+a'}V^{b+b'}N^{c+c'}T^{d+d'},$$

and two such quantities can be added together to form $Q + Q'$ only when $a = a'$, $b = b'$, $c = c'$ and $d = d'$.

The goal of this chapter is to develop a precise mathematical formalism that encapsulates the formal properties of physical quantities as commonly used in practical science and engineering (see [BI96, Chapter 1] for a standard modern reference in dimensional analysis and [Har12] for an attempt at extending the notions of dimensional analysis to linear algebra and calculus). It was argued in our historical note of Section 3.3 that the algebraic building blocks of the standard formulation of modern physical theories, i.e. fields, vector spaces, groups, etc., were originally conceived without any consideration of the natural structure of physical quantities and with an emphasis on endowing abstract sets with total binary operations, i.e. algebraic operations defined for all pairs of elements. As exemplified in the previous paragraph, the two distinguishing formal features of physical quantities are:

- The presence of a set of dimensions or, more concretely, units of measurement, indexing the set of all physical quantities which, furthermore, carries a partial operation given by addition of physical quantities of homogeneous dimensions.

- The set of physical quantities also carries a total operation, given by multiplication, and the set of dimensions carries an abelian group structure, in the example above simply $\mathbb{Z}^4$, with a natural compatibility condition between the two, seen explicitly in the formula for $Q \cdot Q'$ in the example above.

It is no surprise, then, that the structures of conventional algebra, based on single abstract sets endowed with total operations, prove insufficient to capture the essential algebraic properties of physical quantities. The definitions in the sections to follow are directly motivated by these two characteristic features of the formal manipulation of physical quantities.

7.1 Dimensioned Groups, Rings and Modules

Let us begin by introducing the general notion of **dimensioned set** as an abstract set A together with a surjective map $\alpha : A \to D$ onto another abstract set D, called the **dimension set** of A. We will commonly use the notation A_D for a dimensioned set A with dimensions D. The surjective map induces a partition of A by preimages $A_d := \alpha^{-1}(d) \subset A$ that are called the subsets of **homogeneous dimension**. An element $a \in A_d$ is said to have **dimension** $d \in D$. A map between dimensioned sets $\Phi : A_D \to B_E$ is called a **morphism of dimensioned sets** if subsets of homogeneous dimension of A are mapped into subsets of homogeneous dimension of B, in other words, there exists a map between the dimension sets $\phi : D \to E$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ D & \xrightarrow[\phi]{} & E \end{array}$$

the map ϕ will be called the **dimension map**. Clearly, dimensioned sets with the notion of morphism introduced above form a category, we call it **the category of dimensioned sets** and denote it by DimSet.

Groups are the first class of algebraic structures that we generalize to the dimensioned setting. A dimensioned set A_D is called a **dimensioned group** if there is a partial binary operation $\cdot$ on A that, upon restriction, induces a group structure on each of the homogeneous subsets $(A_d, \cdot|_{A_d})$. We shall denote dimensioned groups by $(A_D, \cdot_D)$. A dimensioned morphism between dimensioned groups $\Phi : A_D \to B_E$ is called a **dimensioned group morphism** when the restrictions to the homogeneous subsets $\Phi|_{A_d} : A_d \to B_{\phi(d)}$ are group morphisms for all $d \in D$. The notion of dimensioned group is a direct generalization of the ordinary notion of group as one recovers the defining axioms of group when D is a set with a single element. It clearly follows from our definition that dimensioned groups with dimensioned group morphisms form a subcategory of DimSet that we call **the category of dimensioned groups** and denote it by DimGrp.

Let $(A_D, \cdot_D)$ be a dimensioned group, then the subset $0_D := \{0_d \in (A_d, \cdot_d), \quad d \in D\}$ is called the **zero** of A_D. A subset $S \subset A_D$ is called a **dimensioned subgroup** when $S \cap A_d \subset (A_d, \cdot_d)$ are subgroups for all $d \in D$. A dimensioned subgroup $S \subset A_D$ is clearly a dimensioned group with dimension set given by $\alpha(S)$. We can define the **kernel** of a dimensioned group morphism $\Phi : A_D \to B_E$ in the obvious way

$$\ker(\Phi) := \{a_d \in A_D | \quad \Phi(a_d) = 0_{\phi(d)}\},$$

then clearly $\ker(\Phi)_D \subset A_D$ is a dimensioned subgroup. A dimensioned subgroup $S \subset A_D$ whose homogeneous intersections $S \cap A_d \subset (A_d, \cdot_d)$ are normal subgroups also induces a natural notion of **quotient**:

$$A_D/S := \bigcup_{d \in \alpha(S)} A_d/(S \cap A_d)$$

which has an obvious dimensioned group structure with dimension set $\alpha(S)$. There is also a natural notion of **product** of two dimensioned groups A_D, B_E given by

$$\alpha \times \beta : A \times B \to D \times E$$

with the partial multiplication defined in the obvious way

$$(a_d, b_e) \cdot_{(d,e)} (a'_d, b'_e) := (a_d \cdot_d a'_d, b_e \cdot_e b'_e).$$

A dimensioned group is called **abelian** when all its homogeneous subsets are abelian groups. As an example of a familiar class of objects that displays this kind of structure, we note that vector bundles can be seen as dimensioned groups where the fibre-wise vector addition gives the partial abelian group multiplication and the base manifold is the set of dimensions. An abelian dimensioned group A with set of dimensions D will be denoted by $(A_D, +_D)$. These clearly form a subcategory of DimGrp that we call **the category of dimensioned abelian groups** and denote it by DimAb. Abelian dimensioned groups display structures analogous to those of ordinary abelian groups when we fix a dimension set D and we consider **dimension-preserving morphisms**, i.e. dimensioned group morphisms $\Phi : A_D \to B_D$ for which the induced map on the dimension sets is the identity $\mathrm{id}_D : D \to D$. Abelian dimensioned groups over a fixed dimension set D together with dimension-preserving morphisms form a subcategory $\mathsf{DimAb}_D \subset \mathsf{DimAb}$ that, in addition to the notions of subgroup, kernel and quotient of the general category DimGrp, admits a **direct sum** defined as $A_D \oplus B_D := (A\ {}_\alpha\times_\beta\ B)_D$ with partial multiplication given in the obvious way

$$(a_d, b_d) +_d (a'_d, b'_d) := (a_d +_d a'_d, b_d +_d b'_d).$$

It is easy to prove that this direct sum operation on DimAb_D acts as a product and coproduct for which the notions of kernel and quotient identified in the general category DimGrp satisfy the axioms of abelian category. We call DimAb_D **the category of D-dimensioned abelian groups**.

Another similarity between abelian dimensioned groups and ordinary abelian groups is that the sets of morphisms carry a natural dimensioned group structure. Indeed, if we consider a pair of dimensioned group morphisms between abelian dimensioned groups $\Phi, \Psi : (A_D, +_D) \to (B_E, +_E)$ with dimension maps $\phi, \psi : D \to E$, we could attempt to define

$$(\Phi + \Psi)(a_d) := \Phi(a_d) +_e \Phi(a_d)$$

but we will have to choose $e \in E$ in a way that is consistent with the dimension maps. This is clearly achieved by setting $\phi(d) = e = \psi(d)$, thus addition of dimensioned morphisms is only defined for pairs of dimensioned morphism covering the same dimension map. If we denote by $\mathrm{Map}(D, E)$ the set of maps between the dimension sets, we clearly see that the set of dimensioned group morphisms between A_D and B_E has a structure of a dimensioned group with dimension set given by the set of maps between the dimension sets D and E. This will be called the **dimensioned group of morphisms** and we will denote it by $(\mathrm{Dim}(A_D, B_E)_{\mathrm{Map}(D,E)}, +_{\mathrm{Map}(D,E)})$ or for a single dimensioned group $\mathrm{Dim}(A_D)_{\mathrm{Map}(D)} := \mathrm{Dim}(A_D, A_D)_{\mathrm{Map}(D,D)}$.

Taking inspiration from the definition of a conventional ring as an abelian group with a compatible multiplication operation, we define **dimensioned ring** as a dimensioned abelian group R_G whose dimension set G carries a monoid structure (with multiplication denoted by juxtaposition) and with a total binary operation, called the dimensioned multiplication $\cdot : R_G \times R_G \to R_G$, satisfying the following axioms

1) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
2) $\exists! \quad 1 \in R_G| \quad 1 \cdot a = a = a \cdot 1$,
3) $\rho(a \cdot b) = \rho(a)\rho(b)$,
4) $(a + a') \cdot b = a \cdot b + a' \cdot b$,

for all $a, a', b, c \in R_G$ with $\rho(a) = \rho(a')$ and where $\rho : R \to G$ denotes the surjective dimension map. Note that in order to demand this list of axioms of a multiplication operation $\cdot : R_G \times R_G \to R_G$ in consistency with the dimensioned structure, the presence of a monoid structure on G is necessary. A dimensioned ring will be denoted by $(R_G, +_G, \cdot)$ and we will employ the subindex notation $a_g \in R_G$ to keep track of the dimensions of elements. Using the explicit index notation, axiom 4) reads

$$(a_g +_g a'_g) \cdot b_h = a_g \cdot b_h +_{gh} a'_g \cdot b_h.$$

It follows immediately from these axioms that the dimensioned zero 0_G acts as an absorbent subset in the following sense

$$0_g \cdot a_h = 0_{gh},$$

that the multiplicative identities are mapped to each other

$$\rho(1) = e \in G,$$

and that the homogeneous subset over the monoid identity forms an ordinary ring with the restricted operations $(R_e, +_e, \cdot|_{R_e})$, we call it the **dimensionless ring** of R_G. We see that a dimensioned ring can be simply understood as a dimensioned abelian group with a monoid structure projecting to the dimension set and that is distributive with respect to the dimensioned addition defined on homogeneous subsets. A dimensioned ring is called **commutative** when the monoid structures are commutative. For the reminder of this thesis dimensioned rings will be assumed to be commutative unless otherwise stated.

Let $(R_G, +_G, \cdot)$ and $(P_H, +_H, \cdot)$ be two dimensioned rings, a dimensioned group morphism $\Phi : R_G \to P_H$ is called a **morphism of dimensioned rings** when

$$\Phi(a \cdot b) = \Phi(a) \cdot \Phi(b), \qquad \Phi(1_{R_G}) = 1_{P_H}$$

for all $a, b \in R_G$. The map between the dimension monoids $\phi : G \to H$ is, then, necessarily a monoid morphism. Note that, from the general definition of dimensioned morphism, homogeneous subsets R_G are mapped into homogeneous subsets of P_H:

$$\Phi(R_g) \subset P_{\phi(g)}.$$

In particular, a dimensioned ring morphism induces a morphism of ordinary rings over the identity

$$\Phi|_{R_e} : (R_e, +_e, \cdot|_{R_e}) \to (P_{\phi(e)}, +_{\phi(e)}, \cdot|_{P_{\phi(e)}}).$$

We thus identify dimensioned rings with these morphisms as **the category of dimensioned rings** and denote it by $\mathsf{DimRing}$. We note again that, as in the general case of dimensioned groups above, setting the dimension monoids to the trivial monoid in all the above instances the usual basic definitions and results for ordinary rings are recovered. It is clear then that we can regard the category of ordinary rings as a subcategory of dimensioned rings $\mathsf{Ring} \subset \mathsf{DimRing}$.

An abelian dimensioned subgroup $S \subset R_G$ is called a **dimensioned subring** when $S \cdot S \subset S$ and $1 \in S$. A dimensioned subring $I \subset R_G$ is called a **dimensioned ideal** if for all elements $a_g \in R_G$ and $i_h \in I$ we have

$$a_g \cdot i_h \in I \cap R_{gh}.$$

This condition ensures that the general construction of quotient by an abelian dimensioned subgroup applied to the case of an ideal gives the **dimensioned quotient ring** R_G/I in a natural way. If we denote $I_g := I \cap R_g$, the dimensioned quotient ring multiplication is explicitly checked to be well-defined:

$$(a_g +_g I_g) \cdot (b_h +_h I_h) = a_g \cdot b_h +_{gh} a_g \cdot I_h +_{gh} b_h \cdot I_g +_{gh} I_g \cdot I_h = a_g \cdot b_h +_{gh} I_{gh}.$$

A **choice of units** u in a dimensioned ring R_G is a splitting of the dimension projection

$$\begin{array}{c} R \\ {\scriptstyle\rho}\!\downarrow \curvearrowright^{u} \\ G \end{array} \qquad \rho \circ u = \mathrm{id}_G, \qquad \text{such that} \quad u_{gh} = u_g \cdot u_h, \qquad u_g \neq 0_g$$

for all $g, h \in G$. In other words, a choice of units is a splitting of monoid morphisms $u : G \to R$ with non-zero image. Choices of units can be regarded as the dimensioned generalization of the notion of non-zero element of a ring with the caveat that they may not exist due to the non-vanishing condition being required for all of G. It was noted above that vector bundles give examples of dimensioned rings, then, considering the Moebius band as a dimensioned ring with dimension set the abelian group $\mathrm{U}(1)$ and the zero multiplication operation, we find an explicit example of a dimensioned ring that does not admit choices of unit.

As first examples of dimensioned rings we have already mentioned ordinary rings and vector bundles with the zero multiplication. Another important example is given by pairs of ordinary rings and monoids: let R be a ring and G a monoid, then the Cartesian product $R \times G$ carries a natural dimensioned ring structure, called the **trivial dimensioned ring** R with dimensions in G, defined in the obvious way

$$\mathrm{pr}_2 : R \times G \to G \qquad (a, g) +_g (b, g) := (a + b, g), \qquad (a, g) \cdot (b, h) := (a \cdot b, gh).$$

A dimensioned ring R_G is called a **dimensioned field** when

$$\forall a \notin 0_G \quad \exists a^{-1} | \quad a \cdot a^{-1} = 1 = a^{-1} \cdot a,$$

note that for this requirement to be consistent with the dimension projection the monoid structure of G must be a group structure, for this reason G will be called the dimension group of the dimensioned field. It is easy to see that multiplicative inverses of a dimensioned field are mapped into inverses of the dimension group $\rho(a^{-1}) = \rho(a)^{-1}$.

A direct consequence of the defining condition of dimensioned field is that non-zero elements induce group isomorphisms between homogeneous subsets. Indeed, for a non-zero element $0_g \neq a_g \in R_G$ we have the following maps

$$\begin{aligned} a_g \cdot : R_h &\to R_{gh} \\ b_h &\mapsto a_g \cdot b_h, \end{aligned}$$

which are group morphisms from axiom 4) of dimensioned rings and are invertible with inverse given by $a_g^{-1} \cdot$. These maps allow to prove a general result that confers a role to choices of unit on dimensioned fields similar to that of a trivialization of a fibre bundle.

Proposition 7.1.1 (Choices of Units in Dimensioned Fields)**.** *Let* $(R_G, +_G, \cdot)$ *be a dimensioned field, then a choice of units* $u : G \to R$ *induces an isomorphism with the trivial dimensioned field* R_e *with dimensions in* G*:*

$$R_G \cong R_e \times G.$$

Proof. We can explicitly construct the following map

$$\begin{aligned} \Phi^u : R_e \times G &\to R_G \\ (r, g) &\mapsto u_g \cdot r, \end{aligned}$$

which is a clearly bijective morphism of dimensioned abelian groups from the fact that it is constructed with the group isomorphisms $u_g \cdot$ for all the values of the choice of units u. It only remains to check that it is indeed a dimensioned ring morphism, this follows directly by construction and the fact that u is a morphism of monoids:

$$\begin{aligned} \Phi^u((r_1, g) \cdot (r_2, h)) = \Phi^u((r_1 \cdot r_2, gh)) = u_{gh} \cdot r_1 \cdot r_2 = u_g \cdot u_h \cdot r_1 \cdot r_2 = \\ = (u_g \cdot r_1) \cdot (u_h \cdot r_2) = \Phi^u(u_g \cdot r_1) \cdot \Phi^u(u_h \cdot r_2). \end{aligned}$$

□

This last proposition shows that the dimensioned fields for which choices of units exist are (non-canonically) isomorphic to the trivial dimensioned fields $\mathbb{F} \times G$ with $\mathbb{F}$ an ordinary field and G an abelian group.

Let us consider again a general a dimensioned ring $(R_G, +_G, \cdot)$. Recall from our discussion above that the dimensioned morphisms from R_G into itself form an abelian dimensioned group $(\mathrm{Dim}(R_G)_{\mathrm{Map}(G)}, +_{\mathrm{Map}(G)})$ where $\mathrm{Map}(G)$ denotes the set of maps from G into itself. The presence of the dimensioned ring multiplication allows for the definition of the following module-like structure

$$\cdot : R_G \times \mathrm{Dim}(R_G) \to \mathrm{Dim}(R_G)$$

defined via

$$(a_g \cdot \Phi)(b_h) := a_g \cdot \Phi(b_h).$$

We note that $a_g \cdot \Phi$ is a well-defined dimensioned morphism from the fact G acts naturally on $\mathrm{Map}(G)$ by composition with the left action of G on itself, indeed the if $\phi : G \to G$ is the dimension map of Φ, then $a_g \cdot \Phi$ has dimension map $L_g \circ \phi : G \to G$. It follows directly from the axioms of the dimensioned ring multiplication that that this operation satisfies the usual linearity properties of the conventional notion of R_G-module with the caveat that addition is only partially defined.

Recall now that products and direct sums of general abelian dimensioned groups can be taken so, considering the abelian dimensioned group part of a dimensioned ring $(R_G, +_G, \cdot)$, we can form the product $R_G \times R_G$, which is an abelian dimensioned group with dimension set $G \times G$, or the direct sum $R_G \oplus_G R_G$, which is a dimensioned abelian group with dimension set G. In both cases we can form module-like maps by setting

$$a_g \cdot (b_h, c_k) := (a_g \cdot b_h, a_g \cdot c_k), \qquad a_g \cdot (b_h \oplus c_h) := a_g \cdot b_h \oplus a_g \cdot c_h.$$

These module-like actions preserve the dimensioned structure from the fact that, in the first case, G acts diagonally on $G \times G$ and, in the second case, G acts on itself by left multiplication. Furthermore, from the defining axioms of dimensioned ring, these maps satisfy the usual linearity properties of the conventional notion of R_G-module with the caveat that addition is only partially defined.

These examples motivate the definition of dimensioned modules: let $(R_G, +_G, \cdot)$ be a dimensioned ring and $(A_D, +_D)$ a dimensioned abelian group, then A_D is called a **dimensioned R_G-module** if there is a map

$$\cdot : R_G \times A_D \to A_D$$

that is compatible with the dimensioned structure via a monoid action $G \times D \to D$ (denoted by juxtaposition) in the following sense

$$r_g \cdot a_d = (r \cdot a)_{gd}$$

and that satisfies the following axioms

1) $r_g \cdot (a_d + b_d) = r_g \cdot a_d + r_g \cdot b_d$,

2) $(r_g + p_g) \cdot a_d = r_g \cdot a_d + p_g \cdot a_d$,

3) $(r_g \cdot p_h) \cdot a_d = r_g \cdot (p_h \cdot a_d)$,

4) $1 \cdot a_d = a_d$

for all $r_g, p_h \in R_G$ and $a_d, b_d \in A_D$. Note that these four axioms for a map $\cdot : R_G \times A_D \to A_D$ can only be demanded in consistency with the dimensioned structure in the presence of a monoid action $G \times D \to D$. With this definition at hand, we recover the motivating examples for a dimension ring R_G introduced above: the **dimensioned module of dimensioned morphisms** $\mathrm{Dim}(R_G)$ is clearly a dimensioned R_G-module with dimension set $\mathrm{Map}(G)$; the product $R_G \times R_G$ is a dimensioned R_G-module with dimension set $G \times G$; and the direct sum $R_G \oplus_G R_G$ is a dimensioned R_G-module with dimension set G.

Let $(A_D, +_D)$ and $(B_E, +_E)$ be two dimensioned R_G-modules, a morphism of abelian dimensioned groups $\Phi : A_D \to B_E$ is called **R_G-linear** if

$$\Phi(r_g \cdot a_d) = r_g \cdot \Phi(a_d)$$

for all $r_g \in R_G$ and $a_d \in A_D$. Note that this condition forces the dimension map $\phi : D \to E$ to satisfy

$$\phi(gd) = g\phi(d)$$

for all $g \in G$ and $d \in D$, in other words, the dimension map ϕ must be G-equivariant with respect to the monoid actions of the dimension sets D and E. Let us denote the set of G-equivariant dimension maps as

$$\text{Map}^G(D, E) := \{\phi : D \to E| \quad \phi \circ g = g \circ \phi, \quad \forall g \in G\},$$

then it follows that the dimensioned group of morphisms $\text{Dim}(A_D, B_E)_{\text{Map}(D,E)}$ contains a dimensioned subgroup of morphisms covering G-equivariant dimension maps for which the following dimensioned module map can be defined

$$(r_g \cdot \Phi)(a_d) := r_g \cdot \Phi(a_d) = \Phi(r_g \cdot a_d).$$

This clearly makes $\text{Dim}(A_D, B_E)_{\text{Map}^G(D,E)}$ into a dimensioned R_G-module, we shall call this the **dimensioned module of R_G-linear morphisms**.

Let $(A_D, +_D)$ be a dimensioned R_G-module, an abelian dimensioned subgroup $S \subset A_D$ is called a **dimensioned submodule** if

$$r_g \cdot s_d \in S \cap A_{gd}$$

for all $r_g \in R_G$ and $s_d \in S$. All the notions introduced at the beginning of this section for general abelian dimensioned groups, e.g. direct sums, products, quotients, etc., apply to dimensioned R_G-modules in particular. Furthermore, given a dimensioned submodule $S \subset A_D$ that is a dimensioned I-module for $I \subset R_G$ a dimensioned ideal, there is a natural notion of **dimensioned quotient module** A_D/S with dimensioned ring given by the dimensioned quotient ring R_G/I.

As in the case or ordinary modules, Dimensioned modules admit a tensor product construction: Let $(A_D, +_D)$ and $(B_E, +_E)$ be two dimensioned R_G-modules, then we define their **dimensioned tensor product** as

$$A_D \otimes_{R_G} B_E := R_G \bullet (A_D \times B_E)/\sim$$

where $R_G \bullet (A_D \times B_E)$ denotes the free abelian dimensioned group of pairs (a_d, b_e) with coefficients in R_G and $\sim$ denotes taking a quotient with respect to the following relations within the free abelian dimensioned group

$$(a_d + a'_d, b_e) = (a_d, b_e) + (a'_d, b_e), \qquad (a_d, b_e + b'_e) = (a_d, b_e) + (a_d, b'_e), \qquad (r_g \cdot a_d, b_e) = (a_d, r_g \cdot b_e).$$

Note that these are the same relations used to define the tensor product of ordinary modules with the added caveat that addition is only partially defined. This construction clearly makes $A_D \otimes_{R_G} B_E$ into a dimensioned R_G-module with dimension set $D \times E$ and monoid G-action given by the diagonal action. The dimensioned tensor product so defined makes the abelian category of dimensioned R_G-modules into a monoidal category with the tensor unit given by R_G. Particularly, this definition makes the dimensioned tensor product distributive with respect to the dimensioned direct sum since it easy to check directly from the definition that, for three dimensioned R_G-modules A_D, B_D and C_E, there is a canonical R_G-linear isomorphism

$$(A_D \oplus_D B_D) \otimes_{R_G} C_E \cong A_D \otimes_{R_G} C_E \oplus_{D\times E} B_D \otimes_{R_G} C_E.$$

7.2 Dimensioned Algebras

Let us motivate our discussion on the dimensioned generalization of the notion of algebra for ordinary rings and modules by considering the dimensioned morphisms of a dimensioned ring R_G. In Section 7.1 above it was shown that $\text{Dim}(R_G)_{\text{Map}(G)}$ is a dimensioned R_G-module, we note that the dimension set given by the maps from G into itself $\text{Map}(G)$ carries a natural monoid structure given by composition of maps. Denoting three dimensioned morphisms by $\Phi_\phi, \Theta_\phi, \Psi_\psi : R_G \to R_G$ where $\phi, \psi : G \to G$ are the dimension maps, it follows directly from the defining properties of dimensioned rings that the composition of the dimensioned morphisms is consistent with the monoid structure of the dimension set $\mathrm{Map}(G)$

$$\Phi_\phi \circ \Psi_\psi = (\Phi \circ \Psi)_{\phi\circ\psi},$$

and that it interacts with the R_G-module structure of $\mathrm{Dim}(R_G)$ as a bilinear operation

$$(\Phi_\phi +_\phi \Theta_\phi) \circ \Psi_\psi = \Phi_\phi \circ \Psi_\psi +_{\phi\circ\psi} \Theta_\phi \circ \Psi_\psi \qquad r_g \cdot (\Phi_\phi \circ \Psi_\psi) = (r_g \cdot \Phi_\phi) \circ \Psi_\psi = \Phi_\phi \circ (r_g \cdot \Psi_\psi)$$

for all $r_g \in R_G$. This shows that $(\mathrm{Dim}(R_G), \circ)$ gives a prime example of a bilinear associative operation on a dimensioned module and prompts us to give the following general definition.

Let $(A_D, +_D)$ be a dimensioned R_G-module, a map $M : A_D \times A_D \to A_D$ is called a **dimensioned bilinear multiplication** if it satisfies

$$\begin{aligned} M(a_d +_d b_d, c_e) &= M(a_d, c_e) +_{\mu(d,e)} M(b_d, c_e) \\ M(a_d, b_e +_e c_e) &= M(a_d, b_e) +_{\mu(d,e)} M(a_d, c_e) \\ M(r_g \cdot a_d, s_h \cdot b_e) &= r_g \cdot s_h \cdot M(a_d, b_e) \end{aligned}$$

for all $a_d, b_d, b_e, c_e \in A_D$, $r_g, s_h \in R_G$ and for a **dimension map** $\mu : D \times D \to D$ which is G-equivariant in both entries, i.e.

$$\mu(gd, he) = gh\mu(d, e)$$

for all $g, h \in G$ and $d, e \in D$. When such a map M is present in a dimensioned R_G-module A_D, the pair (A_D, M) is called a **dimensioned R_G-algebra**. The notion of dimensioned tensor product given a the end of Section 7.1 above allows to reformulate the definition of a dimensioned bilinear multiplication $M : A_D \times A_D \to A_D$ as a dimensioned R_G-linear morphism

$$M : A_D \otimes_{R_G} A_D \to A_D.$$

Note that the dimension set of the tensor product $A_D \otimes_{R_G} A_D$ is $D \times D$ with the diagonal G-action induced from the R_G-module structure, then we see that the double G-equivariant condition of μ is reinterpreted now as ordinary G-equivariance with respect to the natural monoid actions.

The natural notions of morphisms and subalgebras of ordinary algebras extend naturally to the dimensioned case. Let (A_D, M) and (B_E, N) be two dimensioned R_G-algebras, a R_G-linear morphism $\Phi : A_D \to B_E$ is called a **morphism of dimensioned algebras** if

$$\Phi(M(a, a')) = N(\Phi(a), \Phi(a')),$$

for all $a, a' \in A_D$. A submodule $S \subset A_D$ such that $M(S, S) \subset S$ is called a **dimensioned subalgebra**.

The dimension map μ of a dimensioned bilinear multiplication in a dimensioned R_G-algebra (A_D, M_μ) is naturally regarded as an binary operation on the set of dimensions D. In a general sense, dimension sets of dimensioned algebras carry the most basic algebraic structures, commonly known as magmas. However, if one wishes to demand specific algebraic properties, such as commutativity or associativity, the algebraic structure present in the dimension magma becomes richer. Let (A_D, M_μ) be a dimensioned R_G-algebra, we say that it is **symmetric** or **antisymmetric** if

$$M(a_d, b_e) = M(b_e, a_d), \qquad M(a_d, b_e) = -M(b_e, a_d)$$

for all $a_d, b_e \in A_D$, respectively. The dimension magmas of symmetric or antisymmetric dimensioned algebras are necessarily commutative, i.e. $\mu(d, e) = \mu(e, d)$ for all $d, e \in D$. The usual 3-element-product properties of ordinary algebras can be demanded for dimensioned algebras in an analogous way, in particular (A_D, M_μ) is called

associative or **Jacobi** if

$$\mathrm{Ass}_M(a_d, b_e, c_f) = 0, \qquad \mathrm{Jac}_M(a_d, b_e, c_f) = 0$$

for all $a_d, b_e, c_f \in A_D$, respectively. The dimension magmas of associative or Jacobi dimensioned algebras are necessarily associative, i.e. $\mu(\mu(d,e),f) = \mu(d,\mu(e,f))$ for all $d,e,f \in D$, making them into semigroups. Returning to the motivating example presented at the beginning of this section, we now see that the dimensioned morphisms of a dimensioned ring R_G give the prime example of dimensioned associative algebra $(\mathrm{Dim}(R_G), \circ)$.

In parallel with the definitions of ordinary algebras, we define **dimensioned commutative algebra** as a symmetric and associative dimensioned algebra and a **dimensioned Lie algebra** as an antisymmetric and Jacobi dimensioned algebra. Note that dimensioned commutative and dimensioned Lie algebras necessarily carry dimension sets that are commutative semigroups.

In keeping with the general philosophy to continue to scrutinize the natural algebraic structure present in the dimensioned module of dimensioned morphisms of a dimensioned ring R_G, let us attempt to find the appropriate dimensioned generalization of the notion of derivations of a ring. Working by analogy, a dimensioned derivation will be a dimensioned morphism $\Delta \in \mathrm{Dim}(R_G)$ covering a dimension map $\delta : G \to G$ satisfying a Leibniz identity with respect to the dimensioned ring multiplication

$$\Delta(r_g \cdot s_h) = \Delta(r_g) \cdot s_h + r_g \cdot \Delta(s_h),$$

for all $r_g, s_h \in R_G$, however, for the right-hand-side to be well-defined, both terms must be of homogeneous dimension, which means that the dimension map must satisfy

$$\delta(gh) = \delta(g)h = g\delta(h)$$

for all $g, h \in G$. Since G is a monoid, this condition is equivalent to the dimension map being given by left (or equivalently due to commutativity, right) multiplication with a monoid element, i.e. $\delta = L_d$ for some element $d \in G$. Following from this observation, we see that there is a natural dimensioned submodule of the dimensioned module of dimensioned morphisms $\mathrm{Dim}(R_G)_G \subset \mathrm{Dim}(R_G)_{\mathrm{Map(G)}}$ given by the dimensioned morphisms whose dimension maps are specified by multiplication with a monoid element. Recall that dimensioned rings are assumed to be commutative and, thus, the dimension monoid has commutative binary operation. This allows for the identification of the first natural example of dimensioned Lie algebra: consider the commutator of the associative dimensioned composition

$$[\Delta, \Delta'] := \Delta \circ \Delta' - \Delta' \circ \Delta,$$

it is easy to check that this bracket is indeed antisymmetric and Jacobi, thus making $(\mathrm{Dim}(R_G)_G, [,])$ into the **dimensioned Lie algebra of dimensioned morphisms** of a dimensioned ring R_G. Notice that this bracket can only be defined on the dimensioned submodule $\mathrm{Dim}(R_G)_G \subset \mathrm{Dim}(R_G)_{\mathrm{Map(G)}}$ since the two terms of the right-hand-side for general dimensioned morphisms will have dimensions given by the composition of maps from G into itself which is a non-commutative binary operation in general. It is then clear that the Leibniz condition proposed above can be demanded in consistency with the dimensioned structure of dimensioned morphisms within the Lie algebra of dimensioned morphisms, so we see the dimensioned Lie algebra of **derivations of a dimensioned ring** R_G as the natural dimensioned Lie subalgebra of the dimensioned morphisms

$$\mathrm{Der}(R_G) \subset (\mathrm{Dim}(R_G)_G, [,]).$$

Derivations covering the monoid identity, i.e. those with dimension map $\mathrm{id}_G : G \to G$, are called **dimensionless derivations** and it is clear by definition that they form an ordinary Lie algebra with the commutator bracket $(\mathrm{Der}(R_G)_0, [,])$. Restricting their action to elements of the dimensionless ring $R_e \subset R_G$ we recover the ordinary Lie algebra of ring derivations, in other words, there is a surjective map of Lie algebras

$$(\mathrm{Der}(R_G)_0, [,]) \to (\mathrm{Der}(R_e), [,]).$$

The example of the dimensioned Lie algebra of derivations of a dimensioned ring illustrates the case of a dimensioned algebra whose space of dimensions is a (commutative) monoid and whose dimension map is simply given by the monoid multiplication. For the reminder of this chapter, the dimension sets of dimensioned modules will be assumed to carry a commutative monoid structure (with multiplication denoted by juxtaposition of elements) unless stated otherwise. Let A_G be a dimensioned module, a dimensioned algebra multiplication $M : A_G \times A_G \to A_G$ is said to be **homogeneous of dimension** m if the dimension map $\mu : G \times G \to G$ is given by monoid multiplication with the element $m \in G$, i.e. $\mu(g, h) = mgh$ for all $g, h \in G$. Assuming a monoid structure on the dimension set of a dimensioned module and considering dimensioned algebra multiplications of homogeneous dimension is particularly useful in order to study several algebra multiplications coexisting on the same set. Indeed, given two homogeneous dimensioned algebra multiplications (A_G, M_1) and (A_G, M_2) with dimensions $m_1 \in G$ and $m_2 \in G$, respectively, the fact that the monoid operation is assumed to be associative and commutative, allows for consistently demanding properties of the interaction of the two dimensioned multiplications involving expressions of the form $M_1(M_2(a, b), c)$ without any further requirements.

Let A_G be a dimensioned R_H-module and let two dimensioned algebra multiplications $* : A_G \times A_G \to A_G$ and $\{,\} : A_G \times A_G \to A_G$ with homogeneous dimensions $p \in G$ and $b \in G$, respectively, the triple $(A_G, *_p, \{,\}_b)$ is called a **dimensioned Poisson algebra** if

1) $(A_G, *_p)$ is a dimensioned commutative algebra,
2) $(A_G, \{,\}_b)$ is a dimensioned Lie algebra,
3) the two multiplications interact via the Leibniz identity

$$\{a, b * c\} = \{a, b\} * c + b * \{a, c\},$$

for all $a, b, c \in A_G$.

Note that the Leibniz condition can be consistently demanded of the two dimensioned algebra multiplications since the dimension projections of each of the terms of the Leibniz identity for $\{a_g, b_h * c_k\}$ are:

$$bgphk, \quad pbghk, \quad phbgk,$$

but they are indeed all equal from the fact that the monoid binary operation is associative and commutative.

A morphism of dimensioned modules between dimensioned Poisson algebras $\Phi : (A_G, *_p, \{,\}_b) \to (B_H, *_r, \{,\}_c)$ is called a **morphism of dimensioned Poisson algebras** if $\Phi : (A_G, *_p) \to (B_H, *_r)$ is a morphism of dimensioned commutative algebras and also $\Phi : (A_G, \{,\}_b) \to (B_H, \{,\}_c)$ is a morphism of dimensioned Lie algebras. A submodule $I \subset A_G$ that is a dimensioned ideal in $(A_G, *_p)$ and that is a dimensioned Lie subalgebra in $(A_G, \{,\}_b)$ is called a **dimensioned coisotrope**.

Proposition 7.2.1 (Dimensioned Poisson Reduction). *Let $(A_G, *_p, \{,\}_b)$ be a dimensioned Poisson algebra and $I \subset A_G$ be a coisotrope, then there is a dimensioned Poisson algebra structure induced in the subquotient*

$$(A'_G := N(I)/I, *'_p, \{,\}'_b)$$

where $N(I)$ denotes the dimensioned Lie idealizer of I regarded as a submodule of the dimensioned Lie algebra.

Proof. We assume without loss of generality that the dimension projection of I is the whole of G, the intersections with the homogeneous subsets are denoted by $I_g := I \cap A_g$. The dimensioned Lie idealizer is defined in the obvious way

$$N(I) := \{n_g \in A_G | \quad \{n_g, i_h\} \in I_{bgh} \quad \forall i_h \in I\}.$$

We clearly see that $N(I)$ is the smallest dimensioned Lie subalgebra that contains I as a dimensioned Lie ideal. The Leibniz identity implies that $N(I)$, furthermore, is a dimensioned commutative subalgebra with respect to

$*_p$ in which I sits as a dimensioned commutative ideal, since it is a commutative ideal in the whole A_G. It follows that we can form the dimensioned quotient commutative algebra $(N(I)/I, *')$ in an entirely analogous way to the construction of the dimensioned quotient ring presented in Section 7.1. The only difference with that case is that commutative multiplication covers a dimension map that is given by the monoid multiplication with a non-identity element $p \in G$, but this has no effect on the quotient construction. To obtain the desired quotient dimensioned Lie bracket we set:

$$\{n_g + I_g, m_h + I_h\}' := \{n_g, m_h\} + I_{bgh}$$

which is easily checked to be well-defined and that inherits the antisymmetry and Jacobi properties directly from dimensioned Lie bracket $\{,\}$ and the fact that $I \subset N(I)$ is a dimensioned Lie ideal. □

7.3 The Potential Functor

In this section we will find a close link between dimensioned algebras and the category of lines Line introduced in Section 2.1.1. The results presented in this section will be instrumental for section 7.6, where we argue that the general notions of dimensioned algebra introduced in sections 7.1 and 7.2 provide the natural language to express constructions of Jacobi manifolds algebraically. Furthermore, these results provide the direct connection of the notion of dimensioned ring with the motivating example of physical quantities with units and the abstract notion of measurand space introduced in Section 3.2.

Recall that in Section 2.1.1 it was shown that the categorical structure of 1-dimensional vector spaces allowed for the construction of abelian groups of tensor powers. More concretely, the **potential** of a line $L \in \mathsf{Line}$ was defined as the set of all tensor powers

$$L^{\odot} := \bigcup_{n\in\mathbb{Z}} L^n$$

where L^n denotes the tensor powers of L for $n > 0$, the tensor powers of the dual line L^* for $n < 0$ and the patron line $\mathbb{R}$ for $n = 0$. This set has than an obvious dimensioned set structure with dimension set $\mathbb{Z}$:

$$\pi : L^{\odot} \to \mathbb{Z}.$$

Since homogeneous subsets are precisely the tensor powers L^n, they carry a natural $\mathbb{R}$-vector space structure thus clearly making the potential of L into an abelian dimensioned group $(L^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}})$. The next proposition shows that the ordinary $\mathbb{R}$-tensor product of vector spaces endows $L^{\odot}$ with a dimensioned field structure.

Proposition 7.3.1 (Dimensioned Ring Structure of the Potential of a Line)**.** *Let $L \in \mathsf{Line}$ be a line and $(L^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}})$ its potential, then the $\mathbb{R}$-tensor product of elements induces a dimensioned multiplication*

$$\odot : L^{\odot} \times L^{\odot} \to L^{\odot}$$

such that $(L^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}}, \odot)$ becomes a dimensioned field.

Proof. The construction of the dimensioned ring multiplication $\odot$ is done simply via the ordinary tensor product of ordinary vectors and taking advantage of the particular properties of 1-dimensional vector spaces. The two main facts that follow from the 1-dimensional nature of lines are: firstly, that linear endomorphisms are simply real numbers

$$\text{End}(L) \cong L^* \otimes L \cong \mathbb{R}$$

which, at the level of elements, means that

$$\text{End}(L) \ni \alpha \otimes a = \alpha(a) \cdot \text{id}_L$$

as it can be easily shown by choosing a basis; and secondly, that the tensor product becomes canonically commutative, since, using the isomorphism above, we can directly check

$$a \otimes b(\alpha, \beta) = \alpha(a)\beta(b) = \alpha(b)\beta(a) = b \otimes a(\alpha, \beta),$$

thus showing

$$a \otimes b = b \otimes a \in L \otimes L = L^2.$$

The binary operation $\odot$ is then explicitly defined for elements $a, b \in L = L^1$, $\alpha, \beta \in L^* = L^{-1}$ and $r, s \in \mathbb{R} = L^0$ by

$$\begin{aligned} a \odot b &:= a \otimes b \\ \alpha \odot \beta &:= \beta \otimes \alpha \\ r \odot s &:= r \otimes s = rs \\ r \odot a &:= ra \\ r \odot \alpha &:= r\alpha \\ \alpha \odot a &:= \alpha(a) = a(\alpha) =: a \odot \alpha \end{aligned}$$

Products of two positive power tensors $a_1 \otimes \cdots \otimes a_q$, $b_1 \otimes \cdots \otimes b_p$ and negative powers $\alpha_1 \otimes \cdots \otimes \alpha_q$, $\beta_1 \otimes \cdots \otimes \beta_p$ are defined by

$$\begin{aligned} (a_1 \otimes \cdots \otimes a_q) \odot (b_1 \otimes \cdots \otimes b_p) &:= a_1 \otimes \cdots \otimes a_q \otimes b_1 \otimes \cdots \otimes b_p \\ (\alpha_1 \otimes \cdots \otimes \alpha_q) \odot (\beta_1 \otimes \cdots \otimes \beta_p) &:= \alpha_1 \otimes \cdots \otimes \alpha_n \otimes \beta_1 \otimes \cdots \otimes \beta_m \end{aligned}$$

and extending by $\mathbb{R}$-linearity. This clearly makes the dimensioned ring product satisfy, for $q, p > 0$,

$$\odot : L^q \times L^p \to L^{q+p}, \qquad \odot : L^{-q} \times L^{-p} \to L^{-q-p}, \qquad \odot : L^0 \times L^0 \to L^0.$$

For products combining positive power tensors $a_1 \otimes \cdots \otimes a_q$ and negative power tensors $\alpha_1 \otimes \cdots \otimes \alpha_p$ we critically make use of the isomorphism $L^* \otimes L \cong \mathbb{R}$ to define, without loss of generality for $p > q > 0$,

$$(a_1 \otimes \cdots \otimes a_q) \odot (\alpha_1 \otimes \cdots \otimes \alpha_p) := \alpha_1(a_1) \cdots \alpha_q(a_q)\alpha_{p-q} \otimes \cdots \otimes \alpha_p.$$

It is then clear that the multiplication $\odot$ satisfies, for all $m, n \in \mathbb{Z}$,

$$\odot : L^m \times L^n \to L^{m+n}$$

and so it is compatible with the dimensioned structure of $L^{\odot}_{\mathbb{Z}}$. The multiplication $\odot$ is clearly associative and bilinear with respect to the addition on each homogeneous subset from the fact that the ordinary tensor product is associative and $\mathbb{R}$-bilinear. Then it follows that $(L^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}}, \odot)$ is a commutative dimensioned ring. It only remains to show that non-zero elements of $L^{\odot}$ have multiplicative inverses. Note that a non-zero element corresponds to some non-vanishing tensor $0 \neq h \in L^n$, but, since L^n is a 1-dimensional vector space for all $n \in \mathbb{Z}$, we can find a unique $\eta \in (L^n)^* = L^{-n}$ such that $\eta(h) = 1$. It follows from the above formula for products of positive and negative tensor powers that, in terms of the dimensioned ring multiplication, this becomes

$$h \odot \eta = 1,$$

thus showing that all non-zero elements have multiplicative inverses, making the dimensioned ring $(L^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}}, \odot)$ into a dimensioned field. □

We now prove that the construction of the potential dimensioned field of a line is, in fact, functorial.

Theorem 7.3.1 (The Potential Functor for Lines)**.** *The assignment of the potential construction to a line is a* *functor*

$$\odot : \mathsf{Line} \to \mathsf{DimRing}.$$

Furthermore, a choice of unit in a line $L \in \mathsf{Line}$ induces a choice of units in the dimensioned field $(L^\odot_\mathbb{Z}, +_\mathbb{Z}, \odot)$ which, since $L^0 = \mathbb{R}$, then gives an isomorphism with the trivial dimensioned field

$$L^\odot \cong \mathbb{R} \times \mathbb{Z}.$$

Proof. To show functoriality we need to define the potential of a factor of lines $B : L_1 \to L_2$

$$B^\odot : L_1^\odot \to L_2^\odot.$$

This can be done explicitly in the obvious way, for $q > 0$

$$\begin{aligned} B^\odot|_{L^q} &:= B \otimes \overset{q}{\cdots} \otimes B : L_1^q \to L_2^q \\ B^\odot|_{L^0} &:= \mathrm{id}_\mathbb{R} : L_1^0 \to L_2^0 \\ B^\odot|_{L^{-q}} &:= (B^{-1})^* \otimes \overset{q}{\cdots} \otimes (B^{-1})^* : L_1^{-q} \to L_2^{-q} \end{aligned}$$

where we have crucially used the invertibility of the factor B. By construction, $B^\odot$ is compatible with the $\mathbb{Z}$-dimensioned structure and since B is a linear map with linear inverse, all the tensor powers act as $\mathbb{R}$-linear maps on the homogeneous sets, thus making $B^\odot : L_1^\odot \to L_2^\odot$ into a morphism of abelian dimensioned groups. Showing that $B^\odot$ is a dimensioned ring morphism follows easily by the explicit construction of the dimensioned ring multiplication $\odot$ given in proposition 7.3.1 above. This is checked directly for products that do not mix positive and negative tensor powers and for mixed products it suffices to note that

$$B^\odot(\alpha) \odot B^\odot(a) = (B^{-1})^*(\alpha) \odot B(a) = \alpha(B^{-1}(B(a))) = \alpha(a) = \mathrm{id}_\mathbb{R}(\alpha(a)) = B^\odot(\alpha \odot a).$$

It follows from the usual properties of tensor products in vector spaces that for another factor $C : L_2 \to L_3$ we have

$$(C \circ B)^\odot = C^\odot \circ B^\odot, \qquad (\mathrm{id}_L)^\odot = \mathrm{id}_{L^\odot},$$

thus making the potential assignment into a functor. Recall that a choice of unit in a line $L \in \mathsf{Line}$ is simply a choice of non-vanishing element $u \in L^\times$. In proposition 7.3.1 we saw that $L^\odot$ is a dimensioned field, so multiplicative inverses exist, let us denote them by $u^{-1} \in (L^*)^\times$. Using the notation for $q > 0$

$$\begin{aligned} u^q &:= u \odot \overset{q}{\cdots} \odot u \\ u^0 &:= 1 \\ u^{-q} &:= u^{-1} \odot \overset{q}{\cdots} \odot u^{-1}, \end{aligned}$$

it is clear that the map

$$\begin{aligned} u : \mathbb{Z} &\to L^\odot \\ n &\mapsto u^n \end{aligned}$$

satisfies

$$u^{n+m} = u^n \odot u^m.$$

By construction, all $u^n \in L^n$ are non-zero, so $u : \mathbb{Z} \to L^\odot$ is a choice of units in the dimensioned field $(L^\odot_\mathbb{Z}, +_\mathbb{Z}, \odot)$. The isomorphism of dimensioned fields $L^\odot \cong \mathbb{R} \times \mathbb{Z}$ follows from proposition 7.1.1 and the observation that, by definition, $(L^\odot)_0 = L^0 = \mathbb{R}$. □

Given a collection of lines $L_1, \ldots, L_k \in \mathsf{Line}$, the above constructions generalize to the following notion of

potential:

$$(L_1, \dots, L_k)^{\odot} := \bigcup_{n_1,\dots n_k\in\mathbb{Z}} L_1^{n_1} \otimes \cdots \otimes L_k^{n_k},$$

which has a natural abelian dimensioned group structure given by $\mathbb{R}$-linear addition and has dimension group $\mathbb{Z}^k$. The dimensioned filed structure generalizes in the obvious way:

$$(a_1 \otimes \cdots \otimes a_k) \odot (b_1 \otimes \cdots \otimes b_k) := a_1 \odot b_1 \otimes \cdots \otimes a_k \odot b_k$$

thus making $((L_1, \dots, L_k)^{\odot}_{\mathbb{Z}^k}, +_{\mathbb{Z}^k}, \odot)$ into a dimensioned field. Note that the potentials of each individual line L_i can be found as dimensioned subfields $L_i^{\odot} \subset (L_1, \dots, L_k)^{\odot}$ since they are simply the dimensional preimages of the natural subgroups $\mathbb{Z} \subset \mathbb{Z}^k$. Furthermore, a choice of unit in each of the individual lines $u_i \in L_i^{\times}$ naturally induces a choice of units for the potential in a natural way

$$U : \mathbb{Z}^k \to (L_1, \dots, L_k)^{\odot}$$
$$(n_1, \dots, n_k) \mapsto u_1^{n_1} \odot \cdots \odot u_k^{n_k}.$$

L-vector spaces were introduced in Section 2.1.1 as the natural generalization of vector spaces when lines are interpreted as "unit-free" fields of numbers. That interpretation, however, was only partial, since a proper generalization of notion of vector space would have to include a module structure with respect to the generalization of the field of numbers. In Section 7.1 it was argued that the notion of dimensioned field indeed captures this generalization and dimensioned modules were introduced in a natural way. The technology developed in this chapter so far allows us to give a proper generalization of vector space in this sense via the notion of **dimensioned vector space over a dimensioned field** defined simply as a dimensioned module over a dimensioned field. These, together with dimensioned morphisms, form a category, DimVect, and the general notions of dimensioned modules introduced in Section 7.1 apply. In particular, if a dimensioned field F_G is fixed, the subcategory of dimensioned vector spaces over it, $\mathsf{DimVect}_{F_G}$, becomes an abelian monoidal category, in complete analogy with the category of ordinary vector spaces over a fixed field.

Our original claim that L-vector bundles represented a valid line generalization of ordinary vector spaces is fully justified by the fact that the datum of a L-vector space $V^L \in \mathsf{LVect}$ gives a dimensioned vector space. This is accomplished explicitly by the construction of the **potential** of $V^L \in \mathsf{LVect}$ which is defined as an abelian dimensioned group with dimensions in $\mathbb{Z}$ in a natural way:

$$V^{\odot L} := \bigcup_{n\in\mathbb{Z}} L^n \otimes V.$$

This abelian dimensioned group carries an obvious dimensioned $L^{\odot}$-module structure that can be defined explicitly by

$$a \cdot (b \otimes v) := (a \odot b) \otimes v$$

for all $a \in L^n$, $b \in L^m$ and $v \in V$, and then extending by linearity. The next proposition shows that. much like in the case of the potential of lines, the potential construction of L-vector spaces is functorial.

Proposition 7.3.2 (The Potential Functor for L-Vector Spaces)**.** *The assignment of the potential construction to a L-vector space is a monoidal functor*

$$\odot : \mathsf{LVect} \to \mathsf{DimVect}$$

compatible with duality. Furthermore, fixing a line $L \in$ Line, the potential assignment

$$\odot : \mathsf{LVect}_L \to \mathsf{DimVect}_{L^{\odot}}$$

becomes an abelian functor.

Proof. To prove functoriality we give the explicit construction of the potential of a L-vector space morphism $\psi^B : V_1^{L_1} \to V_2^{L_2}$ as follows:

$$\psi^{\odot B}(a \otimes v) := B^{\odot}(a) \otimes \psi(v)$$

for all $a \in L^n$ and $v \in V$ and extending by linearity. This clearly clearly makes $\psi^{\odot B}$ into an abelian dimensioned group morphism with identity dimension map

$$\begin{array}{ccc} V_1^{\odot L_1} & \xrightarrow{\psi^{\odot B}} & V_2^{\odot L_2} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow[\mathrm{id}_{\mathbb{Z}}]{} & \mathbb{Z} \end{array}$$

which, by construction, interacts with the dimensioned module morphism as follows

$$\psi^{\odot B}(b \cdot (a \otimes v)) = B^{\odot}(b) \cdot \psi^{\odot B}(a \otimes v).$$

Note that this last expression is the natural generalization of the $\mathbb{R}$-linearity of ordinary vector spaces where the dimensioned ring isomorphism $B^{\odot}$ will be replaced by the particular case of the identity map. Functoriality of $\psi^{\odot B}$ then simply follows by the functoriality of $B^{\odot}$ proved in theorem 7.3.1 and the usual composition of linear maps between ordinary vector spaces. Recall that the L-tensor product is defined as $V_1^{\odot L_1} \otimes V_1^{\odot L_1} := (V_1 \otimes V_2)^{L_1 \otimes L_2}$. We can take the potential of the two lines $(L_1 L_2)^{\odot}$ and define

$$(V_1 \otimes V_2)^{\odot(L_1 \otimes L_2)} := \bigcup_{n_1, n_2 \in \mathbb{Z}} L_1^{n_1} \otimes L_2^{n_2} \otimes V$$

which is a dimensioned vector space over the dimensioned ring $(L_1 L_2)^{\odot}$. Tensor products of L-vector bundle morphisms are clearly sent to tensor products of dimensioned morphism via

$$\odot : \psi^B \otimes \varphi^C \mapsto (\psi \otimes \varphi)^{\odot(B \otimes C)},$$

and thus we see that the potential functor is indeed monoidal. Recall that the L-dual of a L-vector space is defined as $V^{*L} := (V^* \otimes L)^L$, then we observe that, after applying the usual canonical isomorphisms to reorder tensor products, the potential of the L-dual will have homogeneous dimension sets shifted by $+1$. This precisely corresponds to elements of $(V^{\odot L})^*$ being $L^{\odot}$-linear maps of the form $V^{\odot L} \to L^{\odot}$. Exploiting once more the invertibility of line factors, we easily see that the potential of the dual of a L-vector bundle morphism $\psi^{*B} : V_2^{*L_2} \to V_1^{*L_1}$ is an isomorphism of dimensioned vector spaces $\psi^{\odot(*B)} : (V_2^*)^{\odot L_2} \to (V_1^*)^{\odot L_1}$, then we see that the potential construction is compatible with L-duality. Lastly, when we fix a line $L \in \mathsf{Line}$ L-direct sums, subobjects and kernels become well-defined, it is then clear from the $\mathbb{R}$-linearity of all the maps involved, these are preserved under the potential construction, thus showing that it is an abelian functor. □

7.4 Measurand Spaces Revisited

Let us now return to the original question of the formal description of physical quantities and units of measurement but now with the machinery of dimensioned algebra developed in sections 7.1, 7.2 and 7.3 at our disposal. We begin by pointing out that, it should have become obvious by now, the notion of dimensioned field indeed captures the general formal structure of physical quantities. Any concrete example of a scientific model involving quantitative measurements, such us the example of classical thermodynamics presented in the opening of this chapter (7), will involve some finite number k of basic units so the set of all theoretically possible physical quantities form a dimensioned ring $R_{\mathbb{Z}^k}$ with $R_0 = \mathbb{R}$.

In light of the results of Section 7.3, we have not only successfully recovered the algebraic structure of physical quantities, but also given a complete mathematical legitimization to the empirically-motivated definition of measurand spaces proposed in Section 3.2. Indeed, a physical theory will consist of a collection of basic measurable properties, what we called **base measurands**, that are mathematically identified with a collection of lines $L_1, \ldots, L_k \in \mathsf{Line}$. The potential functor $\odot : \mathsf{Line} \to \mathsf{DimRing}$ now gives a mathematically precise meaning to what was defined as the **measurand space** M of the physical theory:

$$M = ((L_1, \ldots, L_k)^{\odot}_{\mathbb{Z}^k}, +_{\mathbb{Z}^k}, \odot)$$

which, following from theorem 7.3.1, has the structure of a dimensioned field. We are now in the position to give a precise mathematical definition of **physical quantity** Q simply as an element in the potentil:

$$Q \in M = (L_1, \ldots, L_k)^{\odot}_{\mathbb{Z}^k}.$$

The term choice of units for a monoid splitting of the dimension projection of a dimensioned ring $u : G \to R_G$ was introduced in Section 7.1 in anticipation of the metrological interpretation that we now give. Units of measurement in applied science an engineering serve as the reference scale for all the measurements of a physical quantity of the same kind. Mathematically, this is conventionally represented by assigning the numerical value 1 to the measurement of the physical quantity applied to the unit of measurement itself, hence the name *unit* of measurement. Since a choice of units assigns an non-zero element to each set of homogeneous dimension, which in the case of a measurand space are simply lines, they form a basis for that set and their component expression is conventionally also given by the numerical value 1. We have thus connected the abstract notion of **choice of units** in a potential of some collection of lines with a **set of units of measurement** that will be used in a physical theory that takes those lines as base measurands.

In dimensional analysis, it is common to recombine basic measurands of a physical theory and express them in terms of products of other measurands, such as in the example of Section 3.2 where area was expressed as the product of lengths. In our formalism, where a physical theory is identified with a measurand space $M = (L_1, \ldots, L_k)^{\odot}_{\mathbb{Z}^k}$, this is simply encapsulated by the notion of dimensioned isomorphism of the measurand space $\Psi : M \to M$ which, in general, will have non-trivial dimension map $\psi : \mathbb{Z}^k \to \mathbb{Z}^k$, corresponding to the recombinations of basic measurands. More concretely, changes in choices of units for the same measurand space are, in virtue of theorem 7.3.1, completely characterized by dimensioned ring isomorphisms of the trivial dimensioned ring

$$\mathbb{R} \times \mathbb{Z}^k \to \mathbb{R} \times \mathbb{Z}^k.$$

7.5 Unit-Free Manifolds Revisited

In Section 2.6.3 we argued that line bundles could be understood as a "unit-free" analogue of ordinary manifolds if one conceptually replaces the role played by the ring of smooth functions with the module of sections of a line bundle. The many results that were proved in that section showing the parallels with the ordinary theory of manifolds attest to the adequacy of this interpretation. However, using the module of sections of a line bundle as the algebraic analogue of the ring of smooth functions meant that the ring multiplication did not have a direct analogue. In this section we will show that this analogue appears as a dimensioned ring structure when we generalize the potential construction of Section 7.3 to line bundles.

Let $\lambda : L \to M$ be a line bundle, in Section 2.6.1 we saw that there is a monoidal structure in the restricted category of line bundles over the same base M. We can use the tensor product in this category in a completely analogous way to the tensor product in the category of lines Line and thus form positive and negative powers of the line bundle L^n, $n \in \mathbb{Z}$. The **potential** of the line bundle L is then defined in a natural way

$$\Gamma(L)^{\odot} := \bigcup_{n \in \mathbb{Z}} \Gamma(L^n).$$

This set carries an obvious dimensioned structure with dimension set $\mathbb{Z}$ and the usual module structure on sections for each power $(\Gamma(L^n), +_n)$ clearly makes $\Gamma(L)^\odot$ into an abelian dimensioned group. Furthermore, the construction of the dimensioned ring product $\odot$ detailed in the proof of proposition 7.3.1 can be reproduced in this case verbatim, thus making the potential of a line bundle into a dimensioned ring $(\Gamma(L)^\odot_\mathbb{Z}, +_\mathbb{Z}, \odot)$. We note that this dimensioned ring encapsulates the usual algebraic structures found in sections of line bundles: indeed, the dimensionless ring of $\Gamma(L)^\odot$ is the ordinary ring of functions of the base manifold $\Gamma(L^0) = \Gamma(\mathbb{R}_M) \cong \mathrm{C}^\infty(M)$ and, for $f \in \Gamma(L^0) = \mathrm{C}^\infty(M)$, $s \in \Gamma(L^1) = \Gamma(L)$ and $\sigma \in \Gamma(L^{-1}) = \Gamma(L^*)$, the dimensioned products

$$f \odot s = f \cdot s, \qquad \sigma \odot s = \sigma(s)$$

amount to the $\mathrm{C}^\infty(M)$-module map and the duality pairing, respectively. We now show that, as was the case for lines, the potential construction of line bundles is functorial.

Proposition 7.5.1 (The Potential Functor for Line Bundles)**.** *The assignment of the potential construction to a line bundle is a contravariant functor*

$$\odot : \mathsf{Line}_{\mathsf{Man}} \to \mathsf{DimRing}.$$

Proof. Let us first define the potential of a factor between line bundles $B : L_1 \to L_2$ covering a smooth map $b : M_1 \to M_2$. We aim to define a dimensioned ring morphism of the form

$$B^\odot : (\Gamma(L_2)^\odot_\mathbb{Z}, +_\mathbb{Z}, \odot) \to (\Gamma(L_1)^\odot_\mathbb{Z}, +_\mathbb{Z}, \odot),$$

our definition will be, furthermore, of a dimensionless morphism, in the sense that it will cover the identity on the dimension group $\mathrm{id}_\mathbb{Z} : \mathbb{Z} \to \mathbb{Z}$, so that it will suffice to provide a collection of maps between the sections of all the tensor powers $B^\odot_n : \Gamma(L_2^n) \to \Gamma(L_1^n)$. The datum provided by the line bundle factor B allows to define three maps

$$\begin{aligned} b^* &: \mathrm{C}^\infty(M_2) \to \mathrm{C}^\infty(M_1) \\ B^* &: \Gamma(L_2) \to \Gamma(L_1) \\ B^* &: \Gamma(L_2^*) \to \Gamma(L_1^*) \end{aligned}$$

where the first is simply the pull-back of the smooth map between base manifolds, the second is the pull-back of sections induced by a factor of line bundles defined point wise by

$$B^*(s_2)(x) := B_x^{-1}(s_2(b(x)))$$

for all $s_2 \in \Gamma(L_2)$, and the third is the usual pull-back of dual forms on general vector bundles, defined point-wise for a general bundle map by

$$B^*\sigma_2(s_1)(x) := \sigma_2(b(x))(B_x(s_1(x)))$$

for all $\sigma_2 \in \Gamma(L_2^*)$, $s_1 \in \Gamma(L_1)$. The maps $B^\odot_n$ are then defined simply as the tensor powers of these pull-backs. Contravariance then follows directly from contravariance of the pull-backs. It is then clear by construction that $B^\odot$ so defined acts as a dimensioned ring morphism for products of positive or negative tensor powers, then it only remains to show that it also acts as such for mixed products of tensor powers. This is readily checked by considering the following observation for sections $s_2 \in \Gamma(L_2)$ and $\sigma_2 \in \Gamma(L_2^*)$:

$$\begin{aligned} B^\odot\sigma_2 \odot B^\odot s_2(x) &= B^*\sigma_2 \odot B^* s_2(x) = \\ &= \sigma_2(b(x))(B_x B_x^{-1}(s_2(b(x)))) = \sigma_2(b(x))(s_2(b(x))) = b^*(\sigma_2(s_2))(x) = B^\odot(\sigma_2 \odot s_2)(x). \end{aligned}$$

□

This last proposition provides the key result for the legitimization of the interpretation of line bundles as unit-free manifolds since we notice the similarity of the potential functor above with the ordinary contravariant functor given by the assignment of the ring of smooth functions to a manifold

$$\mathrm{C}^\infty : \mathsf{Man} \to \mathsf{Ring}.$$

The potential functor $\odot$ is, in fact, a direct generalization of C^∞ as ordinary rings can be regarded as dimensioned rings with dimension set the trivial monoid.

Due to possible topological constraints, the notion of unit of a line, i.e. a non-vanishing element, can be recovered only locally for line bundles. Let $\lambda : L \to M$ be a line bundle and $U \subset M$ an open subset, the potential construction is clearly natural with respect to restrictions since the same prescription used for global sections can be used to define $\Gamma(L|_U)^\odot$. Defining the positive and negative powers of u as it was done for the line case, it is clear that a local unit induces a choice of units for the local potential

$$u : \mathbb{Z} \to \Gamma(L|_U)^\odot.$$

It then follows from the second part of theorem 7.3.1 that a local unit u induces an isomorphism of the local potential with the trivial dimensioned ring of local functions with dimension set $\mathbb{Z}$:

$$\Gamma(L|_U)^\odot \cong \mathrm{C}^\infty(U) \times \mathbb{Z}.$$

To further establish the idea that that potentials of line bundles generalize rings of functions of manifolds, we will discuss the generalization of two aspects of ordinary manifolds for which the ring of functions proves a convenient algebraic tool: derivations and vanishing ideals of submanifolds.

Recall from Section 7.1 that the derivations of a dimensioned ring $\mathrm{Der}(R_G)$ form a dimensioned module with dimension set G that contains the derivations of the dimensionless ring $\mathrm{Der}(R_e)$ as a Lie subalgebra of dimensionless derivations. In the case of the potential of a line bundle $\Gamma(L)^\odot$, this implies that derivations of smooth functions, or, equivalently, vector fields, are recovered as a Lie subalgebra of the dimensionless derivations

$$\mathrm{Der}(\mathrm{C}^\infty(M)) \cong \Gamma(\mathrm{T}M) \subset \mathrm{Der}(\Gamma(L)^\odot).$$

In Section 2.6.3 we argued that the line bundle generalization of the tangent bundle was the der bundle whose sections are the line bundle derivations. The next proposition shows that the derivations of the potentil of a line bundle naturally include the line bundle derivations.

Proposition 7.5.2 (Dimensionless Potential Derivations)**.** *Let $\lambda : L \to M$ be a line bundle and $\Gamma(L)^\odot$ its potential, then there is an isomorphism of Lie algebras*

$$\mathrm{Der}(L) \cong \mathrm{Der}(\Gamma(L)^\odot)_0.$$

Proof. We can give the isomorphism by explicitly specifying two maps. The first sends a dimensionless derivation to its restriction on the homogeneous subsets of dimension 0 and 1

$$\mathrm{Der}(\Gamma(L)^\odot)_0 \ni P \mapsto (P|_{L^0}, P|_{L^1}) =: (X, D)$$

Clearly, from the fact that P is a $\odot$-derivation these satisfy

$$X(fg) := X(f \odot g) = X(f) \odot g + f \odot X(g) = X(f)g + fX(g)$$
$$D(f \cdot s) := D(f \odot s) = X(f) \odot s + f \odot D(s) = X(f) \cdot s + f \cdot D(s),$$

thus showing that D is a line bundle derivation with symbol X. Conversely, given a line bundle derivation $D \in \mathrm{Der}(L)$ with symbol X, we need to define a dimensionless derivation $\mathrm{Der}(\Gamma(L)^\odot)_0$. This is accomplished by extending D as a $\odot$-derivation for non-negative tensor powers of L the following basic identities

$$\begin{aligned} P(f\odot g) &:= X[fg] = X[f]g + fX[g] \\ P(f\odot s) &:= D(f\cdot s) = X[f]\cdot s + f\cdot D(s) \\ P(s\odot r) &:= P(s)\odot r + s\odot P(s). \end{aligned}$$

To account for negative tensor powers we use proposition 2.6.5, which gives an isomorphism $\mathrm{Der}(L)\cong\mathrm{Der}(L^*)$ thus defining a derivation $\Delta\in\mathrm{Der}(L^*)$ from D. This derivation Δ is extended as a $\odot$-derivation for non-positive tensor powers in a similar wat to D. To complete the extension of D to P as a $\odot$-derivation, it only remains to consider products mixing positive and negative tensor powers. This case is accounted for by the following consistency formula that follows from the definition of the dimensioned product in the potentil and the explicit isomorphism $\mathrm{Der}(L)\cong\mathrm{Der}(L^*)$ in the proof of proposition 2.6.5:

$$P(\sigma\odot s) = P(\sigma(s)) = X(\sigma(s)) = D^*(\sigma)(s) + \sigma(D(s)) = D^*(\sigma)\odot s + \sigma\odot D(s) = P(\sigma)\odot s + \sigma\odot P(s).$$

These two maps, which are clearly Lie algebra morphisms since the bracket is simply the commutator, are readily checked to be inverses of each other. □

We remark that dimensionless derivations do not determine all the derivations of a dimensioned ring. General derivations of the potentil of a line bundle are given by collections of differential operators between all the different tensor powers fitting consistently with the $\mathbb{Z}$-dimensioned structure.

Consider now a submanifold $i: S\hookrightarrow M$ of a line bundle $\lambda: L\to M$. We saw in Section 2.6.1 that a line bundle is induced on S by pull-back with an inclusion factor covering the embedding $\iota: L_S\to L$. There, the set of vanishing sections on S, defined formally as the kernel of ι, was shown to be a submodule of the sections of the ambient line bundle $\Gamma_S\subset\Gamma(L)$ that can be seen as (locally) generated by the ideal of vanishing functions $I_S\subset \mathrm{C}^\infty(M)$. The following proposition shows that these two algebraic manifestation of a submanifold in a line bundle fit nicely into the potentil picture.

Proposition 7.5.3 (Vanishing Dimensioned Ideal of a Submanifold)**.** *Let $\lambda: L\to M$ be a line bundle and $i: S\hookrightarrow M$ a submanifold carrying the restricted line bundle L_S. Let us denote the line bundle potentials by $\Gamma(L)^\odot$ and $\Gamma(L_S)^\odot$, then the submanifold defines a dimensioned ideal $I_S\subset\Gamma(L)^\odot$ that allows to characterize (depending on the embedding i, perhaps only locally) the restricted potential as a quotient of dimensioned rings*

$$\Gamma(L_S)^\odot\cong\Gamma(L)^\odot/I_S.$$

Proof. The vanishing dimensioned ideal is simply defined as the set of sections of all the tensor powers that vanish when restricted to S, that is

$$I_S := \{a\in\Gamma(L^n)|\quad a(x) = 0\in L^n_x\quad\forall x\in S\}.$$

Note that this definition is indeed equivalent to the kernel of the potential of the inclusion factor $I_S=\ker(\iota^\odot)$, the dimensioned ideal condition $\Gamma(L)^\odot\odot I_S\subset I_S$ then follows:

$$\iota^\odot(r_n\odot a_m) = (\iota^*)^n r_n\odot(\iota^*)^m a_m = (\iota^*)^n r_n\odot 0 = 0$$

for all $r_n\in\Gamma(L^n)$, $a_m\in I_S$, where the functoriality of the potential construction proved in proposition 7.5.1 has been used. It is clear that, by construction, the ordinary ideal of vanishing functions is the dimensionless component of I_S and that

$$\Gamma_S = I_S\cap\Gamma(L^1).$$

Using a local argument we can see that, similarly to the submodule of vanishing sections, the subsets of homogeneous dimension of I_S can all be generated by elements in the dimensionless component. Just as in the case of the vanishing submodule of sections, it is clear that the quotient $\Gamma(L)^\odot/I_S$ represents the identification of sections of the tensor powers of L_S with extensions in L that differ by a vanishing section of the corresponding tensor power, which then gives the desired the result. □

7.6 Jacobi Manifolds Revisited

In Section 2.7 Jacobi manifolds were presented as the natural generalization of Poisson structures on unit-free manifolds. In light of the results of Section 7.5 above connecting unit-free manifolds to dimensioned algebra, it is natural to ask whether Jacobi structures on line bundles are somehow reflected on the dimensioned rings associated to them under the potential functor. In this section we will see that this is indeed the case as we are now in the position to prove the main theorems of this chapter connecting Jacobi manifolds to dimensioned Poisson algebras in a natural way.

We begin by identifying a dimensioned Poisson algebra structure on the potential of a Jacobi manifold.

Theorem 7.6.1 (Dimensioned Poisson Algebra associated with a Jacobi Manifold)**.** *Let $\lambda : L \to M$ be a line bundle and $(\Gamma(L), \{,\})$ a Jacobi structure, then there exists a unique dimensioned Poisson algebra of dimension -1 on the potential dimensioned ring $(\Gamma(L)^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}}, \odot_0, \{,\}_{-1})$ such that the brackets combining elements of dimensions $+1$, 0, and -1 are determined by the Jacobi bracket $\{,\}$, its symbol X and its squiggle Λ.*

Proof. We give an explicit construction of the dimensioned Poisson algebra on the potential $(\Gamma(L)^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}}, \odot_0)$, that we regard here as a dimensioned commutative algebra over the real numbers with dimension set $\mathbb{Z}$ and dimensionless commutative multiplication $\odot_0$. Since the Jacobi bracket maps pairs of sections into sections $\{,\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$, we aim to extend it to all the tensor powers of the potential as a dimensioned algebra bracket of dimension $-1 \in \mathbb{Z}$:

$$\{,\}_{-1} : \Gamma(L^n) \times \Gamma(L^m) \to \Gamma(L^{n+m-1}).$$

It is clear that we obtain a partial Lie bracket for all positive tensor powers simply by extending the Jacobi bracket as $\odot$-derivations in each argument, i.e. setting $\{a, b\}_{-1} := \{a, b\}$ and generating all the brackets between higher powers from the basic identity:

$$\{a, b \odot c\}_{-1} := \{a, b\} \odot c + b \odot \{a, c\}$$

for all $a, b, c \in \Gamma(L^1) = \Gamma(L)$. Note that this is analogous to using the isomorphism $\mathrm{Der}(L) \cong \mathrm{Der}(\Gamma(L)^{\odot})_0$ of proposition 7.5.2 to regard the Hamiltonian derivation of the Jacobi bracket D_a as a dimensionless derivation of the potential. The symbol-squiggle identity of the Jacobi bracket written in terms of the potential dimensioned multiplication reads

$$\{f \odot a, g \odot b\}_{-1} = f \odot g \odot \{a, b\} + f \odot X_a[g] \odot b - g \odot X_b[f] \odot a + \Lambda(df \otimes a, dg \otimes b)$$

for $f, g \in \Gamma(L^0) \cong \mathrm{C}^\infty(M)$, $a, b \in \Gamma(L^1) = \Gamma(L)$, then we can extract the definition of the dimensioned Poisson bracket for non-negative tensor powers by reading off the above formula interpreted as a Leibniz rule of the $\odot$ multiplication:

$$\{a, f\}_{-1} := X_a[f] = -\{f, a\}_{-1}$$
$$\{f, g\}_{-1}(a) := \Lambda^\sharp(df \otimes a)[g] = -\{g, f\}_{-1}(a)$$

Note that the second bracket has been defined on a generic argument since

$$\{,\}_{-1} : \Gamma(L^0) \times \Gamma(L^0) \to \Gamma(L^{-1}) = \Gamma(L^*).$$

To define the bracket on negative tensor powers we first use the isomorphism $R : \mathrm{Der}(L) \to \mathrm{Der}(L^*)$ proved in proposition 2.6.5 to define the Hamiltonian derivation on dual sections $\Delta_a := R(D_a) \in \mathrm{Der}(L^*)$ and set

$$\{a, \alpha\}_{-1} := \Delta_a(\alpha) = -\{\alpha, a\}_{-1}.$$

Note that this definition is consistent with the previous definitions of brackets of non-negative tensor powers as we readily check that it acts as a $\odot$-derivation in both arguments:

$$\{a, \alpha \odot b\}_{-1} = \{a, \alpha(b)\}_{-1} = X_a[\alpha(b)] = \Delta_a(\alpha)(b) + \alpha(D_a(b)) = \{a, \alpha\}_{-1} \odot b + \alpha \odot \{a, b\}_{-1}.$$

With the brackets defined so far for non-negative tensor powers and the mixed bracket above, we can expand the expression $\{f \odot a, \alpha \odot b\}$ by $\odot$-derivations (full details of the computation shown in appendix B) to find the only non-yet defined bracket:

$$\{f, \alpha, \}_{-1}(a, b) := \Lambda^\sharp(df \otimes a)[\alpha(b)] + X_b[f]\alpha(a).$$

Similarly, expanding the bracket $\{\alpha \odot a, \beta \odot b\}$ (again, full details in appendix B) we find:

$$\{\alpha, \beta\}_{-1}(a, b, c) := \Lambda^\sharp(d\alpha(a) \otimes b)[\beta(c)] + X_c[\alpha(a)]\beta(b) - \alpha(b)X_a[\beta(c)] + \alpha(b)\beta(\{a, c\}).$$

With these partial brackets we can now define the brackets of combinations of positive and negative tensor powers via extension as $\odot$-derivations. Clearly, following from the observation made at the end of the proof of proposition 7.5.1, the Jacobi identity of bracket for the negative tensor powers so defined will be directly dependent on the Jacobi identity for the bracket at dimensions +1 and 0. In appendix B the Jacobi identities for brackets of all the combinations of the tensor powers +1 and 0 are shown to follow directly from the basic identities satisfied by the symbol X and squiggle Λ of the original Jacobi structure. □

The next theorem shows that, much like how the functor $\mathrm{C}^\infty : \mathsf{Man} \to \mathsf{Ring}$ characterizes Poisson manifolds as a subcategory of Poisson algebras, the potential functor allows to regard the category of Jacobi manifolds as a subcategory of dimensioned Poisson algebras.

Theorem 7.6.2 (The Potential Functor for Jacobi Manifolds)**.** *The assignment of the potential of a line bundle restricted to the category of Jacobi manifolds with Jacobi maps gives a contravariant functor*

$$\odot : \mathsf{Jac}_{\mathsf{Man}} \to \mathsf{DimPoissAlg}.$$

Proof. In proposition 7.6.2 it was shown that a line bundle factor $B : L \to L'$ covering a smooth map $b : M \to M'$. is mapped to a dimensioned ring morphism $B^\odot : \Gamma(L')^\odot \to \Gamma(L)^\odot$ under the potential contravariant functor, then it will suffice to show that when B is a Jacobi map, i.e.

$$B^*\{a, b\}' = \{B^*a', B^*b'\}$$

for all $a, b \in \Gamma(L')$, then the potential map is a dimensioned Lie algebra morphism, i.e.

$$B^\odot\{s, r\}'_{-1} = \{B^\odot s, B^\odot r\}_{-1}$$

for all $s, r \in \Gamma(L')^\odot$. Note that $B^\odot$ was defined in proposition 7.5.1 as the tensor powers of the pull-backs of sections of L' , its dual and the smooth functions, then it is clear from the fact that the dimensioned Lie brackets $\{,\}_{-1}$, $\{,\}'_{-1}$ are defined by extension as $\odot$-derivations that the dimensioned Lie algebra morphism condition for brackets of positive and negative tensor powers is dependent on the same condition for all the bracket combinations of elements in dimensions +1, 0 and −1. These conditions are checked directly using the definitions. For the bracket of a pair of elements of dimension +1, the condition of dimensioned Lie algebra morphism for $B^\odot$ is precisely the condition that B is a Jacobi map. By considering a bracket of the form $\{a, f \cdot b\}'$ we can see that the fact that B is a Jacobi map and the basic properties of the pull-backs of line bundle factors imply the morphism condition for the bracket of elements of dimension +1 and 0:

$$B^\odot\{a, f\}'_{-1} = b^*X'_a[f] = X_{B^*a}[b^*f] = \{B^*a, b^*f\}_{-1} = \{B^\odot a, B^\odot f\}_{-1}.$$

From similar considerations for a bracket of the form $\{f\cdot a, g\cdot b\}$, it follows that

$$B^{\odot}\{f,g\}'_{-1}(c,d) = B^*\Lambda'(df\otimes c, dg\otimes d) = \Lambda(db^*f\otimes c, db^*g\otimes d) = \{b^*f,b^*g\}_{-1}(c,d) = \{B^{\odot}f,B^{\odot}g\}_{-1}(c,d)$$

for all $c,d\in\Gamma(L)$. To account for brackets containing elements of dimension -1 we first consider the defining formula of the isomorphism $\mathrm{Der}(L)\cong\mathrm{Der}(L^*)$ under pull-back

$$\begin{aligned} B^*(\Delta'_a(\alpha)(c)) &= b^*X'_a[\alpha(c)] - b^*\alpha(\{a,c\}') \\ &= X_{B^*a}[b^*\alpha(c)] - B^*\alpha(B^*\{a,c\}) \\ &= X_{B^*a}[B^*\alpha(B^*c)] - B^*\alpha(\{B^*a,B^*c\}) \\ &= \Delta_{B^*a}(B^*\alpha)(B^*c). \end{aligned}$$

Which clearly implies, in particular, the dimensioned Lie morphism condition for the bracket of mixed tensor powers

$$B^{\odot}\{a,\alpha\}'_{-1} = B^*\Delta'_a(\alpha) = \Delta_{B^*a}(B^*\alpha) = \{B^*a,B^*\alpha\}_{-1} = \{B^{\odot}a,B^{\odot}\alpha\}_{-1}.$$

In the proof of theorem 7.6.1 it was shown that the brackets $\{f,\alpha\}_{-1}$ and $\{\alpha,\beta\}_{-1}$ were determined by extending the previously defined brackets between elements of dimensions $+1$, 0 and -1 as $\odot$-derivations, thus the dimensioned Lie algebra morphism condition for these follows from the fact that $B^{\odot}$ is defined as the tensor powers of B^*. □

In our discussion of the problem of reduction for Jacobi manifolds in Section 2.7, vanishing submodules of coisotropic submanifolds were recognized to be analogous to the coisotropes of algebraic Poisson reduction, however, the lack of a commutative multiplication structure on the module of sections of a line bundle made it impossible to make the analogy more precise. Our identification of the dimensioned Poisson algebra on the potential of a Jacobi manifold does this precisely as the next proposition shows that coisotropic submanifolds induce dimensioned coisotropes.

Proposition 7.6.1 (Coisotropic Submanifolds induce Dimensioned Coisotropes)**.** *Let $\lambda: L\to M$ be a line bundle and $(\Gamma(L),\{,\})$ a Jacobi structure, then the vanishing dimensioned ideal of a coistropic submanifold $i: S\hookrightarrow M$ is a dimensioned coisotrope of the dimensioned Poisson algebra on the potential*

$$I_S\subset(\Gamma(L)^{\odot}_{\mathbb{Z}},+_{\mathbb{Z}},\odot_0,\{,\}_{-1}).$$

Proof. Proposition 7.5.3 shows that $I_S\subset\Gamma(L)^{\odot}$ is a dimensioned $\odot$-ideal for any submanifold S, then it will suffice to show that I_S is a dimensioned Lie subalgebra. The vanishing ideal is generated by the $\odot$-products of elements of dimension $+1$, 0 and -1, then, by the Leibniz identity of the dimensioned Poisson bracket, it suffices to check the dimensioned Lie subalgebra conditions for elements of those dimensions. For this we will use characterizations 2, 3 and 4 of coisotropic submanifolds of a Jacobi manifold given in proposition 2.7.3. Clearly the condition on the brackets of positive powers $\{a,b\}_{-1}$ for $a,b\in\Gamma(L)$ is the fact that the vanishing submodule of sections of S forms a Lie subalgebra of the Jacobi structure, characterization 3. For the bracket $\{a,f\}_{-1}=X_a[f]$ it is the fact that Hamiltonian vector fields of vanishing sections are tangent to the submanifold, i.e. $X_{\Gamma_S}[I_S]\subset I_S$, characterization 4. For the bracket $\{f,g\}_{-1}$ it is the condition $\Lambda^\sharp(dI_S\otimes L)\subset \mathrm{T}S$, characterization 2. From the observation that

$$a\in I_S\cap\Gamma(L^1)\Rightarrow \alpha\odot a=\alpha(a)\in I_S$$

for any $\alpha\in\Gamma(L^*)$, we check the dimensioned Lie subalgebra condition for the brackets $\{f,\alpha\}_{-1}$ and $\{\alpha,\beta\}_{-1}$ by writing the explicit defining formulas presented in the proof of theorem 7.6.1 and using characterizations 2 and 4 from proposition 2.7.3 combined. □

Lastly, when a coisotropic submanifold furthermore fits in a reduction scheme of Jacobi manifolds, the associated dimensioned Poisson algebras fit in a dimensioned algebra reduction scheme, again in direct analogy with the ordinary Poisson case.

Theorem 7.6.3 (Coisotropic Reduction induces Dimensioned Poisson Reduction)**.** *Let $\lambda : L \to M$ be a line bundle with a Jacobi structure $(\Gamma(L), \{,\})$ and let $i : S \hookrightarrow M$ be a coisotropic submanifold satisfying the assumptions of proposition 2.7.6 so that there is a a reduced Jacobi structure $(\Gamma(L'), \{,\}')$ fitting in the reduction diagram:*

$$\begin{array}{ccccc} L_S & \xrightarrow{\iota} & L & & \\ \downarrow{\scriptstyle \pi} & \searrow & & \searrow & \\ & & S & \xrightarrow[i]{} & M \\ L' & & \downarrow{\scriptstyle p} & & \\ & \searrow & M' & & \end{array}$$

then there is an isomorphism of dimensioned Poisson algebras between the potential of the reduced Jacobi structure and the algebraic dimensioned Poisson reduction by the vanishing dimensioned coisotrope:

$$\Gamma(L')^{\odot} \cong N(I_S)/I_S.$$

Proof. Recall from Section 2.7 that the Jacobi reducibility condition was given explicitly in terms of the brackets as

$$\pi^*\{a_1, a_2\}' = \iota^*\{A_1, A_2\}$$

for all $a_i \in \Gamma(L')$ and $A_i \in \Gamma(L)$ extensions satisfying $\pi^* a_i = \iota^* A_i$. The definition of the potential of a line bundle factor of proposition 7.5.1 and the explicit definition of the dimensioned bracket $\{,\}_{-1}$ clearly show that the reducibility condition translates into the potential setting verbatim as one finds that the dimensioned Poisson brackets on the potentials of L and L' are related by the following condition

$$\pi^{\odot}\{a_1, a_2\}'_{-1} = \iota^{\odot}\{A_1, A_2\}_{-1}$$

for all $a_i \in \Gamma(L')^{\odot}$ and $A_i \in \Gamma(L)^{\odot}$ extensions satisfying $\pi^{\odot} a_i = \iota^{\odot} A_i$. We aim to relate the dimensioned Lie idealizer of the vanishing dimensioned ideal $N(I_S)$ to the submersion factor $\pi : L_S \to L'$ in a natural way. This will follow by the compatibility condition assumed in proposition 2.7.6 for the coistropic submanifold:

$$\delta(\ker(\mathrm{D}\pi)) = \Lambda^{\sharp}((\mathrm{T}S)^{0L})$$

which, exploiting the jet sequence of the Jacobi structure, can be rewritten as

$$\ker(\mathrm{T}p) = (\tilde{\Lambda}^{\sharp} \circ i)(\mathrm{T}^0 S \otimes L_S).$$

This equation gives the point-wise condition that the p-fibration on S is a foliation integrating the tangent distribution of Hamiltonian vector fields of the vanishing sections. It follows from the observation made at the end of the proof of proposition 7.6.1 that the Lie idealizer $N(I_S)$ is generated by the brackets of elements of dimension $+1$ and 0, then it suffices to identify the elements satisfying the dimensioned Lie idealizer defining condition of these dimensions. These will be $f \in \Gamma(L^0) \cong \mathrm{C}^\infty(M)$ and $s \in \Gamma(L^1) = \Gamma(L)$ such that

$$\{f, g\}_{-1} \in I_S, \qquad \{s, g\}_{-1} \in I_S, \qquad \{s, a\}_{-1} \in I_S$$

for all $g \in I_S \cap \Gamma(L^0)$ and $a \in I_S \cap \Gamma(L^1)$. From the explicit formulas given for the dimensioned bracket $\{,\}_{-1}$ given in the proof of theorem 7.6.1 we clearly see that the compatibility condition of the coisotropic submanifold with the submersion factor gives a point-wise identification of elements in the idealizer and the infinitesimal description of the p-fibration with the restricted line bundles on S. Since, by assumption, the submersion factor fits in a reduction scheme of (smooth) Jacobi manifolds this infinitesimal identification carries over globally to allow the identification of $N(I_S)$ with the line bundle π-fibration. Quotienting by I_S, as seen in proposition 7.5.3, amounts to restricting to the submanifold S, thus we find the equivalent description of the reduced bracket $\{,\}'_{-1}$ as the canonical dimensioned Poisson bracket on the quotient $N(I_S)/I_S$. □