is taken by "unrolling" M. M and M' are observationally indistinguishable since no observational past of any future-inextendible curve in either extends beyond the excision barriers. But only M' admits a global time function.

Notes

1. See also Clark Glymour, "Topology, Cosmology and Convention," *Synthèse* 24 (1972): 195–218.

2. A comprehensive treatment of work on the global structure of (relativistic) space-times is given in: S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press, 1973). See also Roger Penrose, *Techniques of Differential Topology in Relativity* (Philadelphia: Society for Industrial and Applied Mathematics, 1972). More accessible than either is Robert Geroch, "Space-Time Structure from a Global Viewpoint," in B. K. Sachs, ed., *General Relativity and Cosmology* (New York: Academic Press, 1971).

3. A future end point need not be a point on the curve. The definition is this: If M is a space-time, I a connected subset of R, and $\sigma: I \rightarrow M$ a future-directed causal curve, a point x is the *future end point* of σ if for every neighborhood O of x there is a $t_0 \in I$ such that $\sigma(t) \in$ O for all $t \in I$ where $t > t_0$, i.e., σ enters and remains in every neighborhood of x.

4. A space-time is *strongly causal* if, given any point x and any neighborhood O of x, there is always a subneighborhood $O' \subset O$ of x such that no future-directed timelike curve which leaves O$'$ ever returns to it.

5. A countable cover of this form can be found in *any* space-time M, strongly causal or not. Since M is without boundary, for every y in M there is an x in M such that $y << x$, i.e., $y \in I^-(x)$. So the set $\{I^-(x): x \in M\}$ is an open cover of M. But M has a countable basis for its topology (Robert Geroch, "Spinor Structure of Space-Times in General Relativity I," *Journal of Mathematical Physics* 9 (1968): 1739–1744.) So by the Lindelöf Theorem there is a countable subset of $\{I^-(x): x \in M\}$ which covers M.

6. Robert Geroch, "Limits of Spacetimes," *Communications in Mathematical Physics* 13 (1969): 180–193.

7. See John Earman, "Laplacian Determinism in Classical Physics" (to appear) and Robert Geroch's paper in this volume.

8. Robert Geroch, "Domain of Dependence," *Journal of Mathematical Physics* 11 (1970): 437–449. (A somewhat different but equivalent definition of global hyperbolicity is used.)

9. There is a problem of how to define observational indistinguishability in a nontemporally orientable space-time (the definition given presupposed temporal orientation). But under any plausible candidate, M and M' in the example would come out observationally indistinguishable. One could associate with every inextendible timelike curve σ all the points that are connected with some point on the curve by another timelike curve. (In a temporally oriented space-time this would be the union $I^+[\sigma] \cup I^-[\sigma]$.) *Even these* sets in M and M' would find isometric counterparts in the other.

10. A space-time is *stably causal* if there are no closed causal curves *and* if there are no closed causal curves with respect to any metric close to the original. (This can be made precise by putting an appropriate topology on the set of all metrics on the space-time manifold.) Note that in the space-time M of the following example the slightest flattening of the light cones would allow timelike curves to scoot around the barriers. The equivalence is proven in S. W. Hawking, "The Existence of Cosmic Time Functions," *Proceedings of the Royal Society A*, 308 (1968): 433–435.

ROBERT GEROCH

Prediction in General Relativity

1. Introduction

There are at least two contexts within which one might place a discussion of the possibilities for making predictions in physics. In the first, one is concerned only with the actual physical world: one imagines that he has somehow learned what some physical system is like now, and one wishes to determine what that system will be like in the future. In the second, one is concerned only with the internal structure of some particular physical theory: one wishes to state and prove, within the mathematical formalism of the theory, theorems that can be interpreted physically in terms of possibilities for making predictions.

Of the two, the second context certainly seems to be the simpler and the more direct. Indeed, it is perhaps not even clear what the first context means. One's only guide in making a prediction in the physical world is one's past experiences in the relationship between the present and the future. But it is precisely the collection of these experiences, systematized and formalized, which makes up what is called a physical theory. That is to say, one seems to be led naturally from the first context to the second. One would perhaps even be tempted to conclude that the two contexts are essentially the same thing, were it not for the fact that it seems never to be the case in practice that one's past experiences lead in any sense uniquely to a physical theory; one must, at some point, make a choice from among several competing theories in order to discuss prediction. Thus one might divide a discussion of prediction in physics into two parts: (1) the choice of a physical theory and (2) the establishment and interpretation of certain theorems within the mathematical formalism of that theory.

Consider, as an example, Newtonian mechanics. Suppose that we wish to describe within this theory our solar system, which we idealize as

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follows: the sun and planets are represented by ten mass points, subject to Newtonian gravitational forces. Because of the structure of the differential equations of the theory, one can determine, given the positions and velocities of these points at any one instant of time, their positions and velocities at all later times. Such predictions are of course made routinely in the case of the solar system and are later confirmed, with remarkable agreement, observationally. Let us now attempt to express this activity in terms of some theorem in Newtonian mechanics. We take, as the statement of our theory, the following: "The world is described by points in Euclidean space, each of which is assigned a mass, and which move with time according to a given law of force between them." One might conjecture, within this theory, a mathematical result of the following general form: "Given the positions and velocities of some collection of mass points at some particular time in some region of Euclidean space, there is one and only one solution of the equations of motion, in that region, for later times." But this particular conjecture, at least, is false, for one has the option of having additional mass points, initially outside the given region (and hence not included in the initial data), which subsequently move into the region and influence the motion of our original mass points. In fact, our conjecture is not even true if we further demand that the fixed region be all of Euclidean space. One can, within Newtonian mechanics, construct a solution representing two rocket ships which bounce between them a mass point with ever increasing speed. The result is that the rocket ships accelerate in opposite directions; if the speeds are adjusted correctly, the ships can be made to escape to the "edge" of our Euclidean space in finite time, leaving nothing behind. The time-reverse of this situation, then, is also a solution of the equations, a solution which allows objects to "rush in from infinity," influencing the later development of our system without ever having been included in the initial data.

In fact, there seems to be no theorem in ordinary Newtonian mechanics that suggests possibilities for prediction. Our conjecture above would, presumably, be true if we required in the conjecture that the fixed region be all of Euclidean space and, furthermore, that no information come into the system from infinity. But this result would not, at least to me, suggest prediction, for it constrains both the initial state of the system and its future behavior. One might, instead, consider an alternative theory, e.g., that above, but with the additional proviso that the only admissible solutions are those in which the total number of mass points remains the same

with time. (In fact, there are apparently some technical difficulties with such a theory. For example, one must restrict the class of allowed force laws to guarantee existence of admissible solutions, and the passage to a more realistic version in which mass points are replaced by a continuous mass distribution may be tricky.) We emphasize that the new theory is identical with the old as far as observational evidence in our World is concerned, for exotic systems such as that described above have not been observed. Nonetheless one can easily imagine that the two theories will differ markedly in terms of what theorems, suggestive of the possibilities of prediction, they will admit.

The purpose of this paper is to introduce and discuss a few issues relating to the question of prediction in the general theory of relativity.[1] The remarks above are intended to justify the rather narrow framework in which we shall operate. Our theory is standard general relativity. We have a smooth, connected, four-dimensional manifold M, whose points represent "events" (occurrences in the physical world having extension in neither space nor time). There is on this manifold a smooth metric g_{ab} of Lorentz signature, which describes certain results of measuring spatial distances and elapsed times between pairs of nearby events. To simplify the discussion, we shall suppose also that our space-time M, g is strongly causal, i.e., that every point has a small neighborhood through which no timelike curve passes more than once. Observers are described by timelike curves in space-time, light rays by null geodesics, etc. Other physical phenomena are described by tensor fields on space-time, subject to differential equations (e.g., electromagnetic phenomena by the Maxwell field, subject to Maxwell's equations). Our goal is to formulate definitions and theorems within this mathematical framework.

2. Domain of Dependence

It is clear that the difficulties associated with Newtonian mechanics arise from the feature of that theory that it does not restrict the speeds of particles. There is, however, such a restriction in relativity, in which the limiting speed is that of light. One might guess, therefore, that it will actually be easier to discuss prediction in relativity than in Newtonian mechanics. This turns out to be the case, a fact which finds expression in the notion of the domain of dependence.

Let M, g be a space-time. Let S be a three-dimensional, achronal (i.e., no two points of S may be joined by a timelike curve) surface in M. The (future) domain of dependence[2] of S, $D^+(S)$, is the collection of all points p of M such that every future-directed timelike curve in M, having future endpoint p and no past endpoint, meets S. For example, if S is a three-dimensional, spacelike disk in Minkowski space (Figure 1), then $D^+(S)$ is the "cone-shaped region" shown. The point q is not in $D^+(S)$, for the future-directed timelike curve γ in the figure fails to meet S.

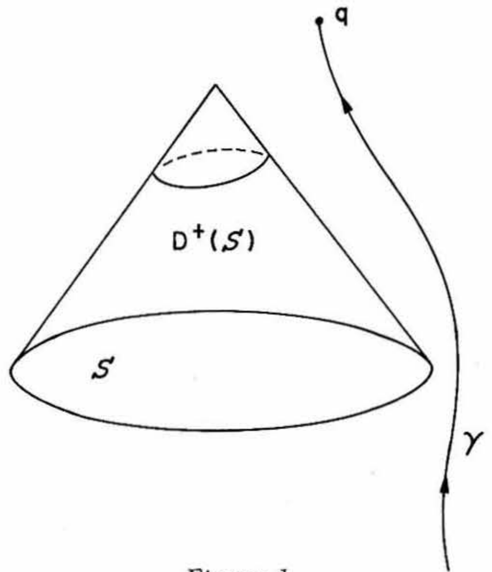

Figure 1

The physical meaning of this definition is the following. The surface S represents "a region of space at some instant of time." Signals in general relativity travel along timelike curves. For q in $D^+(S)$, every such curve to q must have met S, i.e., in physical terms, every signal which could possibly influence the state of affairs at q must have been registered, in some sense, on S. For q not in $D^+(S)$, signals could reach and hence influence the physics at q without having been registered on S. In short, one expects that a sufficiently detailed knowledge of what is happening on S (i.e., at the "initial time") should determine completely what is happening at each point of $D^+(S)$. This physical picture is in fact supported by a collection of theorems in general relativity. The detailed statement depends on the type of matter or fields considered; as an example, we take electromagnetic fields. Electromagnetism is represented by an antisymmetric, second-rank tensor field on space-time, subject to Maxwell's equations (say, without sources). The theorem, in this case, reads as follows: Given the electromagnetic field on S, there is at most one extension of that field to $D^+(S)$, subject to Maxwell's equations. That is to say, the physical situation (in this example, the electromagnetic field) is uniquely determined at any point q of $D^+(S)$, given the situation on S.

We emphasize that the domain of dependence is essentially a relativistic concept. For example, for S of "spatial size" one light-year, $D^+(S)$ will "extend into the future" for about one year in time, i.e., only until signals from outside S have time to move into our region. That there is no analogous notion in Newtonian mechanics is the source of the examples in the previous section.

It is tempting to conclude that this definition essentially exhausts what can be said within our theory: what can be determined from initial data (on S) is precisely what is in $D^+(S)$, and so all that remains is to work out the properties of this $D^+(S)$, its dependence on S, etc. That the situation is not so simple can be seen in the following example. Let M, g be Minkowski space-time, and let S be a spacelike, three-dimensional plane in M. Then $D^+(S)$ is the entire region to the future of S, as shown in Figure 2. We next consider a second space-time, M', g', which is Minkowski space-time with a small, closed, spherical "hole" removed, and a similar surface S' in this space-time. Then $D^+(S')$ is as shown in the figure. The point is that these two space-times, both legitimate within our theory, look identical in the immediate vicinities of their respective surfaces, although they are of course quite different in the large. For example, the only solution of Maxwell's equations in M, g that vanishes on S is the solution that also vanishes to the future of S, while there are solutions of Maxwell's equations in M', g' that vanish on S' and yet do not vanish to the future of S'. (Such a solution must, as already noted, vanish in $D^+(S')$, but it need not vanish in the region indicated in the figure because, physically, "electromagnetic radiation can emerge from the hole.") Suppose, then, that one has decided that our universe, at some time, looks like a neighborhood of S in M, g and that there are no electromagnetic fields present. Could one conclude that no electromagnetic fields will later be seen? Clearly, from this example, one could not. Similar, but more elaborate, examples can be constructed for other situations. In what sense, then, can one make any physical predictions within the general theory of relativity?

It is clear that the mechanism of the example above is the fact that,

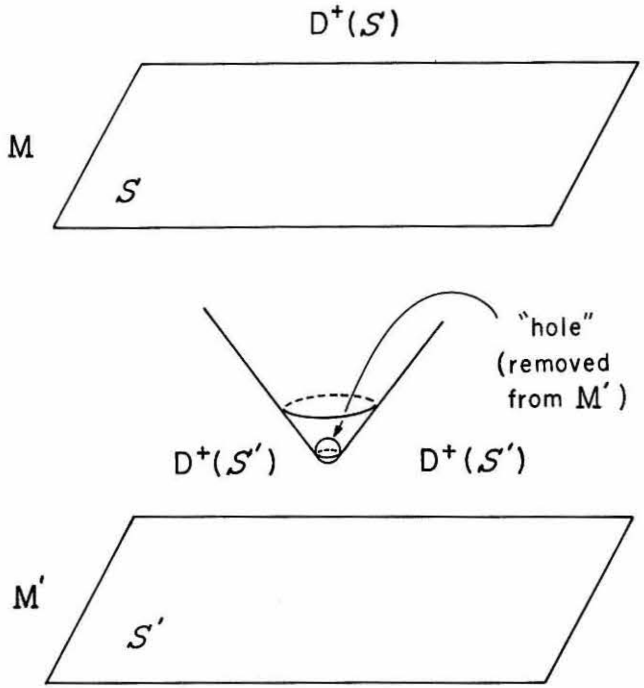

Figure 2

although S determines what happens in $D^+(S)$, what this $D^+(S)$ will be, and in particular how "large" it is, requires knowledge not only of S, but also of the space-time M, g in which S is embedded. Even the imposition of Einstein's equation (which we ignored in the example above) permits only the determination of the geometry in $D^+(S)$, and hence does not prohibit the construction of similar examples by "cutting holes in space-time." Apparently, the situation is that, although the notion of the domain of dependence expresses well what there is of the relationship "present determines future" in general relativity, it is nonetheless difficult to find therein a totally satisfactory formulation, from the physical viewpoint, of this relationship.

Thus general relativity, which seemed at first as though it would admit a natural and powerful statement at prediction, apparently does not. It seems to me that the only cure is to attempt to do for general relativity what we discussed earlier for Newtonian mechanics—change the theory.

We here describe, as an example of the possibilities available along these lines, one such.

Call a space-time M, g *hole-free* if it has the following property: given any achronal, three-dimensional surface S in M, and any metric-preserving embedding Ψ of $D^+(S)$ into some other space-time M', g', then $\Psi(D^+(S)) = D^+(\Psi(S))$. That is to say, we require that the domain of dependence, in M', of the surface $\Psi(S)$ in M' be the same as the image by Ψ of the domain of dependence of S in M. Minkowski space-time, for example, is hole-free (as, indeed, are the standard exact solutions in general relativity). On the other hand, Minkowski space-time, with a hole as in Figure 2, is not hole-free. (Let Ψ be a metric-preserving mapping from $D^+(S')$ in that example to Minkowski space-time.) This definition, then, provides an intrinsic characterization of space-times that have been constructed by cutting holes (although an imperfect one: Minkowski space-time to the past of a null plane is hole-free by this definition). Note that one could not accomplish the same objective by simply insisting that space-times not be constructed by cutting holes in given space-times, for this characterization involves not only the space-time itself but also its mode of presentation. Similarly, "maximally extended" is no substitution for "hole-free," for there are space-times that satisfy the former and not the latter.

One might now modify general relativity as follows: the new theory is to be general relativity, but with the additional condition that only hole-free space-times are permitted. As far as observational consequences in our world are concerned, the two theories are identical, since non-hole-free space-times never arise in any practical applications. The new theory, however, admits a simple and natural theorem which suggests prediction: if S and S' are achronal, three-dimensional surfaces in hole-free space-times M, g and M', g', respectively, and if there is a mapping from S to S' which preserves all fields, then there is such a mapping from $D^+(S)$ to $D^+(S')$. This result is in fact practically a restatement of the definition.

It might be of interest to understand better the strength and role of this definition, as well as the scope of other possibilities.

3. Prediction

In the previous section, we were concerned with the relationship between what is happening in one region of space-time (the present) and what is happening in some other region (the future). The word "predict," however, suggests not only the existence of such relationships but also the existence of some agent who gathers the initial data and actually makes a claim about the future. In the present section, we describe within the theory such agents.

Consider first the following example. Let S be a small, three-dimensional, spacelike disk in Minkowski space-time (Figure 3). Then, as we have remarked, initial data on S determine the physical fields in $D^+(S)$, in particular at point p. Let us now introduce an observer, represented by timelike curve γ, who is to actually make this prediction regarding p. In order to make his prediction, our observer must first collect the data from S, a task he carries out as follows. At point r, our observer sends out a swarm of other observers, who fan out, experience, and record

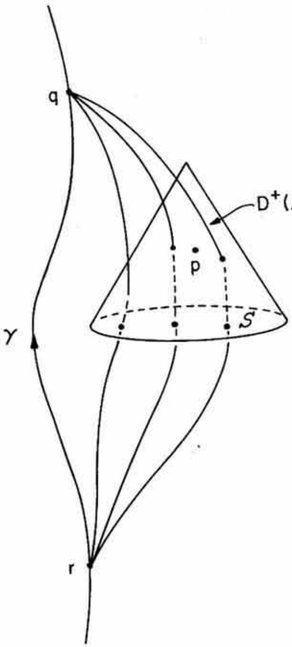

Figure 3

every part of S. They then return to the original observer with this information, meeting him at point q. Thus by point q our observer has assembled all the relevant information and is prepared to make his prediction regarding point p. But note that p lies to the past and not to the future of q. In physical terms, by the time our observer gets around to making his prediction regarding p, p has already happened; he makes a retrodiction rather than a prediction.

It is clear that the problem in the example above arises because the other observers cannot exceed the speed of light in returning to the original observer, whence they arrive too late for a genuine prediction. It is also clear why in Newtonian mechanics, with no limit on the speeds of signals, no distinction need be made between "determination" and "prediction."

Other choices of S and q in Minkowski space-time lead to the same result: retrodiction. Indeed, it is perhaps not immediately clear whether or not one can construct any examples in which genuine prediction is possible in the theory. It turns out that there are such examples. Let M, g be the space-time obtained by removing from Minkowski space-time two small, spacelike, three-dimensional disks, as shown in Figure 4, and identifying[3] the lower edge of disk A with the upper edge of disk B. Thus, for example, a timelike curve entering A from below will re-emerge from the top of B. Let the surface S, the point p, and the timelike curve γ, representing our observer, be those shown. Then, since every future-directed timelike curve to p meets S, p is in $D^+(S)$. Our observer, however, can now gather his initial data by point q, where p is not in the past of q. At the still later point v, our observer can finally learn of point p and so can then check his prediction observationally. In this space-time, then, predictions are possible.

It is interesting to note that it is an essential feature of the example above that the observer verifies his prediction only indirectly—by reaching a point v to the future of p—rather than directly by passing through p. That direct verification is also possible is shown by the following example. Let the space-time M, g be the Einstein universe, so the spatial sections are three-spheres and time is the real line (Figure 5). Our observer has, at q, collected the data from S, while $D^+(S)$ is the entire future of S. Thus all the experiences of the observer beyond q could have been predicted at q.

It is convenient to isolate the essential features of these examples by means of a definition. Let M, g be a space-time. For x any point of M,

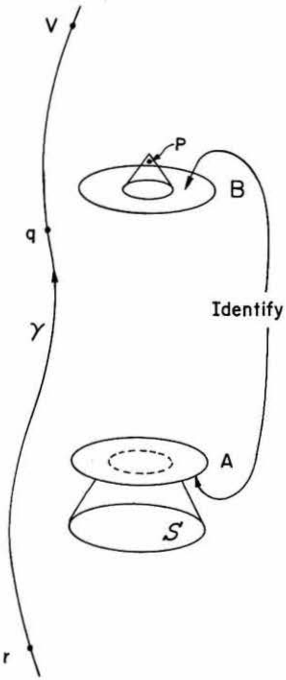

Figure 4

denote by $I^{-}(x)$, the past of x, the set of points that can be reached from x by past-directed timelike curves. Now fix any point q of M, and denote by $P(q)$ the set of all points x such that every past-directed timelike curve from x, without past end point, enters $I^{-}(q)$, but such that $I^{-}(x)$ is not a subset of $I^{-}(q)$. We shall call this set $P(q)$ the domain of prediction of q. For example, for q any point of Minkowski space-time, every point x either has the property that $I^{-}(x) \subset I^{-}(q)$ or has the property that some past-directed timelike curve from x fails to meet $I^{-}(q)$. Hence the domain of prediction of each point q in Minkowski space is empty.

The physical meaning of this definition is as follows. The point q repre-

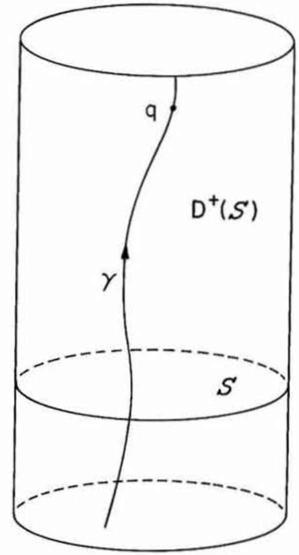

Figure 5

sents the point (of our predicting observer) at which all the information has been collected. Then the set $I^{-}(q)$ represents that region of space-time from which information could reach q. The first condition for membership of x in $P(q)$ requires, physically, that every signal that could affect x must have come from $I^{-}(q)$, i.e., that every such signal could have been recorded and carried to q. The second condition requires essentially that x not be in $I^{-}(q)$, i.e., that we have a prediction at x rather than a retrodiction. This interpretation is supported by the following, easily proved, result: point x is in $P(q)$ if and only if $I^{-}(x) \not\subset I^{-}(q)$, and, in addition, there is a three-dimensional, achronal surface S in $I^{-}(q)$ with x in $D^{+}(S)$. It follows immediately, for example, that, in the examples of Figures 4 and 5, p is a point of $P(q)$. Thus we interpret the domain of prediction of q as "the region of space-time that can be predicted from q."

Assuming that the definition above accurately reflects the physical notion of "making predictions in general relativity," what remains is to study its consequences. We give one example. We saw in the example of Figure 4 that $P(q)$ is nonempty but contains no points to the future of q ("predictions could not be verified directly"). In the example of Figure 5, on the other hand, $P(q)$ includes points to the future of q, and, in that example, we have a "closed universe." In fact these observations are a special case of a more general result, namely: Given a space-time M, g, and a point q of M such that $P(q)$ contains a point to the future of q, then M, g is a closed universe, in the sense that it admits a compact spacelike surface. (This result is essentially a corollary of a theorem[4] of Earman's that a Cauchy surface to the past of a point must be compact.) In physical terms, "predictions which can be verified directly arise only in closed universes." Why should this strange result follow from just the basic principles of general relativity? Are there any other similar theorems about the domain of prediction?

4. Conclusion

We can conveniently summarize by comparing general relativity and Newtonian mechanics. For our purposes, there are apparently two essential differences between the two theories: (1) signal speeds are unlimited in Newtonian mechanics but limited in general relativity; and (2) the space-time framework is fixed once and for all in Newtonian mechanics (Euclidean space plus time) but not in general relativity. The notion "future from present" seems to arise far more simply and naturally in general relativity than in Newtonian mechanics because of the limitation on signal speeds in the former. On the other hand, the freedom in the space-time model in general relativity leads to new difficulties not present in Newtonian mechanics. Finally, the question of the collection of initial data, while irrelevant in Newtonian mechanics, leads, because of limitations on signal speeds, to additional complications in general relativity.

The notion of observational indistinguishability[5] leads to a classification of properties of space-times according to their interaction with this notion. In a similar way, one could classify properties as deterministic and nondeterministic, and as predictive and nonpredictive.

Notes

1. For a careful and thorough discussion of this issue, see J. Earman, "Laplacian Determinism in Classical Physics," preprint.

2. See, for example, R. Geroch, "Domain of Dependence," *Journal of Mathematical Physics* 11 (1970): 437; S. W. Hawking, G. Ellis, *The Large-Scale Structure of Space-time* (Cambridge: Cambridge University Press, 1974).

3. For a discussion of this construction, see, for example, R. Geroch, "Space-Time Structure from a Global Viewpoint," in R. Sachs, ed., *Relativity and Cosmology* (New York: Academic Press, 1971), p. 71.

4. J. Earman, private communication.

5. C. Glymour, this volume; D. Malament, this volume.