

3 The Category of Propositional Theories

One of the primary goals of this book is to provide a formal model of “the universe of all scientific theories.” In the twentieth century, mathematics stepped up another level of abstraction, and it began to talk of structured collections of mathematical objects – e.g., the category of groups, topological spaces, manifolds, Hilbert spaces, or sets. This maneuver can be a little bit challenging for foundationally oriented thinkers, viz. philosophers, because we are now asked to consider collections that are bigger than any set. However, mathematicians know very well how to proceed in this manner without falling into contradictions (e.g., by availing themselves of Grothendieck universes).

We want to follow the lead of the mathematicians, but instead of talking about the category of groups, or manifolds, or Hilbert spaces, etc., we want to talk about the **category of all theories**. In the present chapter, we work out one special case: the category of all propositional theories. Of course, this category is too simple to serve as a good model for the category of all scientific theories. However, already for predicate logic, the category of theories becomes extremely complex, almost to the point of mathematical intractability. In subsequent chapters, we will make some headway with that case; for the remainder of this chapter, we restrict ourselves to the propositional case.

After defining the relevant category **Th** of propositional theories, we will show that **Th** is equivalent to the category **Bool** of Boolean algebras. We then prove a version of the famous Stone duality theorem, which shows that **Bool** is dual to a certain category **Stone** of topological spaces. This duality shows that each propositional theory corresponds to a unique topological space, viz. the space of its models, and each translation between theories corresponds to a continuous mapping between their spaces of models.

3.1 Basics

DEFINITION 3.1.1 We let **Th** denote the category whose objects are propositional theories and whose arrows are translations between theories. We say that two translations $f, g : T \rightrightarrows T'$ are equal, written $f \simeq g$, just in case $T' \vdash f(\phi) \leftrightarrow g(\phi)$ for every $\phi \in \text{Sent}(\Sigma)$. (Note well: equality between translations is weaker than set-theoretic equality.)

DEFINITION 3.1.2 We say that a translation $f : T \rightarrow T'$ is **conservative** just in case, for any $\phi \in \text{Sent}(\Sigma)$, if $T' \vdash f(\phi)$ then $T \vdash \phi$.

PROPOSITION 3.1.3 *A translation $f : T \rightarrow T'$ is conservative if and only if f is a monomorphism in the category **Th**.*

Proof Suppose first that f is conservative, and let $g, h : T'' \rightarrow T$ be translations such that $f \circ g = f \circ h$. That is, $T' \vdash fg(\phi) \leftrightarrow fh(\phi)$ for every sentence ϕ of Σ'' . Since f is conservative, $T \vdash g(\phi) \leftrightarrow h(\phi)$ for every sentence ϕ of Σ'' . Thus, $g = h$, and f is a monomorphism in **Th**.

Conversely, suppose that f is a monomorphism in the category **Th**. Let ϕ be a Σ sentence such that $T' \vdash f(\phi)$. Thus, $T' \vdash f(\phi) \leftrightarrow f(\psi)$, where ψ is any Σ sentence such that $T \vdash \psi$. Now let T'' be the empty theory in signature $\Sigma'' = \{p\}$. Define $g : \Sigma'' \rightarrow \mathbf{Sent}(\Sigma)$ by $g(p) = \phi$, and define $h : \Sigma'' \rightarrow \mathbf{Sent}(\Sigma)$ by $h(p) = \psi$. It's easy to see then that $f \circ g = f \circ h$. Since f is monic, $g = h$, which means that $T \vdash g(p) \leftrightarrow h(p)$. Therefore, $T \vdash \phi$, and f is conservative. \square

DEFINITION 3.1.4 We say that a translation $f : T \rightarrow T'$ is **essentially surjective** just in case for any sentence ϕ of Σ' , there is a sentence ψ of Σ such that $T' \vdash \phi \leftrightarrow f(\psi)$. (Sometimes we use the abbreviation “eso” for essentially surjective.)

PROPOSITION 3.1.5 *If $f : T \rightarrow T'$ is essentially surjective, then f is an epimorphism in **Th**.*

Proof Suppose that $f : T \rightarrow T'$ is eso. Let $g, h : T' \rightrightarrows T''$ such that $g \circ f = h \circ f$. Let ϕ be an arbitrary Σ' sentence. Since f is eso, there is a sentence ψ of Σ such that $T' \vdash \phi \leftrightarrow f(\psi)$. But then $T'' \vdash g(\phi) \leftrightarrow h(\phi)$. Since ϕ was arbitrary, $g = h$. Therefore, f is an epimorphism. \square

What about the converse of this proposition? Are all epimorphisms in **Th** essentially surjective? The answer is yes, but the result is not easy to prove. We'll prove it later on, by means of the correspondence that we establish between theories, Boolean algebras, and Stone spaces.

PROPOSITION 3.1.6 *Let $f : T \rightarrow T'$ be a translation. If f is conservative and essentially surjective, then f is a homotopy equivalence.*

Proof Let $p \in \Sigma'$. Since f is eso, there is some $\phi_p \in \mathbf{Sent}(\Sigma)$ such that $T' \vdash p \leftrightarrow f(\phi_p)$. Define a reconstrual $g : \Sigma' \rightarrow \mathbf{Sent}(\Sigma)$ by setting $g(p) = \phi_p$. As usual, g extends naturally to a function from $\mathbf{Sent}(\Sigma')$ to $\mathbf{Sent}(\Sigma)$, and it immediately follows that $T' \vdash \psi \leftrightarrow fg(\psi)$, for every sentence ψ of Σ' .

We claim now that g is a translation from T' to T . Suppose that $T' \vdash \psi$. Since $T' \vdash \psi \leftrightarrow fg(\psi)$, it follows that $T' \vdash fg(\psi)$. Since f is conservative, $T \vdash g(\psi)$. Thus, for all sentences ψ of Σ' , if $T' \vdash \psi$ then $T \vdash g(\psi)$, which means that $g : T' \rightarrow T$ is a translation. By the previous paragraph, $1_{T'} \simeq fg$.

It remains to show that $1_T \simeq gf$. Let ϕ be an arbitrary sentence of Σ . Since f is conservative, it will suffice to show that $T' \vdash f(\phi) \leftrightarrow fgf(\phi)$. But by the previous paragraph, $T' \vdash \psi \leftrightarrow fg(\psi)$ for all sentences ψ of Σ' . Therefore, $1_T \simeq gf$, and f is a homotopy equivalence. \square

Before proceeding, let's remind ourselves of some of the motivations for these technical investigations.

The category **Sets** is, without a doubt, extremely useful. However, a person who is familiar with **Sets** might have developed some intuitions that could be misleading when applied to other categories. For example, in **Sets**, if there are injections $f : X \rightarrow Y$ and $g : Y \rightarrow X$, then there is a bijection between X and Y . Thus, it's tempting to think, for example, that if there are embeddings $f : T \rightarrow T'$ and $g : T' \rightarrow T$ of theories, then T and T' are equivalent. (Here an embedding between theories is a monomorphism in **Th**, i.e., a conservative translation.) Similarly, in **Sets**, if there is an injection $f : X \rightarrow Y$ and a surjection $g : X \rightarrow Y$, then there is a bijection between X and Y . However, in **Th**, the analogous result fails to hold.

TECHNICAL ASIDE 3.1.7 For those familiar with the category **Vect** of vector spaces: **Vect** is similar to **Sets** in that mutually embeddable vector spaces are isomorphic. That is, if $f : V \rightarrow W$ and $g : W \rightarrow V$ are monomorphisms (i.e., injective linear maps), then V and W have the same dimension and, hence, are isomorphic. The categories **Sets** and **Vect** share in common the feature that the objects can be classified by cardinal numbers. In the case of sets, if $|X| = |Y|$, then $X \cong Y$. In the case of vector spaces, if $\dim(V) = \dim(W)$, then $V \cong W$.

In Exercise 1.4.7, you showed that if $f : T \rightarrow T'$ is a translation, and if v is a model of T' , then $v \circ f$ is a model of T . Let $M(T)$ be the set of all models of T , and define a function $f^* : M(T') \rightarrow M(T)$ by setting $f^*(v) = v \circ f$.

PROPOSITION 3.1.8 *Let $f : T \rightarrow T'$ be a translation. If $f^* : M(T') \rightarrow M(T)$ is surjective, then f is conservative.*

Proof Suppose that f^* is surjective, and suppose that ϕ is a sentence of Σ such that $T \not\vdash \phi$. Then there is a $v \in M(T)$ such that $v(\phi) = 0$. (Here we have invoked the completeness theorem, but we haven't proven it yet. Note that our proof of the completeness theorem, page 79, does not cite this result or any that depend on it.) Since f^* is surjective, there is a $w \in M(T')$ such that $f^*(w) = v$. But then

$$w(f(\phi)) = f^*w(\phi) = v(\phi) = 0,$$

from which it follows that $T' \not\vdash f(\phi)$. Therefore, f is conservative. \square

Example 3.1.9 Let $\Sigma = \{p_0, p_1, \dots\}$, and let T be the empty theory in Σ . Let $\Sigma' = \{q_0, q_1, \dots\}$, and let T' be the theory with axioms $q_0 \rightarrow q_i$, for $i = 0, 1, \dots$. We will show that there are conservative translations $f : T \rightarrow T'$ and $g : T' \rightarrow T$.

Define $f : \Sigma \rightarrow \text{Sent}(\Sigma')$ by $f(p_i) = q_{i+1}$. Since T is the empty theory, f is a translation. Then for any valuation v of Σ' , we have

$$f^*v(p_i) = v(f(p_i)) = v(q_{i+1}).$$

Furthermore, for any sequence of zeros and ones, there is a valuation v of Σ' that assigns that sequence to q_1, q_2, \dots . Thus, f^* is surjective, and f is conservative.

Now define $g : \Sigma' \rightarrow \text{Sent}(\Sigma)$ by setting $g(q_i) = p_0 \vee p_i$. Since $T \vdash p_0 \vee p_0 \rightarrow p_0 \vee p_i$, it follows that g is a translation. Furthermore, for any valuation v of Σ , we have

$$g^*v(q_i) = v(g(q_i)) = v(p_0 \vee p_i).$$

Recall that $M(T')$ splits into two parts: (1) a singleton set containing the valuation z where $z(q_i) = 1$ for all i , and (2) the infinitely many other valuations that assign 0 to q_0 . Clearly, $z = g^*v$, where v is any valuation such that $v(p_0) = 1$. Furthermore, for any valuation w of Σ' such that $w(p_0) = 0$, we have $w = g^*v$, where $v(p_i) = w(q_i)$. Therefore, g^* is surjective, and g is conservative. \perp

EXERCISE 3.1.10 In Example 3.1.9, show that f and g are not essentially surjective.

Example 3.1.11 Let T and T' be as in the previous example. Now we'll show that there are essentially surjective (eso) translations $k : T \rightarrow T'$ and $h : T' \rightarrow T$. The first is easy: the translation $k(p_i) = q_i$ is obviously eso. For the second, define $h(q_0) = \perp$, where \perp is some contradiction, and define $h(q_i) = p_{i-1}$ for $i > 0$. \perp

Let's pause to think about some of the questions we might want to ask about theories. We arrange these in roughly decreasing order of technical tractability.

1. Does **Th** have the **Cantor–Bernstein property**? That is, if there are monomorphisms $f : T \rightarrow T'$ and $g : T' \rightarrow T$, then is there an isomorphism $h : T \rightarrow T'$?
2. Is **Th** balanced, in the sense that if $f : T \rightarrow T'$ is both a monomorphism and an epimorphism, then f is an isomorphism?
3. If there is both a monomorphism $f : T \rightarrow T'$ and an epimorphism $g : T' \rightarrow T$, then are T and T' homotopy equivalent?
4. Can an arbitrary theory T be embedded into a theory T_0 that has no axioms? Quine and Goodman (1940) present a proof of this claim – and they argue that it undercuts the analytic-synthetic distinction. They are right about the technical claim (see 3.7.10), but have perhaps misconstrued its philosophical implications.
5. If theories have the same number of models, then are they equivalent? If not, then can we determine whether T and T' are equivalent by inspecting $M(T)$ and $M(T')$?
6. How many theories (up to isomorphism) are there with n models?
7. (Does supervenience imply reduction?) Suppose that the truth value of a sentence ψ **supervenes** on the truth value of some other sentences ϕ_1, \dots, ϕ_n , i.e., for any valuations v, w of the propositional constants occurring in $\phi_1, \dots, \phi_n, \psi$, if $v(\phi_i) = w(\phi_i)$, for $i = 1, \dots, n$, then $v(\psi) = w(\psi)$. Does it follow then that $\vdash \psi \leftrightarrow \theta$, where θ contains only the propositional constants that occur in ϕ_1, \dots, ϕ_n ? We will return to this issue in Section 6.7.
8. Suppose that $f : T \rightarrow T'$ is conservative. Suppose also that every model of T extends uniquely to a model of T' . Does it follow that $T \cong T'$?
9. Suppose that T and T' are consistent in the sense that there is no sentence θ in $\Sigma \cap \Sigma'$ such that $T \vdash \theta$ and $T' \vdash \neg\theta$. Is there a unified theory T'' that extends both T and T' ? (The answer is yes, as shown by **Robinson's theorem**.)
10. What does it mean for one theory to be **reducible** to another? Can we explicate this notion in terms of a certain sort of translation between the relevant theories? Some philosophers have claimed that the reduction relation ought to be treated

semantically, rather than syntactically. In other words, they would have us consider functions from $M(T')$ to $M(T)$, rather than translations from T to T' . In light of the Stone duality theorem proved later in the chapter, it appears that syntactic and semantic approaches are equivalent to each other.

11. Consider various formally definable notions of theoretical equivalence. What are the advantages and disadvantages of the various notions? Is homotopy equivalence too liberal? Is it too conservative?

3.2 Boolean Algebras

DEFINITION 3.2.1 A **Boolean algebra** is a set B together with a unary operation \neg , two binary operations \wedge and \vee , and designated elements $0 \in B$ and $1 \in B$, which satisfy the following equations:

1. Top and Bottom

$$a \wedge 1 = a \vee 0 = a$$

2. Idempotence

$$a \wedge a = a \vee a = a$$

3. De Morgan's Rules

$$\neg(a \wedge b) = \neg a \vee \neg b, \quad \neg(a \vee b) = \neg a \wedge \neg b$$

4. Commutativity

$$a \wedge b = b \wedge a, \quad a \vee b = b \vee a$$

5. Associativity

$$(a \wedge b) \wedge c = a \wedge (b \wedge c), \quad (a \vee b) \vee c = a \vee (b \vee c)$$

6. Distribution

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

7. Excluded Middle

$$a \wedge \neg a = 0, \quad a \vee \neg a = 1$$

Here we are implicitly universally quantifying over a, b, c .

Example 3.2.2 Let 2 denote the Boolean algebra of subsets of a one-point set. Note that 2 looks just like the truth-value set Ω . Indeed, Ω is equipped with operations \wedge, \vee , and \neg that make it into a Boolean algebra. ┘

Example 3.2.3 Let $\Sigma = \{p\}$. Define an equivalence relation \simeq on sentences of Σ by $\phi \simeq \psi$ just in case $\vdash \phi \leftrightarrow \psi$. If we let F denote the set of equivalence classes, then it's not hard to see that F has four elements: $0, 1, [p], [\neg p]$. Define $[\phi] \wedge [\psi] = [\phi \wedge \psi]$,

where the \wedge on the right is the propositional connective, and the \wedge on the left is a newly defined binary function on F . Perform a similar construction for the other logical connectives. Then F is a Boolean algebra. \square

We now derive some basic consequences from the axioms for Boolean algebras. The first two results are called the **absorption laws**.

1. $a \wedge (a \vee b) = a$
 $a \wedge (a \vee b) = (a \vee 0) \wedge (a \vee b) = a \vee (0 \wedge b) = a \vee 0 = a.$
2. $a \vee (a \wedge b) = a$
 $a \vee (a \wedge b) = (a \wedge 1) \vee (a \wedge b) = a \wedge (1 \vee b) = a \wedge 1 = a.$
3. $a \vee 1 = 1$
 $a \vee 1 = a \vee (a \vee \neg a) = a \vee \neg a = 1.$
4. $a \wedge 0 = 0$
 $a \wedge 0 = a \wedge (a \wedge \neg a) = a \wedge \neg a = 0.$

DEFINITION 3.2.4 If B is a Boolean algebra and $a, b \in B$, we write $a \leq b$ when $a \wedge b = a$.

Since $a \wedge 1 = a$, it follows that $a \leq 1$, for all $a \in B$. Since $a \wedge 0 = 0$, it follows that $0 \leq a$, for all $a \in B$. Now we will show that \leq is a partial order, i.e., reflexive, transitive, and asymmetric.

PROPOSITION 3.2.5 *The relation \leq on a Boolean algebra B is a partial order.*

Proof (Reflexive) Since $a \wedge a = a$, it follows that $a \leq a$.

(Transitive) Suppose that $a \wedge b = a$ and $b \wedge c = b$. Then

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a,$$

which means that $a \leq c$.

(Asymmetric) Suppose that $a \wedge b = a$ and $b \wedge a = b$. By commutativity of \wedge , it follows that $a = b$. \square

We now show how \leq interacts with \wedge, \vee , and \neg . In particular, we show that if \leq is thought of as implication, then \wedge behaves like conjunction, \vee behaves like disjunction, \neg behaves like negation, 1 behaves like a tautology, and 0 behaves like a contradiction.

PROPOSITION 3.2.6 $c \leq a \wedge b$ iff $c \leq a$ and $c \leq b$.

Proof Since $a \wedge (a \wedge b) = a \wedge b$, it follows that $a \wedge b \leq a$. By similar reasoning, $a \wedge b \leq b$. Thus, if $c \leq a \wedge b$, then transitivity of \leq entails that both $c \leq a$ and $c \leq b$.

Now suppose that $c \leq a$ and $c \leq b$. That is, $c \wedge a = c$ and $c \wedge b = c$. Then $c \wedge (a \wedge b) = (c \wedge a) \wedge (c \wedge b) = c \wedge c = c$. Therefore, $c \leq a \wedge b$. \square

Notice that \leq and \wedge interact precisely as implication and conjunction interact in propositional logic. The elimination rule says that $a \wedge b$ implies a and b . Hence, if c implies $a \wedge b$, then c implies a and b . The introduction rule says that a and b imply $a \wedge b$. Hence, if c implies a and b , then c implies $a \wedge b$.

PROPOSITION 3.2.7 $a \leq c$ and $b \leq c$ iff $a \vee b \leq c$

Proof Suppose first that $a \leq c$ and $b \leq c$. Then

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = a \vee b.$$

Therefore, $a \vee b \leq c$.

Suppose now that $a \vee b \leq c$. By the absorption law, $a \wedge (a \vee b) = a$, which implies that $a \leq a \vee b$. By transitivity, $a \leq c$. Similarly, $b \leq a \vee b$, and by transitivity, $b \leq c$. \square

Now we show that the connectives \wedge and \vee are monotonic.

PROPOSITION 3.2.8 If $a \leq b$, then $a \wedge c \leq b \wedge c$, for any $c \in B$.

Proof

$$(a \wedge c) \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c.$$

\square

PROPOSITION 3.2.9 If $a \leq b$, then $a \vee c \leq b \vee c$, for any $c \in B$.

Proof

$$(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c = a \vee c.$$

\square

PROPOSITION 3.2.10 If $a \wedge b = a$ and $a \vee b = a$, then $a = b$.

Proof $a \wedge b = a$ means that $a \leq b$. We now claim that $a \vee b = a$ iff $b \wedge a = b$ iff $b \leq a$. Indeed, if $a \vee b = a$, then

$$b \wedge a = b \wedge (a \vee b) = (0 \vee b) \wedge (a \vee b) = (0 \wedge a) \vee b = b.$$

Conversely, if $b \wedge a = b$, then

$$a \vee b = a \vee (a \wedge b) = (a \wedge 1) \vee (a \wedge b) = a \wedge (1 \vee b) = a.$$

Thus, if $a \wedge b = a$ and $a \vee b = a$, then $a \leq b$ and $b \leq a$. By asymmetry of \leq , it follows that $a = b$. \square

We now show that $\neg a$ is the unique complement of a in B .

PROPOSITION 3.2.11 If $a \wedge b = 0$ and $a \vee b = 1$, then $b = \neg a$.

Proof Since $b \vee a = 1$, we have

$$b = b \vee 0 = b \vee (a \wedge \neg a) = (b \vee a) \wedge (b \vee \neg a) = b \vee \neg a.$$

Since $b \wedge a = 0$, we also have

$$b = b \wedge 1 = b \wedge (a \vee \neg a) = (b \wedge a) \vee (b \wedge \neg a) = b \wedge \neg a.$$

By the preceding proposition, $b = \neg a$. \square

PROPOSITION 3.2.12 $\neg 1 = 0$.

Proof We have $1 \wedge 0 = 0$ and $1 \vee 0 = 1$. By the preceding proposition, $0 = \neg 1$. \square

PROPOSITION 3.2.13 *If $a \leq b$, then $\neg b \leq \neg a$.*

Proof Suppose that $a \leq b$, which means that $a \wedge b = a$, and, equivalently, $a \vee b = b$. Thus, $\neg a \wedge \neg b = \neg(a \vee b) = \neg b$, which means that $\neg b \leq \neg a$. \square

PROPOSITION 3.2.14 $\neg\neg a = a$.

Proof We have $\neg a \vee \neg\neg a = 1$ and $\neg a \wedge \neg\neg a = 0$. By Proposition 3.2.11, it follows that $\neg\neg a = a$. \square

DEFINITION 3.2.15 Let A and B be Boolean algebras. A **homomorphism** is a map $\phi : A \rightarrow B$ such that $\phi(0) = 0$, $\phi(1) = 1$, and for all $a, b \in A$, $\phi(\neg a) = \neg\phi(a)$, $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$ and $\phi(a \vee b) = \phi(a) \vee \phi(b)$.

It is easy to see that if $\phi : A \rightarrow B$ and $\psi : B \rightarrow C$ are homomorphisms, then $\psi \circ \phi : A \rightarrow C$ is also a homomorphism. Moreover, $1_A : A \rightarrow A$ is a homomorphism, and composition of homomorphisms is associative.

DEFINITION 3.2.16 We let **Bool** denote the category whose objects are Boolean algebras and whose arrows are homomorphisms of Boolean algebras.

Since **Bool** is a category, we have notions of **monomorphisms**, **epimorphisms**, **isomorphisms**, etc. Once again, it is easy to see that an injective homomorphism is a monomorphism and a surjective homomorphism is an epimorphism.

PROPOSITION 3.2.17 *Monomorphisms in **Bool** are injective.*

Proof Let $f : A \rightarrow B$ be a monomorphism, and let $a, b \in A$. Let F denote the Boolean algebra with four elements, and let p denote one of the two elements in F that is neither 0 nor 1. Define $\hat{a} : F \rightarrow A$ by $\hat{a}(p) = a$, and define $\hat{b} : F \rightarrow A$ by $\hat{b}(p) = b$. It is easy to see that \hat{a} and \hat{b} are uniquely defined by these conditions, and that they are Boolean homomorphisms. Suppose now that $f(a) = f(b)$. Then $f\hat{a} = f\hat{b}$, and, since f is a monomorphism, $\hat{a} = \hat{b}$, and, therefore, $a = b$. Therefore, f is injective. \square

It is also true that epimorphisms in **Bool** are surjective. However, proving that fact is no easy task. We will return to it later in the chapter.

PROPOSITION 3.2.18 *If $f : A \rightarrow B$ is a homomorphism of Boolean algebras, then $a \leq b$ only if $f(a) \leq f(b)$.*

Proof $a \leq b$ means that $a \wedge b = a$. Thus,

$$f(a) \wedge f(b) = f(a \wedge b) = f(a),$$

which means that $f(a) \leq f(b)$. \square

DEFINITION 3.2.19 A homomorphism $\phi : B \rightarrow 2$ is called a **state** of B .

3.3 Equivalent Categories

We now have two categories on the table: the category **Th** of theories and the category **Bool** of Boolean algebras. Our next goal is to show that these categories are **structurally identical**. But what do we mean by this? What we mean is that they are **equivalent categories**. In order to explain what that means, we need a few more definitions.

DEFINITION 3.3.1 Suppose that **C** and **D** are categories. We let \mathbf{C}_0 denote the objects of **C**, and we let \mathbf{C}_1 denote the arrows of **C**. A (covariant) **functor** $F : \mathbf{C} \rightarrow \mathbf{D}$ consists of a pair of maps: $F_0 : \mathbf{C}_0 \rightarrow \mathbf{D}_0$, and $F_1 : \mathbf{C}_1 \rightarrow \mathbf{D}_1$ with the following properties:

1. F_0 and F_1 are compatible in the sense that if $f : X \rightarrow Y$ in **C**, then $F_1(f) : F_0(X) \rightarrow F_0(Y)$ in **D**.
2. F_1 preserves identities and composition in the following sense: $F_1(1_X) = 1_{F_0(X)}$, and $F_1(g \circ f) = F_1(g) \circ F_1(f)$.

When no confusion can result, we simply use F in place of F_0 and F_1 .

NOTE 3.3.2 There is also a notion of a **contravariant functor**, where F_1 reverses the direction of arrows: if $f : X \rightarrow Y$ in **C**, then $F_1(f) : F_0(Y) \rightarrow F_0(X)$ in **D**. Contravariant functors will be especially useful for examining the relation between a theory and its set of models. We've already seen that a translation $f : T \rightarrow T'$ induces a function $f^* : M(T') \rightarrow M(T)$. In Section 3.7, we will see that $f \mapsto f^*$ is part of a contravariant functor.

Example 3.3.3 For any category **C**, there is a functor $1_{\mathbf{C}}$ that acts as the identity on both objects and arrows. That is, for any object X of **C**, $1_{\mathbf{C}}(X) = X$. And for any arrow f of **C**, $1_{\mathbf{C}}(f) = f$. ⌋

DEFINITION 3.3.4 Let $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{C} \rightarrow \mathbf{D}$ be functors. A **natural transformation** $\eta : F \Rightarrow G$ consists of a family $\{\eta_X : F(X) \rightarrow G(X) \mid X \in \mathbf{C}_0\}$ of arrows in **D**, such that for any arrow $f : X \rightarrow Y$ in **C**, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow \eta_X & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

DEFINITION 3.3.5 A natural transformation $\eta : F \Rightarrow G$ is said to be a **natural isomorphism** just in case each arrow $\eta_X : F(X) \rightarrow G(X)$ is an isomorphism. In this case, we write $F \cong G$.

DEFINITION 3.3.6 Let $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{D} \rightarrow \mathbf{C}$ be functors. We say that F and G are a **categorical equivalence** just in case $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$.

3.4 Propositional Theories Are Boolean Algebras

In this section, we show that there is a one-to-one correspondence between theories (in propositional logic) and Boolean algebras. We first need some preliminaries.

DEFINITION 3.4.1 Let Σ be a propositional signature (i.e., a set), let B be a Boolean algebra, and let $f : \Sigma \rightarrow B$ be an arbitrary function. (Here we use \cap, \cup and $-$ for the Boolean operations in order to avoid confusion with the logical connectives \wedge, \vee and \neg .) Then f naturally extends to a map $f : \mathbf{Sent}(\Sigma) \rightarrow B$ as follows:

1. $f(\phi \wedge \psi) = f(\phi) \cap f(\psi)$
2. $f(\phi \vee \psi) = f(\phi) \cup f(\psi)$
3. $f(\neg\phi) = -f(\phi)$.

Now let T be a theory in Σ . We say that f is an **interpretation** of T in B just in case: for all sentences ϕ , if $T \vdash \phi$ then $f(\phi) = 1$.

DEFINITION 3.4.2 Let $f : T \rightarrow B$ be an interpretation. We say that

1. f is **conservative** just in case: for all sentences ϕ , if $f(\phi) = 1$ then $T \vdash \phi$.
2. f **surjective** just in case: for each $a \in B$, there is a $\phi \in \mathbf{Sent}(\Sigma)$ such that $f(\phi) = a$.

LEMMA 3.4.3 Let $f : T \rightarrow B$ be an interpretation. Then the following are equivalent:

1. f is conservative.
2. For any $\phi, \psi \in \mathbf{Sent}(\Sigma)$, if $f(\phi) = f(\psi)$ then $T \vdash \phi \leftrightarrow \psi$.

Proof Note first that $f(\phi) = f(\psi)$ if and only if $f(\phi \leftrightarrow \psi) = 1$. Suppose then that f is conservative. If $f(\phi) = f(\psi)$, then $f(\phi \leftrightarrow \psi) = 1$, and hence $T \vdash \phi \leftrightarrow \psi$. Suppose now that (2) holds. If $f(\phi) = 1$, then $f(\phi) = f(\phi \vee \neg\phi)$, and hence $T \vdash (\phi \vee \neg\phi) \leftrightarrow \phi$. Therefore, $T \vdash \phi$, and f is conservative. \square

LEMMA 3.4.4 If $f : T \rightarrow B$ is an interpretation, and $g : B \rightarrow A$ is a homomorphism, then $g \circ f$ is an interpretation.

Proof This is almost obvious. \square

LEMMA 3.4.5 If $f : T \rightarrow B$ is an interpretation, and $g : T' \rightarrow T$ is a translation, then $f \circ g : T' \rightarrow B$ is an interpretation.

Proof This is almost obvious. \square

LEMMA 3.4.6 Suppose that T is a theory, and $e : T \rightarrow B$ is a surjective interpretation. If $f, g : B \rightarrow A$ are homomorphisms such that $fe = ge$, then $f = g$.

Proof Suppose that $fe = ge$, and let $a \in B$. Since e is surjective, there is a $\phi \in \mathbf{Sent}(\Sigma)$ such that $e(\phi) = a$. Thus, $f(a) = fe(\phi) = ge(\phi) = g(a)$. Since a was arbitrary, $f = g$. \square

Let T' and T be theories, and let $f, g : T' \rightrightarrows T$ be translations. Recall that we defined identity between translations as follows: $f = g$ if and only if $T \vdash f(\phi) \leftrightarrow g(\phi)$ for all $\phi \in \mathbf{Sent}(\Sigma')$.

LEMMA 3.4.7 *Suppose that $m : T \rightarrow B$ is a conservative interpretation. If $f, g : T' \rightrightarrows T$ are translations such that $mf = mg$, then $f = g$.*

Proof Let $\phi \in \mathbf{Sent}(\Sigma')$, where Σ' is the signature of T' . Then $mf(\phi) = mg(\phi)$. Since m is conservative, $T \vdash f(\phi) \leftrightarrow g(\phi)$. Since this holds for all sentences, it follows that $f = g$. \square

PROPOSITION 3.4.8 *For each theory T , there is a Boolean algebra $L(T)$ and a conservative, surjective interpretation $i_T : T \rightarrow L(T)$ such that for any Boolean algebra B and interpretation $f : T \rightarrow B$, there is a unique homomorphism $\bar{f} : L(T) \rightarrow B$ such that $\bar{f}i_T = f$.*

$$\begin{array}{ccc} T & \xrightarrow{i_T} & L(T) \\ & \searrow f & \downarrow \bar{f} \\ & & B \end{array}$$

We define an equivalence relation \equiv on the sentences of Σ :

$$\phi \equiv \psi \quad \text{iff} \quad T \vDash \phi \leftrightarrow \psi,$$

and we let

$$E_\phi := \{\psi \mid \phi \equiv \psi\}.$$

Finally, let

$$L(T) := \{E_\phi \mid \phi \in \mathbf{Sent}(\Sigma)\}.$$

We now equip $L(T)$ with the structure of a Boolean algebra. To this end, we need the following facts, which correspond to easy proofs in propositional logic.

FACT 3.4.9 *If $E_\phi = E_{\phi'}$ and $E_\psi = E_{\psi'}$, then:*

1. $E_{\phi \wedge \psi} = E_{\phi' \wedge \psi'}$
2. $E_{\phi \vee \psi} = E_{\phi' \vee \psi'}$
3. $E_{\neg \phi} = E_{\neg \phi'}$.

We then define a unary operation $-$ on $L(T)$ by

$$-E_\phi := E_{\neg \phi},$$

and we define two binary operations on $L(T)$ by

$$E_\phi \cap E_\psi := E_{\phi \wedge \psi}, \quad E_\phi \cup E_\psi := E_{\phi \vee \psi}.$$

Finally, let ϕ be an arbitrary Σ sentence, and let $0 = E_{\phi \wedge \neg \phi}$ and $1 = E_{\phi \vee \neg \phi}$. The proof that $\langle L(T), \cap, \cup, -, 0, 1 \rangle$ is a Boolean algebra requires a series of straightforward

verifications. For example, let's show that $1 \cap E_\psi = E_\psi$, for all sentences ψ . Recall that $1 = E_{\phi \vee \neg\phi}$ for some arbitrarily chosen sentence ϕ . Thus,

$$1 \cap E_\psi = E_{\phi \vee \neg\phi} \cap E_\psi = E_{(\phi \vee \neg\phi) \wedge \psi}.$$

Moreover, $T \vdash \psi \leftrightarrow ((\phi \vee \neg\phi) \wedge \psi)$, from which it follows that $E_{(\phi \vee \neg\phi) \wedge \psi} = E_\psi$. Therefore, $1 \cap E_\psi = E_\psi$.

Consider now the function $i_T : \Sigma \rightarrow L(T)$ given by $i_T(\phi) = E_\phi$, and its natural extension to $\mathbf{Sent}(\Sigma)$. A quick inductive argument, using the definition of the Boolean operations on $L(T)$, shows that $i_T(\phi) = E_\phi$ for all $\phi \in \mathbf{Sent}(\Sigma)$. The following shows that i_T is a conservative interpretation of T in $L(T)$.

PROPOSITION 3.4.10 $T \vdash \phi$ if and only if $i_T(\phi) = 1$.

Proof $T \vdash \phi$ iff $T \vdash (\psi \vee \neg\psi) \leftrightarrow \phi$ iff $i_T(\phi) = E_\phi = E_{\psi \vee \neg\psi} = 1$. \square

Since $i_T(\phi) = E_\phi$, the interpretation i_T is also surjective.

PROPOSITION 3.4.11 Let B be a Boolean algebra, and let $f : T \rightarrow B$ be an interpretation. Then there is a unique homomorphism $\bar{f} : L(T) \rightarrow B$ such that $\bar{f}i_T = f$.

Proof If $E_\phi = E_\psi$, then $T \vdash \phi \leftrightarrow \psi$, and so $f(\phi) = f(\psi)$. Thus, we may define $\bar{f}(E_\phi) = f(\phi)$. It is straightforward to verify that \bar{f} is a Boolean homomorphism, and it is clearly unique. \square

DEFINITION 3.4.12 The Boolean algebra $L(T)$ is called the **Lindenbaum algebra** of T .

PROPOSITION 3.4.13 Let B be a Boolean algebra. There is a theory T_B and a conservative, surjective interpretation $e_B : T_B \rightarrow B$ such that for any theory T and interpretation $f : T \rightarrow B$, there is a unique interpretation $\bar{f} : T \rightarrow T_B$ such that $e_B\bar{f} = f$.

$$\begin{array}{ccc} T_B & \xrightarrow{e_B} & B \\ \bar{f} \uparrow & \nearrow f & \\ T & & \end{array}$$

Proof Let $\Sigma_B = B$ be a signature. (Recall that a propositional signature is just a set where each element represents an elementary proposition.) We define $e_B : \Sigma_B \rightarrow B$ as the identity and use the symbol e_B also for its extension to $\mathbf{Sent}(\Sigma_B)$. We define a theory T_B on Σ_B by $T_B \vdash \phi$ if and only if $e_B(\phi) = 1$. Thus, $e_B : T_B \rightarrow B$ is automatically a conservative interpretation of T_B in B .

Now let T be some theory in signature Σ , and let $f : T \rightarrow B$ be an interpretation. Since $\Sigma_B = B$, f automatically gives rise to a reconstrual $f : \Sigma \rightarrow \Sigma_B$, which we will rename \bar{f} for clarity. And since e_B is just the identity on $B = \Sigma_B$, we have $f = e_B\bar{f}$.

Finally, to see that $\bar{f} : T \rightarrow T_B$ is a translation, suppose that $T \vdash \phi$. Since f is an interpretation of T , $f(\phi) = 1$, which means that $e_B(\bar{f}(\phi)) = 1$. Since e_B is conservative, $T_B \vdash \bar{f}(\phi)$. Therefore, \bar{f} is a translation. \square

We have shown that each propositional theory T corresponds to a Boolean algebra $L(T)$ and each Boolean algebra B corresponds to a propositional theory T_B . We will now show that these correspondences are functorial. First we show that a morphism $f : B \rightarrow A$ in **Bool** naturally gives rise to a morphism $T(f) : T_B \rightarrow T_A$ in **Th**. Indeed, consider the following diagram:

$$\begin{array}{ccc} T_B & \xrightarrow{T(f)} & T_A \\ \downarrow e_B & & \downarrow e_A \\ B & \xrightarrow{f} & A \end{array}$$

Since $f e_B$ is an interpretation of T_B in A , Prop. 3.4.13 entails that there is a unique translation $T(f) : T_B \rightarrow T_A$ such that $e_A T(f) = f e_B$. The uniqueness clause also entails that T commutes with composition of morphisms, and maps identity morphisms to identity morphisms. Thus, $T : \mathbf{Bool} \rightarrow \mathbf{Th}$ is a functor.

Let's consider this translation $T(f) : T_B \rightarrow T_A$ more concretely. First of all, recall that translations from T_B to T_A are actually equivalence classes of maps from Σ_B to $\mathbf{Sent}(\Sigma_A)$. Thus, there's no sense to the question, "which function is $T(f)$?" However, there's a natural choice of a representative function. Indeed, consider f itself as a function from $\Sigma_B = B$ to $\Sigma_A = A$. Then, for $x \in \Sigma_B = B$, we have

$$(e_A \circ T(f))(x) = e_A(f(x)) = f(x) = f(e_B(x)),$$

since e_A is the identity on Σ_A , and e_B is the identity on Σ_B . In other words, $T(f)$ is the equivalence class of f itself. [But recall that translations, while initially defined on the signature Σ_B , extend naturally to all elements of $\mathbf{Sent}(\Sigma_B)$. From this point of view, $T(f)$ has a larger domain than f .]

A similar construction can be used to define the functor $L : \mathbf{Th} \rightarrow \mathbf{Bool}$. In particular, let $f : T \rightarrow T'$ be a morphism in **Th**, and consider the following diagram:

$$\begin{array}{ccc} T & \xrightarrow{f} & T' \\ \downarrow i_T & & \downarrow i_{T'} \\ L(T) & \xrightarrow{L(f)} & L(T') \end{array}$$

Since $i_{T'} f$ is an interpretation of T in $L(T')$, Prop. 3.4.8 entails that there is a unique homomorphism $L(f) : L(T) \rightarrow L(T')$ such that $L(f) i_T = i_{T'} f$.

More explicitly,

$$L(f)(E_\phi) = L(f)(i_T(\phi)) = i_{T'} f(\phi) = E_{f(\phi)}.$$

Recall, however, that identity of arrows in **Th** is *not* identity of the corresponding functions, in the set-theoretic sense. Rather, $f \simeq g$ just in case $T' \vdash f(\phi) \leftrightarrow g(\phi)$, for all $\phi \in \mathbf{Sent}(\Sigma)$. Thus, we must verify that if $f \simeq g$ in **Th**, then $L(f) = L(g)$. Indeed, since $i_{T'}$ is an interpretation of T' , we have $i_{T'}(f(\phi)) = i_{T'}(g(\phi))$; and since the diagram above commutes, $L(f) \circ i_T = L(g) \circ i_T$. Since i_T is surjective, $L(f) = L(g)$. Thus, $f \simeq g$ only if $L(f) = L(g)$. Finally, the uniqueness clause in Prop. 3.4.8

entails that L commutes with composition and maps identities to identities. Therefore, $L : \mathbf{Th} \rightarrow \mathbf{Bool}$ is a functor.

We will soon show that the functor $L : \mathbf{Th} \rightarrow \mathbf{Bool}$ is an equivalence of categories, from which it follows that L preserves all categorically definable properties. For example, a translation $f : T \rightarrow T'$ is monic if and only if $L(f) : L(T) \rightarrow L(T')$ is monic, etc. However, it may be illuminating to prove some such facts directly.

PROPOSITION 3.4.14 *Let $f : T \rightarrow T'$ be a translation. Then f is conservative if and only if $L(f)$ is injective.*

Proof Suppose first that f is conservative. Let $E_\phi, E_\psi \in L(T)$ such that $L(f)(E_\phi) = L(f)(E_\psi)$. Using the definition of $L(f)$, we have $E_{f(\phi)} = E_{f(\psi)}$, which means that $T' \vdash f(\phi) \leftrightarrow f(\psi)$. Since f is conservative, $T \vdash \phi \leftrightarrow \psi$, from which $E_\phi = E_\psi$. Therefore, $L(f)$ is injective.

Suppose now that $L(f)$ is injective. Let ϕ be a Σ sentence such that $T' \vdash f(\phi)$. Since $f(\top) = \top$, we have $T' \vdash f(\top) \leftrightarrow f(\phi)$, which means that $L(f)(E_\top) = L(f)(E_\phi)$. Since $L(f)$ is injective, $E_\top = E_\phi$, from which $T \vdash \phi$. Therefore, f is conservative. \square

PROPOSITION 3.4.15 *For any Boolean algebra B , there is a natural isomorphism $\eta_B : B \rightarrow L(T_B)$.*

Proof Let $e_B : T_B \rightarrow B$ be the interpretation from Prop. 3.4.13, and let $i_{T_B} : T_B \rightarrow L(T_B)$ be the interpretation from Prop. 3.4.8. Consider the following diagram:

$$\begin{array}{ccc} T_B & \xrightarrow{i_{T_B}} & L(T_B) \\ & \searrow e_B & \downarrow \eta_B \\ & & B \end{array}$$

By Prop. 3.4.8, there is a unique homomorphism $\eta_B : L(T_B) \rightarrow B$ such that $e_B = \eta_B i_{T_B}$. Since e_B is the identity on Σ_B ,

$$\eta_B(E_x) = \eta_B i_{T_B}(x) = e_B(x) = x,$$

for any $x \in B$. Thus, if η_B has an inverse, it must be given by the map $x \mapsto E_x$. We claim that this map is a Boolean homomorphism. To see this, recall that $\Sigma_B = B$. Moreover, for $x, y \in B$, the Boolean meet $x \cap y$ is again an element of B , hence an element of the signature Σ_B . By the definition of T_B , we have $T_B \vdash (x \cap y) \leftrightarrow (x \wedge y)$, where the \wedge symbol on the right is conjunction in $\mathbf{Sent}(\Sigma_B)$. Thus,

$$E_{x \cap y} = E_{x \wedge y} = E_x \cap E_y.$$

A similar argument shows that $E_{-x} = -E_x$. Therefore, $x \mapsto E_x$ is a Boolean homomorphism, and η_B is an isomorphism.

It remains to show that η_B is natural in B . Consider the following diagram:

$$\begin{array}{ccccc}
 T_B & \xrightarrow{T_f} & T_A & & \\
 \searrow e_B & & \searrow e_A & & \\
 & & B & \xrightarrow{f} & A \\
 \searrow i_{T_B} & & \uparrow \eta_B & & \uparrow \eta_A \\
 & & L(T_B) & \xrightarrow{LT(f)} & L(T_A)
 \end{array}$$

The top square commutes by the definition of the functor T . The triangles on the left and right commute by the definition of η . And the outmost square commutes by the definition of the functor L . Thus we have

$$\begin{aligned}
 f \circ \eta_B \circ i_{T_B} &= f \circ e_B \\
 &= e_A \circ T_f \\
 &= \eta_A \circ i_{T_A} \circ T_f \\
 &= \eta_A \circ LT(f) \circ i_{T_B}.
 \end{aligned}$$

Since i_{T_B} is surjective, it follows that $f \circ \eta_B = \eta_A \circ LT(f)$, and, therefore, η is a natural transformation. \square

DISCUSSION 3.4.16 Consider the algebra $L(T_B)$, which we have just proved is isomorphic to B . This result is hardly surprising. For any $x, y \in \Sigma_B$, we have $T_B \vdash x \leftrightarrow y$ if and only if $x = e_B(x) = e_B(y) = y$. Thus, the equivalence class E_x contains x and no other element from Σ_B . (That's why $\eta_B(E_x) = x$ makes sense.) We also know that for every $\phi \in \mathbf{Sent}(\Sigma_B)$, there is an $x \in \Sigma_B = B$ such that $T_B \vdash x \leftrightarrow \phi$. In particular, $T_B \vdash e_B(\phi) \leftrightarrow \phi$. Thus, $E_\phi = E_x$, and there is a natural bijection between elements of $L(T_B)$ and elements of B .

PROPOSITION 3.4.17 For any theory T , there is a natural isomorphism $\epsilon_T : T \rightarrow T_{L(T)}$.

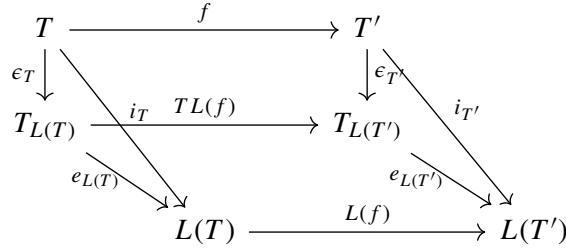
Proof Consider the following diagram:

$$\begin{array}{ccc}
 T_{L(T)} & \xrightarrow{e_{L(T)}} & L(T) \\
 \uparrow \epsilon_T & \nearrow i_T & \\
 T & &
 \end{array}$$

By Prop. 3.4.13, there is a unique interpretation $\epsilon_T : T \rightarrow T_{L(T)}$ such that $e_{L(T)}\epsilon_T = i_T$. We claim that ϵ_T is an isomorphism. To see that ϵ_T is conservative, suppose that $T_{L(T)} \vdash \epsilon_T(\phi)$. Since $e_{L(T)}$ is an interpretation, $e_{L(T)}\epsilon_T(\phi) = 1$ and hence $i_T(\phi) = 1$. Since i_T is conservative, $T \vdash \phi$. Therefore ϵ_T is conservative.

To see that ϵ_T is essentially surjective, suppose that $\psi \in \mathbf{Sent}(\Sigma_{L(T)})$. Since i_T is surjective, there is a $\phi \in \mathbf{Sent}(\Sigma)$ such that $i_T(\phi) = e_{L(T)}(\psi)$. Thus, $e_{L(T)}(\epsilon_T(\phi)) = e_{L(T)}(\psi)$. Since $e_{L(T)}$ is conservative, $T_{L(T)} \vdash \epsilon_T(\phi) \leftrightarrow \psi$. Therefore, ϵ_T is essentially surjective.

It remains to show that ϵ_T is natural in T . Consider the following diagram:



The triangles on the left and the right commute by the definition of ϵ . The top square commutes by the definition of L , and the bottom square commutes by the definition of T . Thus, we have

$$\begin{aligned}
 e_{L(T')} \circ \epsilon_{T'} \circ f &= i_{T'} \circ f \\
 &= L(f) \circ i_T \\
 &= L(f) \circ e_{L(T)} \circ \epsilon_T \\
 &= e_{L(T')} \circ TL(f) \circ \epsilon_T.
 \end{aligned}$$

Since $e_{L(T')}$ is conservative, $\epsilon_{T'} \circ f = TL(f) \circ \epsilon_T$. Therefore, ϵ_T is natural in T . \square

DISCUSSION 3.4.18 Recall that ϵ_T doesn't denote a unique function; it denotes an equivalence class of functions. One representative of this equivalence class is the function $\epsilon_T : \Sigma \rightarrow \Sigma_{L(T)}$ given by $\epsilon_T(p) = E_p$. In this case, a straightforward inductive argument shows that $T_{L(T)} \vdash E_\phi \leftrightarrow \epsilon_T(\phi)$, for all $\phi \in \mathbf{Sent}(\Sigma)$.

We know that ϵ_T has an inverse, which itself is an equivalence class of functions from $\Sigma_{L(T)}$ to $\mathbf{Sent}(\Sigma)$. We can define a representative f of this equivalence class by choosing, for each $E \in \Sigma_{L(T)} = L(T)$, some $\phi \in E$, and setting $f(E) = \phi$. Another straightforward argument shows that if we made a different set of choices, the resulting function f' would be equivalent to f – i.e., it would correspond to the same translation from $T_{L(T)}$ to T .

Since there are natural isomorphisms $\epsilon : \mathbf{1}_{\mathbf{Th}} \Rightarrow TL$ and $\eta : \mathbf{1}_{\mathbf{Bool}} \Rightarrow LT$, we have the following result:

Lindenbaum Theorem

The categories **Th** and **Bool** are equivalent.

3.5 Boolean Algebras Again

The Lindenbaum theorem would deliver everything we wanted – if we had a perfectly clear understanding of the category **Bool**. However, there remain questions about **Bool**.

For example, are all epimorphisms in **Bool** surjections? In order to shed even more light on **Bool**, and hence on **Th**, we will show that **Bool** is dual to a certain category of topological spaces. This famous result is called the **Stone duality theorem**. But in order to prove it, we need to collect a few more facts about Boolean algebras.

DEFINITION 3.5.1 Let B be a Boolean algebra. A subset $F \subseteq B$ is said to be a **filter** just in case

1. If $a, b \in F$, then $a \wedge b \in F$.
2. If $a \in F$ and $a \leq b$, then $b \in F$.

If, in addition, $F \neq B$, then we say that F is a **proper filter**. We say that F is an **ultrafilter** just in case F is maximal among proper filters – i.e., if $F \subseteq F'$ where F' is a proper filter, then $F = F'$.

DISCUSSION 3.5.2 Consider the Boolean algebra B as a theory. Then a filter $F \subseteq B$ can be thought of as supplying an update of information. The first condition says that if we learn a and b , then we've learned $a \wedge b$. The second condition says that if we learn a , and $a \leq b$, then we've learned b . In particular, an ultrafilter supplies maximal information.

EXERCISE 3.5.3 Let F be a filter. Show that F is proper if and only if $0 \notin F$.

DEFINITION 3.5.4 Let $F \subseteq B$ be a filter, and let $a \in B$. We say that a is **compatible** with F just in case $a \wedge x \neq 0$ for all $x \in F$.

LEMMA 3.5.5 Let $F \subseteq B$ be a proper filter, and let $a \in B$. Then either a or $\neg a$ is compatible with F .

Proof Suppose for reductio ad absurdum that neither a nor $\neg a$ is compatible with F . That is, there is an $x \in F$ such that $x \wedge a = 0$, and there is a $y \in F$ such that $y \wedge \neg a = 0$. Then

$$x \wedge y = (x \wedge y) \wedge (a \vee \neg a) = (x \wedge y \wedge a) \vee (x \wedge y \wedge \neg a) = 0.$$

Since $x, y \in F$, it follows that $0 = x \wedge y \in F$, contradicting the assumption that F is proper. Therefore, either a or $\neg a$ is compatible with F . \square

PROPOSITION 3.5.6 Let F be a proper filter on B . Then the following are equivalent:

1. F is an ultrafilter.
2. For all $a \in B$, either $a \in F$ or $\neg a \in F$.
3. For all $a, b \in B$, if $a \vee b \in F$, then either $a \in F$ or $b \in F$.

Proof (1 \Rightarrow 2) Suppose that F is an ultrafilter. By Lemma 3.5.5, either a or $\neg a$ is compatible with F . Suppose first that a is compatible with F . Then the set

$$F' = \{y : x \wedge a \leq y, \text{ some } x \in F\},$$

is a proper filter that contains F and a . Since F is an ultrafilter, $F' = F$, and hence $a \in F$. By symmetry, if $\neg a$ is compatible with F , then $\neg a \in F$.

(2 \Rightarrow 3) Suppose that $a \vee b \in F$. By 2, either $a \in F$ or $\neg a \in F$. If $\neg a \in F$, then $\neg a \wedge (a \vee b) \in F$. But $\neg a \wedge (a \vee b) \leq b$, and so $b \in F$.

(3 \Rightarrow 1) Suppose that F' is a filter that contains F , and let $a \in F' - F$. Since $a \vee \neg a = 1 \in F$, it follows from (3) that $\neg a \in F$. But then $0 = a \wedge \neg a \in F'$; that is, $F' = B$. Therefore F is an ultrafilter. \square

PROPOSITION 3.5.7 *There is a bijective correspondence between ultrafilters in B and homomorphisms from B into 2 . In particular, for any homomorphism $f : B \rightarrow 2$, the subset $f^{-1}(1)$ is an ultrafilter in B .*

Proof Let U be an ultrafilter on B . Define $f : B \rightarrow 2$ by setting $f(a) = 1$ iff $a \in U$. Then

$$\begin{aligned} f(a \wedge b) = 1 & \text{ iff } a \wedge b \in U \\ & \text{ iff } a \in U \text{ and } b \in U \\ & \text{ iff } f(a) = 1 \text{ and } f(b) = 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} f(\neg a) = 1 & \text{ iff } \neg a \in U \\ & \text{ iff } a \notin U \\ & \text{ iff } f(a) = 0. \end{aligned}$$

Therefore, f is a homomorphism.

Now suppose that $f : B \rightarrow 2$ is a homomorphism, and let $U = f^{-1}(1)$. Since $f(a) = 1$ and $f(b) = 1$ only if $f(a \wedge b) = 1$, it follows that U is closed under conjunction. Since $a \leq b$ only if $f(a) \leq f(b)$, it follows that U is closed under implication. Finally, since $f(a) = 0$ iff $f(\neg a) = 1$, it follows that $a \notin U$ iff $\neg a \in U$. \square

DEFINITION 3.5.8 For $a, b \in B$, define

$$a \rightarrow b := \neg a \vee b,$$

and define

$$a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a).$$

It's straightforward to check that \rightarrow behaves like the conditional from propositional logic. The next lemma gives a Boolean algebra version of modus ponens.

LEMMA 3.5.9 *Let F be a filter. If $a \rightarrow b \in F$ and $a \in F$, then $b \in F$.*

Proof Suppose that $\neg a \vee b = a \rightarrow b \in F$ and $a \in F$. We then compute

$$b = b \vee 0 = b \vee (a \wedge \neg a) = (a \vee b) \wedge (\neg a \vee b).$$

Since $a \in F$ and $a \leq a \vee b$, we have $a \vee b \in F$. Since F is a filter, $b \in F$. \square

EXERCISE 3.5.10

1. Let B be a Boolean algebra, and let $a, b, c \in B$. Show that the following hold:
 - (a) $(a \rightarrow b) = 1$ iff $a \leq b$
 - (b) $(a \wedge b) \leq c$ iff $a \leq (b \rightarrow c)$
 - (c) $a \wedge (a \rightarrow b) \leq b$
 - (d) $(a \leftrightarrow b) = (b \leftrightarrow a)$
 - (e) $(a \leftrightarrow a) = 1$
 - (f) $(a \leftrightarrow 1) = a$
2. Let $\mathcal{P}N$ be the powerset of the natural numbers, and let \mathcal{U} be an ultrafilter on $\mathcal{P}N$. Show that if \mathcal{U} contains a finite set F , then \mathcal{U} contains a singleton set.

DEFINITION 3.5.11 Let B be a Boolean algebra, and let R be an equivalence relation on the underlying set of B . We say that R is a **congruence** just in case R is compatible with the operations on B in the following sense: if aRa' and bRb' , then $(a \wedge b)R(a' \wedge b')$, and $(a \vee b)R(a' \vee b')$, and $(\neg a)R(\neg a')$.

In a category \mathbf{C} with limits (products, equalizers, pullbacks, etc.), it's possible to formulate the notion of an equivalence relation in \mathbf{C} . Thus, in **Bool**, an equivalence relation R on B is a subalgebra R of $B \times B$ that satisfies the appropriate analogues of reflexivity, symmetry, and transitivity. Since R is a subalgebra of $B \times B$, it follows in particular that if $\langle a, b \rangle \in R$, and $\langle a', b' \rangle \in R$, then $\langle a \wedge a', b \wedge b' \rangle \in R$. Continuing this reasoning, it's not difficult to see that congruences, as defined earlier, are precisely the equivalence relations in the category **Bool** of Boolean algebras. Thus, in the remainder of this chapter, when we speak of an equivalence relation on a Boolean algebra B , we mean an equivalence relation in **Bool** – in other words, a congruence. (To be clear, not every equivalence relation on the set B is an equivalence relation on the Boolean algebra B .)

Now suppose that \mathbf{C} is a category in which equivalence relations are definable, and let $p_0, p_1 : R \rightrightarrows B$ be an equivalence relation. (Here p_0 and p_1 are the projections of R , considered as a subobject of $B \times B$.) Then we can ask, do these two maps p_0 and p_1 have a coequalizer? That is, is there an object B/R , and a map $q : B \rightarrow B/R$, with the relevant universal property? In the case of **Bool**, a coequalizer can be constructed directly. We merely note that the Boolean operations on B can be used to induce Boolean operations on the set B/R of equivalence classes.

DEFINITION 3.5.12 (Quotient algebra) Suppose that R is an equivalence relation on B . For each $a \in B$, let E_a denote its equivalence class, and let $B/R = \{E_a \mid a \in B\}$. We then define $E_a \wedge E_b = E_{a \wedge b}$, and similarly for $E_a \vee E_b$ and $\neg E_a$. Since R is a congruence (i.e., an equivalence relation on **Bool**), these operations are well defined. It then follows immediately that B/R is a Boolean algebra, and the quotient map $q : B \rightarrow B/R$ is a surjective Boolean homomorphism.

LEMMA 3.5.13 Let $R \subseteq B \times B$ be an equivalence relation. Then $q : B \rightarrow B/R$ is the coequalizer of the projection maps $p_0 : R \rightarrow B$ and $p_1 : R \rightarrow B$. In particular, q is a regular epimorphism.

Proof It is obvious that $qp_0 = qp_1$. Now suppose that A is another Boolean algebra and $f : B \rightarrow A$ such that $fp_0 = fp_1$. Define $g : B/R \rightarrow A$ by setting $g(E_x) = f(x)$. Since $fp_0 = fp_1$, g is well defined. Furthermore,

$$g(E_x \wedge E_y) = g(E_{x \wedge y}) = f(x \wedge y) = f(x) \wedge f(y) = g(E_x) \wedge g(E_y).$$

Similarly, $g(\neg E_x) = \neg g(E_x)$. Therefore, g is a Boolean homomorphism. Since q is an epimorphism, g is the unique homomorphism such that $gq = f$. Therefore, $q : B \rightarrow B/R$ is the coequalizer of p_0 and p_1 . \square

The category **Bool** has further useful structure: there is a one-to-one correspondence between equivalence relations and filters.

LEMMA 3.5.14 *Suppose that $R \subseteq B \times B$ is an equivalence relation. Let $F = \{a \in B \mid aR1\}$. Then F is a filter, and $R = \{\langle a, b \rangle \in B \times B \mid a \leftrightarrow b \in F\}$.*

Proof Suppose that $a, b \in F$. That is, $aR1$ and $bR1$. Since R is a congruence, $(a \wedge b)R(1 \wedge 1)$ and, therefore, $(a \wedge b)R1$. That is, $a \wedge b \in F$. Now suppose that x is an arbitrary element of B such that $a \leq x$. That is, $x \vee a = x$. Since R is a congruence, $(x \vee a)R(x \vee 1)$ and so $(x \vee a)R1$, from which it follows that $xR1$. Therefore, $x \in F$, and F is a filter.

Now suppose that aRb . Since R is reflexive, $(a \vee \neg a)R1$, and, thus, $(b \vee \neg a)R1$. Similarly, $(a \vee \neg b)R1$, and, therefore, $(a \leftrightarrow b)R1$. That is, $a \leftrightarrow b \in F$. \square

LEMMA 3.5.15 *Suppose that F is a filter on B . Let $R = \{\langle a, b \rangle \in B \times B \mid a \leftrightarrow b \in F\}$. Then R is an equivalence relation, and $F = \{a \in B \mid aR1\}$.*

Proof Showing that R is an equivalence relation requires several straightforward verifications. For example, $a \leftrightarrow a = 1$, and $1 \in F$; therefore, aRa . We leave the remaining verifications to the reader.

Now suppose that $a \in F$. Since $a = (a \leftrightarrow 1)$, it follows that $a \leftrightarrow 1 \in F$, which means that $aR1$. \square

DEFINITION 3.5.16 (Quotient algebra) Let F be a filter on B . Given the correspondence between filters and equivalence relations, we write B/F for the corresponding algebra of equivalence classes.

PROPOSITION 3.5.17 *Let F be a proper filter on B . Then B/F is a two-element Boolean algebra if and only if F is an ultrafilter.*

Proof Suppose first that $B/F \cong 2$. That is, for any $a \in B$, either $a \leftrightarrow 1 \in F$ or $a \leftrightarrow 0 \in F$. But $a \leftrightarrow 1 = a$ and $a \leftrightarrow 0 = \neg a$. Therefore, either $a \in F$ or $\neg a \in F$, and F is an ultrafilter.

Suppose now that F is an ultrafilter. Then for any $a \in B$, either $a \in F$ or $\neg a \in F$. In the former case, $a \leftrightarrow 1 \in F$. In the latter case, $a \leftrightarrow 0 \in F$. Therefore, $B/F \cong 2$. \square

EXERCISE 3.5.18 (This exercise presupposes knowledge of measure theory.) Let Σ be the Boolean algebra of Borel subsets of $[0, 1]$, and let μ be Lebesgue measure on $[0, 1]$. Let $\mathcal{F} = \{S \in \Sigma \mid \mu(S) = 1\}$. Show that \mathcal{F} is a filter, and describe the equivalence relation on Σ corresponding to \mathcal{F} .

According to our motivating analogy, a Boolean algebra B is like a theory, and a homomorphism $\phi : B \rightarrow 2$ is like a model of this theory. We say that the algebra B is **syntactically consistent** just in case $0 \neq 1$. (In fact, we defined Boolean algebras so as to require syntactic consistency.) We say that the algebra B is **semantically consistent** just in case there is a homomorphism $\phi : B \rightarrow 2$. Then semantic consistency clearly implies syntactic consistency. But does syntactic consistency imply semantic consistency?

It's at this point that we have to invoke a powerful theorem – or, more accurately, a powerful set-theoretic axiom. In short, if we use the axiom of choice, or some equivalent such as Zorn's lemma, then we can prove that every syntactically consistent Boolean algebra is semantically consistent. However, we do not actually need the full power of the Axiom of Choice. As set-theorists know, the Boolean ultrafilter axiom (UF for short) is strictly weaker than the Axiom of Choice.

PROPOSITION 3.5.19 *The following are equivalent:*

1. **Boolean ultrafilter axiom (UF):** For any Boolean algebra B , there is a homomorphism $f : B \rightarrow 2$.
2. For any Boolean algebra B , and proper filter $F \subseteq B$, there is a homomorphism $f : B \rightarrow 2$ such that $f(a) = 1$ when $a \in F$.
3. For any Boolean algebra B , if $a, b \in B$ such that $a \neq b$, then there is a homomorphism $f : B \rightarrow 2$ such that $f(a) \neq f(b)$.
4. For any Boolean algebra B , if $\phi(a) = 1$ for all $\phi : B \rightarrow 2$, then $a = 1$.
5. For any two Boolean algebras A, B , and homomorphisms $f, g : A \rightarrow B$, if $\phi f = \phi g$ for all $\phi : B \rightarrow 2$, then $f = g$.

Proof (1 \Rightarrow 2) Suppose that F is a proper filter in B . Then there is a homomorphism $q : B \rightarrow B/F$ such that $q(a) = 1$ for all $a \in F$. By UF, there is a homomorphism $\phi : B/F \rightarrow 2$. Therefore, $\phi \circ q : B \rightarrow 2$ is a homomorphism such that $(\phi \circ q)(a) = 1$ for all $a \in F$.

(1 \Rightarrow 3) Suppose that $a, b \in B$ with $a \neq b$. Then either $\neg a \wedge b \neq 0$ or $a \wedge \neg b \neq 0$. Without loss of generality, we assume that $\neg a \wedge b \neq 0$. In this case, the filter F generated by $\neg a \wedge b$ is proper. By UF, there is a homomorphism $\phi : B \rightarrow 2$ such that $\phi(x) = 1$ when $x \in F$. In particular, $\phi(\neg a \wedge b) = 1$. But then $\phi(a) = 0$ and $\phi(b) = 1$.

(2 \Rightarrow 4) Suppose that $\phi(a) = 1$ for all $\phi : B \rightarrow 2$. Now let F be the filter generated by $\neg a$. If F is proper, then by (2), there is a $\phi : B \rightarrow 2$ such that $\phi(\neg a) = 1$, a contradiction. Thus, $F = B$, which implies that $\neg a = 0$ and $a = 1$.

(4 \Rightarrow 5) Let $f, g : A \rightarrow B$ be homomorphisms, and suppose that for all $\phi : B \rightarrow 2$, $\phi f = \phi g$. That is, for each $a \in A$, $\phi(f(a)) = \phi(g(a))$. But then $\phi(f(a) \leftrightarrow g(a)) = 1$ for all $\phi : B \rightarrow 2$. By (4), $f(a) \leftrightarrow g(a) = 1$ and, therefore, $f(a) = g(a)$.

(5 \Rightarrow 3) Let B be a Boolean algebra, and $a, b \in B$. Suppose that $\phi(a) = \phi(b)$ for all $\phi : B \rightarrow 2$. Let F be the four element Boolean algebra, with generator p . Then there is a homomorphism $\hat{a} : F \rightarrow B$ such that $\hat{a}(p) = a$, and a homomorphism $\hat{b} : F \rightarrow B$ such that $\hat{b}(p) = b$. Thus, $\phi \hat{a} = \phi \hat{b}$ for all $\phi : B \rightarrow 2$. By (5), $\hat{a} = \hat{b}$, and therefore $a = b$.

(3 \Rightarrow 1) Let B be an arbitrary Boolean algebra. Since $0 \neq 1$, (3) implies that there is a homomorphism $\phi : B \rightarrow 2$. \square

We are finally in a position to prove the completeness of the propositional calculus. The following result assumes the Boolean ultrafilter axiom (UF).

Completeness Theorem

If $T \models \phi$, then $T \vdash \phi$.

Proof Suppose that $T \not\models \phi$. Then in the Lindenbaum algebra $L(T)$, we have $E_\phi \neq 1$. In this case, there is a homomorphism $h : L(T) \rightarrow 2$ such that $h(E_\phi) = 0$. Hence, $h \circ i_T$ is a model of T such that $(h \circ i_T)(\phi) = h(E_\phi) = 0$. Therefore, $T \not\models \phi$. \square

EXERCISE 3.5.20 Let $\mathcal{P}N$ be the powerset of the natural numbers. We say that a subset E of N is **cofinite** just in case $N \setminus E$ is finite. Let $\mathcal{F} \subseteq \mathcal{P}N$ be the set of cofinite subsets of N . Show that \mathcal{F} is a filter, and show that there are infinitely many ultrafilters containing \mathcal{F} .

3.6 Stone Spaces

If we're going to undertake an exact study of "possible worlds," then we need to make a proposal about what structure this space carries. But what do I mean here by "structure"? Isn't the collection of possible worlds just a bare set? Let me give you a couple of reasons why it's better to think of possible worlds as forming a **topological space**.

Suppose that there are infinitely many possible worlds, which we represent by elements of a set X . As philosophers are wont to do, we then represent **propositions** by subsets of X . But should we think that all $2^{|X|}$ subsets of X correspond to genuine propositions? What would warrant such a claim?

There is another reason to worry about this approach. For a person with training in set theory, it is not difficult to build a collection C_1, C_2, \dots of subsets of X with the following features: (1) each C_i is nonempty, (2) $C_{i+1} \subseteq C_i$ for all i , and (3) $\bigcap_i C_i$ is empty. Intuitively speaking, $\{C_i \mid i \in \mathbb{N}\}$ is a family of propositions that are individually consistent (since nonempty) and that are becoming more and more specific, and yet there is no world in X that makes all C_i true. Why not? It seems that X is missing some worlds! Indeed, here's a description of a new world w that does not belong to X : for each proposition ϕ , let ϕ be true in w if and only if $\phi \cap C_i$ is nonempty for all i . It's not difficult to see that w is, in fact, a truth valuation on the set of all propositions – i.e., it is a possible world. But w is not represented by a point in X . What we have here is a mismatch between the set X of worlds and the set of propositions describing these worlds.

The idea behind logical topology is that not all subsets of X correspond to propositions. A designation of a topology on X is tantamount to saying which subsets of X correspond to propositions. However, the original motivation for the study of topology comes from geometry (and analysis), not from logic. Recall high school mathematics, where you learned that a continuous function is one where you don't have to lift your pencil from the paper in order to draw the graph. If your high school class was really good, or if you studied calculus in college, then you will have learned that there is a more rigorous definition of a continuous function – a definition involving epsilons and deltas. In the early twentieth century, it was realized that the essence of continuity is even more abstract than epsilons and deltas would suggest: all we need is a notion of nearness of points, which we can capture in terms of a notion of a neighborhood of a point. The idea then is that a function $f : X \rightarrow Y$ is continuous at a point x just in case for any neighborhood V of $f(x)$, there is some neighborhood U of x such that $f(U) \subseteq V$. Intuitively speaking, f preserves closeness of points.

Notice, however, that if X is an arbitrary set, then it's not obvious what "closeness" means. To be able to talk about closeness of points in X , we need specify which subsets of X count as the neighborhoods of points. Thus, a **topology** on X is a set of subsets of X that satisfies certain conditions.

DEFINITION 3.6.1 A **topological space** is a set X and a family \mathcal{F} of subsets of X satisfying the following conditions:

1. $\emptyset \in \mathcal{F}$ and $X \in \mathcal{F}$.
2. If $U, V \in \mathcal{F}$ then $U \cap V \in \mathcal{F}$.
3. If \mathcal{F}_0 is a subfamily of \mathcal{F} , then $\bigcup_{U \in \mathcal{F}_0} U \in \mathcal{F}$.

The sets in \mathcal{F} are called **open subsets** of the space (X, \mathcal{F}) . If $p \in U$ with U an open subset, we say that U is a **neighborhood** of p .

There are many familiar examples of topological spaces. In many cases, however, we only know the open sets indirectly, by means of certain nice open sets. For example, in the case of the real numbers, not every open subset is an interval. However, every open subset is a union of intervals. In that case, we call the open intervals in \mathbb{R} a **basis** for the topology.

PROPOSITION 3.6.2 Let \mathcal{B} be a family of subsets of X with the property that if $U, V \in \mathcal{B}$, then $U \cap V \in \mathcal{B}$. Then there is a unique smallest topology \mathcal{F} on X containing \mathcal{B} .

Proof Let \mathcal{F} be the collection obtained by taking all unions of sets in \mathcal{B} , and then taking finite intersections of the resulting collection. Clearly \mathcal{F} is a topology on X , and any topology on X containing \mathcal{B} also contains \mathcal{F} . \square

DEFINITION 3.6.3 If \mathcal{B} is a family of subsets of X that is closed under intersection, and if \mathcal{F} is the topology generated by \mathcal{B} , then we say that \mathcal{B} is a **basis** for \mathcal{F} .

PROPOSITION 3.6.4 Let (X, \mathcal{F}) be a topological space. Let \mathcal{F}_0 be a subfamily of \mathcal{F} with the following properties: (1) \mathcal{F}_0 is closed under finite intersections, and (2) for

each $x \in X$ and $U \in \mathcal{F}_0$ with $x \in U$, there is a $V \in \mathcal{F}_0$ such that $x \in V \subseteq U$. Then \mathcal{F}_0 is a basis for the topology \mathcal{F} .

Proof We need only show that each $U \in \mathcal{F}$ is a union of elements in \mathcal{F}_0 . And that follows immediately from the fact that if $x \in U$, then there is $V \in \mathcal{F}_0$ with $x \in V \subseteq U$. \square

DEFINITION 3.6.5 Let X be a topological space. A subset C of X is called **closed** just in case $C = X \setminus U$ for some open subset U of X . The intersection of closed sets is closed. Hence, for each subset E of X , there is a unique smallest closed set \overline{E} containing E , namely the intersection of all closed supersets of E . We call \overline{E} the **closure** of E .

PROPOSITION 3.6.6 Let $p \in X$ and let $S \subseteq X$. Then $p \in \overline{S}$ if and only if every open neighborhood U of p has nonempty intersection with S .

Proof Exercise. \square

DEFINITION 3.6.7 Let S be a subset of X . We say that S is **dense** in X just in case $\overline{S} = X$.

DEFINITION 3.6.8 Let $E \subseteq X$. We say that p is a **limit point** of E just in case for each open neighborhood U of p , $U \cap E$ contains some point besides p . We let E' denote the set of all limit points of E .

LEMMA 3.6.9 $E' \subseteq \overline{E}$.

Proof Let $p \in E'$, and let C be a closed set containing E . If $p \in X \setminus C$, then p is contained in an open set that has empty intersection with E . Thus, $p \in C$. Since C was an arbitrary closed superset of E , it follows that $p \in \overline{E}$. \square

PROPOSITION 3.6.10 $\overline{E} = E \cup E'$.

Proof The previous lemma gives $E' \subseteq \overline{E}$. Thus, $E \cup E' \subseteq \overline{E}$.

Suppose now that $p \notin E$ and $p \notin E'$. Then there is an open neighborhood U of p such that $U \cap E$ is empty. Then $E \subseteq X \setminus U$, and since $X \setminus U$ is closed, $\overline{E} \subseteq X \setminus U$. Therefore, $p \notin \overline{E}$. \square

DEFINITION 3.6.11 A topological space X is said to be

- T_1 , or **Frechet**, just in case all singleton subsets are closed.
- T_2 , or **Hausdorff**, just in case, for any $x, y \in X$, if $x \neq y$, then there are disjoint open neighborhoods of x and y .
- T_3 , or **regular**, just in case for each $x \in X$, and for each closed $C \subseteq X$ such that $x \notin C$, there are open neighborhoods U of x , and V of C , such that $U \cap V = \emptyset$.
- T_4 , or **normal**, just in case any two disjoint closed subsets of X can be separated by disjoint open sets.

Clearly we have the implications

$$(T_1 + T_4) \Rightarrow (T_1 + T_3) \Rightarrow T_2 \Rightarrow T_1.$$

A discrete space satisfies all of the separation axioms. A nontrivial indiscrete space satisfies none of the separation axioms. A useful heuristic here is that the stronger the separation axiom, the closer the space is to discrete. In this book, most of the spaces we consider are very close to discrete (which means that all subsets are open).

EXERCISE 3.6.12

1. Show that X is regular iff for each $x \in X$ and open neighborhood U of x , there is an open neighborhood V of x such that $\overline{V} \subseteq U$.
2. Show that if $E \subseteq F$, then $\overline{E} \subseteq \overline{F}$.
3. Show that $\overline{\overline{E}} = \overline{E}$.
4. Show that the intersection of two topologies is a topology.
5. Show that the infinite distributive law holds:

$$U \cap \left(\bigcup_{i \in I} V_i \right) = \bigcup_{i \in I} (U \cap V_i).$$

DEFINITION 3.6.13 Let $S \subseteq X$. A family \mathcal{C} of open subsets of X is said to **cover** S just in case $S \subseteq \bigcup_{U \in \mathcal{C}} U$. We say that S is **compact** just in case for every open cover \mathcal{C} of S , there is a finite subcollection \mathcal{C}_0 of \mathcal{C} that also covers S . We say that the space X is compact just in case it's compact as a subset of itself.

DEFINITION 3.6.14 A collection \mathcal{C} of subsets of X is said to satisfy the **finite intersection property** if for every finite subcollection C_1, \dots, C_n of \mathcal{C} , the intersection $C_1 \cap \dots \cap C_n$ is nonempty.

DISCUSSION 3.6.15 Suppose that X is the space of possible worlds, so that we can think of subsets of X as propositions. If $A \cap B$ is nonempty, then the propositions A and B are consistent – i.e., there is a world in which they are both true. Thus, a collection \mathcal{C} of propositions has the finite intersection property just in case it is finitely consistent.

Recall that compactness of propositional logic states that if a set \mathcal{C} of propositions is finitely consistent, then \mathcal{C} is consistent. The terminology here is no accident; a topological space is compact just in case finite consistency entails consistency.

PROPOSITION 3.6.16 A space X is compact if and only if for every collection \mathcal{C} of closed subsets of X , if \mathcal{C} satisfies the finite intersection property, then $\bigcap \mathcal{C}$ is nonempty.

Proof (\Rightarrow) Assume first that X is compact, and let \mathcal{C} be a family of closed subsets of X . We will show that if \mathcal{C} satisfies the finite intersection property, then the intersection of all sets in \mathcal{C} is nonempty. Assume the negation of the consequent, i.e., that $\bigcap_{C \in \mathcal{C}} C$ is empty. Let $\mathcal{C}' = \{C' : C \in \mathcal{C}\}$, where $C' = X \setminus C$ is the complement of C in X . (Warning: this notation can be confusing. Previously we used E' to denote the set of limit points of E . This C' has nothing to do with limit points.) Each C' is open, and

$$\left(\bigcup_{C \in \mathcal{C}} C' \right)' = \bigcap_{C \in \mathcal{C}} C,$$

which is empty. It follows then that \mathcal{C}' is an open cover of X . Since X is compact, there is a finite subcover \mathcal{C}'_0 of \mathcal{C}' . If we let \mathcal{C}_0 be the complements of sets in \mathcal{C}'_0 , then \mathcal{C}_0 is a finite collection of sets in \mathcal{C} whose intersection is empty. Therefore, \mathcal{C} does not satisfy the finite intersection property.

(\Leftarrow) Assume now that X is not compact. In particular, suppose that \mathcal{U} is an open cover with no finite subcover. Let $\mathcal{C} = \{X \setminus U \mid U \in \mathcal{U}\}$. For any finite subcollection $X \setminus U_1, \dots, X \setminus U_n$ of \mathcal{C} , we have

$$U_1 \cup \dots \cup U_n \neq X,$$

and hence

$$(X \setminus U_1) \cap \dots \cap (X \setminus U_n) \neq \emptyset.$$

Thus, \mathcal{C} has the fip. Nonetheless, since \mathcal{U} covers X , the intersection of all sets in \mathcal{C} is empty. \square

PROPOSITION 3.6.17 *In a compact space, closed subsets are compact.*

Proof Let \mathcal{C} be an open cover of S , and consider the cover $\mathcal{C}' = \mathcal{C} \cup \{X \setminus S\}$ of X . Since X is compact, there is a finite subcover \mathcal{C}'_0 of \mathcal{C}' . Removing $X \setminus S$ from \mathcal{C}'_0 gives a finite subcover of the original cover \mathcal{C} of S . \square

PROPOSITION 3.6.18 *Suppose that X is compact, and let U be an open set in X . Let $\{F_i\}_{i \in I}$ be a family of closed subsets of X such that $\bigcap_{i \in I} F_i \subseteq U$. Then there is a finite subset J of I such that $\bigcap_{i \in J} F_i \subseteq U$.*

Proof Let $C = X \setminus U$, which is closed. Thus, the hypotheses of the proposition say that the family $\mathcal{C} := \{C\} \cup \{F_i : i \in I\}$ has empty intersection. Since X is compact, \mathcal{C} also fails to have the finite intersection property. That is, there are $i_1, \dots, i_k \in I$ such that $C \cap F_{i_1} \cap \dots \cap F_{i_k} = \emptyset$. Therefore, $F_{i_1} \cap \dots \cap F_{i_k} \subseteq U$. \square

PROPOSITION 3.6.19 *If X is compact Hausdorff, then X is regular.*

Proof Let $x \in X$, and let $C \subseteq X$ be closed. For each $y \in C$, let U_y be an open neighborhood of x , and V_y an open neighborhood of y such that $U_y \cap V_y = \emptyset$. The V_y form an open cover of C . Since C is closed and X is compact, C is compact. Hence, there is a finite subcollection V_{y_1}, \dots, V_{y_n} that cover C . But then $U = \bigcap_{i=1}^n U_{y_i}$ is an open neighborhood of x , and $V = \bigcup_{i=1}^n V_{y_i}$ is an open neighborhood of C , such that $U \cap V = \emptyset$. Therefore, X is regular. \square

PROPOSITION 3.6.20 *In Hausdorff spaces, compact subsets are closed.*

Proof Let p be a point of X that is not in K . Since X is Hausdorff, for each $x \in K$, there are open neighborhoods U_x of x and V_x of p such that $U_x \cap V_x = \emptyset$. The family $\{U_x : x \in K\}$ covers K . Since K is compact, it is covered by a finite subcollection U_{x_1}, \dots, U_{x_n} . But then $\bigcap_{i=1}^n V_{x_i}$ is an open neighborhood of p that is disjoint from K . It follows that $X \setminus K$ is open, and K is closed. \square

DEFINITION 3.6.21 Let X, Y be topological spaces. A function $f : X \rightarrow Y$ is said to be **continuous** just in case for each open subset U of Y , $f^{-1}(U)$ is an open subset of X .

Example 3.6.22 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function that is constantly 0 on $(-\infty, 0)$, and 1 on $[0, \infty)$. Then f is not continuous: $f^{-1}(\frac{1}{2}, \frac{3}{2}) = [0, \infty)$, which is not open. \square

In the exercises, you will show that a function f is continuous if and only if $f^{-1}(C)$ is closed whenever C is closed. Thus, in particular, if C is a clopen subset of Y , then $f^{-1}(C)$ is a clopen subset of X .

PROPOSITION 3.6.23 Let **Top** consist of the class of topological spaces and continuous maps between them. For $X \xrightarrow{f} Y \xrightarrow{g} Z$, define $g \circ f$ to be the composition of g and f . Then **Top** is a category.

Proof It needs to be confirmed that if f and g are continuous, then $g \circ f$ is continuous. We leave this to the exercises. Since composition is associative, **Top** is a category. \square

PROPOSITION 3.6.24 Suppose that $f : X \rightarrow Y$ is continuous. If K is compact in X , then $f(K)$ is compact in Y .

Proof Let \mathcal{G} be a collection of open subsets of Y that covers $f(K)$. Let

$$\mathcal{G}' = \{f^{-1}(U) : U \in \mathcal{G}\}.$$

When \mathcal{G}' is an open cover of K . Since K is compact, \mathcal{G}' has a finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_n)$. But then U_1, \dots, U_n is a finite subcover of \mathcal{G} . \square

We remind the reader of the category theoretic definitions:

- f is a **monomorphism** just in case $fh = fk$ implies $h = k$.
- f is an **epimorphism** just in case $hf = kf$ implies $h = k$.
- f is an **isomorphism** just in case there is a $g : Y \rightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$.

For historical reasons, isomorphisms in **Top** are usually called **homeomorphisms**. It is easy to show that a continuous map $f : X \rightarrow Y$ is monic if and only if f is injective. It is also true that $f : X \rightarrow Y$ is epi if and only if f is surjective (but the proof is somewhat subtle). In contrast, a continuous bijection is not necessarily an isomorphism in **Top**. For example, if we let X be a two-element set with the discrete topology, and if we let Y be a two-element set with the indiscrete topology, then any bijection $f : X \rightarrow Y$ is continuous but is not an isomorphism.

EXERCISE 3.6.25

1. Show that if f and g are continuous, then $g \circ f$ is continuous.
2. Suppose that $f : X \rightarrow Y$ is a surjection. Show that if E is dense in X , then $f(E)$ is dense in Y .
3. Show that $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(C)$ is closed whenever C is closed.
4. Let Y be a Hausdorff space, and let $f, g : X \rightarrow Y$ be continuous. Show that if f and g agree on a dense subset of X , then $f = g$.

EXERCISE 3.6.26 Show that $f^{-1}(V) \subseteq U$ if and only if $V \subseteq Y \setminus f(X \setminus U)$.

DEFINITION 3.6.27 A continuous mapping $f : X \rightarrow Y$ is said to be **closed** just in case for every closed set $C \subseteq X$, the image $f(C)$ is closed in Y . Similarly, $f : X \rightarrow Y$ is said to be **open** just in case for every open set $U \subseteq X$, the image $f(U)$ is open in Y .

PROPOSITION 3.6.28 Let $f : X \rightarrow Y$ be continuous. Then the following are equivalent.

1. f is closed.
2. For every open set $U \subseteq X$, the set $\{y \in Y \mid f^{-1}\{y\} \subseteq U\}$ is open.
3. For every $y \in Y$, and every neighborhood U of $f^{-1}\{y\}$, there is a neighborhood V of y such that $f^{-1}(V) \subseteq U$.

Proof (2 \Leftrightarrow 3) The equivalence of (2) and (3) is straightforward, and we leave its proof as an exercise.

(3 \Rightarrow 1) Suppose that f satisfies condition (3), and let C be a closed subset of X . To show that $f(C)$ is closed, assume that $y \in Y \setminus f(C)$. Then $f^{-1}\{y\} \subseteq X \setminus C$. Since $X \setminus C$ is open, there is a neighborhood U of $f^{-1}\{y\}$ such that $f^{-1}(U) \subseteq X \setminus C$. Then

$$U \subseteq Y \setminus f(X \setminus U) = Y \setminus f(C).$$

Since y was an arbitrary element of $Y \setminus f(C)$, it follows that $Y \setminus f(C)$ is open, and $f(C)$ is closed.

(1 \Rightarrow 3) Suppose that f is closed. Let $y \in Y$, and let U be a neighborhood of $f^{-1}\{y\}$. Then $X \setminus U$ is closed, and $f(X \setminus U)$ is also closed. Let $V = Y \setminus f(X \setminus U)$. Then V is an open neighborhood of y and $f^{-1}(V) \subseteq U$. \square

PROPOSITION 3.6.29 Suppose that X and Y are compact Hausdorff. If $f : X \rightarrow Y$ is continuous, then f is a closed map.

Proof Let B be a closed subset of X . By Proposition 3.6.17, B is compact. By Proposition 3.6.24, $f(B)$ is compact. And by Proposition 3.6.20, $f(B)$ is closed. Therefore, f is a closed map. \square

PROPOSITION 3.6.30 Suppose that X and Y are compact Hausdorff. If $f : X \rightarrow Y$ is a continuous bijection, then f is an isomorphism.

Proof Let $f : X \rightarrow Y$ be a continuous bijection. Thus, there is function $g : Y \rightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$. We will show that g is continuous. By Proposition 3.6.29, f is closed. Moreover, for any closed subset B of X , we have $g^{-1}(B) = f(B)$. Thus, g^{-1} preserves closed subsets, and hence g is continuous. \square

DEFINITION 3.6.31 A topological space X is said to be **totally separated** if for any $x, y \in X$, if $x \neq y$ then there is a closed and open (clopen) subset of X containing x but not y .

DEFINITION 3.6.32 We say that X is a **Stone space** if X is compact and totally separated. We let **Stone** denote the full subcategory of **Top** consisting of Stone spaces.

To say that **Stone** is a full subcategory means that the arrows between two Stone spaces X and Y are just the arrows between X and Y considered as topological spaces – i.e., continuous functions.

NOTE 3.6.33 Let E be a clopen subset of X . Then there is a continuous function $f : X \rightarrow \{0, 1\}$ such that $f(x) = 1$ for $x \in E$, and $f(x) = 0$ for $x \in X \setminus E$. Here we are considering $\{0, 1\}$ with the discrete topology.

PROPOSITION 3.6.34 Let X and Y be Stone spaces. If $f : X \rightarrow Y$ is an epimorphism, then f is surjective.

Proof Suppose that f is not surjective. Since X is compact, the image $f(X)$ is compact in Y , hence closed. Since f is not surjective, there is a $y \in Y \setminus f(X)$. Since Y is a regular space, there is a clopen neighborhood U of y such that $U \cap f(X) = \emptyset$. Define $g : Y \rightarrow \{0, 1\}$ to be constantly 0. Define $h : Y \rightarrow \{0, 1\}$ to be 1 on U , and 0 on $Y \setminus U$. Then $g \circ f = h \circ f$, but $g \neq h$. Therefore, f is not an epimorphism. \square

PROPOSITION 3.6.35 Let X and Y be Stone spaces. If $f : X \rightarrow Y$ is both a monomorphism and an epimorphism, then f is an isomorphism.

Proof By Proposition 3.6.34, f is surjective. Therefore, f is a continuous bijection. By Proposition 3.6.30, f is an isomorphism. \square

3.7 Stone Duality

In this section, we show that the category **Bool** is dual to the category **Stone** of Stone spaces. To say that categories are “dual” means that the first is equivalent to the mirror image of the second.

DEFINITION 3.7.1 We say that categories **C** and **D** are **dual** just in case there are contravariant functors $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{D} \rightarrow \mathbf{C}$ such that $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$. To see that this definition makes sense, note that if F and G are contravariant functors, then GF and FG are covariant functors. If **C** and **D** are dual, we write $\mathbf{C} \cong \mathbf{D}^{op}$, to indicate that **C** is equivalent to the opposite category of **D** – i.e., the category that has the same objects as **D**, but arrows running in the opposite direction.

The Functor from Bool to Stone

We now define a contravariant functor $S : \mathbf{Bool} \rightarrow \mathbf{Stone}$. For reasons that will become clear later, the functor S is sometimes called the **semantic functor**.

Consider the set $\text{hom}(B, 2)$ of two-valued homomorphisms of the Boolean algebra B . For each $a \in B$, define

$$C_a = \{\phi \in \text{hom}(B, 2) \mid \phi(a) = 1\}.$$

Clearly, the family $\{C_a \mid a \in B\}$ forms a basis for a topology on $\text{hom}(B, 2)$. We let $S(B)$ denote the resulting topological space. Note that $S(B)$ has a basis of clopen sets. Thus, if $S(B)$ is compact, then $S(B)$ is a Stone space.

LEMMA 3.7.2 *If B is a Boolean algebra, then $S(B)$ is a Stone space.*

Proof Let $\mathcal{B} = \{C_a \mid a \in B\}$ denote the chosen basis for the topology on $S(B)$. To show that $S(B)$ is compact, it will suffice to show that for any subfamily \mathcal{C} of \mathcal{B} , if \mathcal{C} has the finite intersection property, then $\bigcap \mathcal{C}$ is nonempty. Now let F be the set of $b \in B$ such that

$$C_{a_1} \cap \cdots \cap C_{a_n} \subseteq C_b,$$

for some $C_{a_1}, \dots, C_{a_n} \in \mathcal{C}$. Since \mathcal{C} has the finite intersection property, F is a filter in B . Thus, UF entails that F is contained in an ultrafilter U . This ultrafilter U corresponds to a $\phi : B \rightarrow 2$, and we have $\phi(a) = 1$ whenever $C_a \in \mathcal{C}$. In other words, $\phi \in C_a$, whenever $C_a \in \mathcal{C}$. Therefore, $\bigcap \mathcal{C}$ is nonempty, and $S(B)$ is compact. \square

Let $f : A \rightarrow B$ be a homomorphism, and let $S(f) : S(B) \rightarrow S(A)$ be given by $S(f) = \text{hom}(f, 2)$; that is,

$$S(f)(\phi) = \phi \circ f, \quad \forall \phi \in S(B).$$

We claim now that $S(f)$ is a continuous map. Indeed, for any basic open subset C_a of $S(A)$, we have

$$S(f)^{-1}(C_a) = \{\phi \in S(B) \mid \phi(f(a)) = 1\} = C_{f(a)}. \quad (3.1)$$

It is straightforward to verify that $S(1_A) = 1_{S(A)}$, and that $S(g \circ f) = S(f) \circ S(g)$. Therefore, $S : \mathbf{Bool} \rightarrow \mathbf{Stone}$ is a contravariant functor.

The Functor from Stone to Bool

Let X be a Stone space. Then the set $K(X)$ of clopen subsets of X is a Boolean algebra, and is a basis for the topology on X . We now show that K is the object part of a contravariant functor $K : \mathbf{Stone} \rightarrow \mathbf{Bool}$. For reasons that will become clear later, K is sometimes called the **syntactic functor**.

Indeed, if X, Y are Stone spaces, and $f : X \rightarrow Y$ is continuous, then for each clopen subset U of Y , $f^{-1}(U)$ is a clopen subset of X . Moreover, f^{-1} preserves union, intersection, and complement of subsets; thus $f^{-1} : K(Y) \rightarrow K(X)$ is a Boolean homomorphism. We define the mapping K on arrows by $K(f) = f^{-1}$. Obviously, $K(1_X) = 1_{K(X)}$, and $K(g \circ f) = K(f) \circ K(g)$. Therefore, K is a contravariant functor.

Now we will show that KS is naturally isomorphic to the identity on \mathbf{Bool} , and SK is naturally isomorphic to the identity on \mathbf{Stone} . For each Boolean algebra B , define $\eta_B : B \rightarrow KS(B)$ by

$$\eta_B(a) = C_a = \{\phi \in S(B) \mid \phi(a) = 1\}.$$

LEMMA 3.7.3 *The map $\eta_B : B \rightarrow KS(B)$ is an isomorphism of Boolean algebras.*

Proof We first verify that $a \mapsto C_a$ is a Boolean homomorphism. For $a, b \in B$, we have

$$\begin{aligned}
C_{a \wedge b} &= \{\phi \mid \phi(a \wedge b) = 1\} \\
&= \{\phi \mid \phi(a) = 1 \text{ and } \phi(b) = 1\} \\
&= C_a \wedge C_b.
\end{aligned}$$

A similar calculation shows that $C_{\neg a} = X \setminus C_a$. Therefore, $a \mapsto C_a$ is a Boolean homomorphism.

To show that $a \mapsto C_a$ is injective, it will suffice to show that $C_a = \emptyset$ only if $a = 0$. In other words, it will suffice to show that for each $a \in B$, if $a \neq 0$ then there is some $\phi : B \rightarrow 2$ such that $\phi(a) = 1$. Thus, the result follows from UF.

Finally, to see that η_B is surjective, let U be a clopen subset of $S(B)$. Since U is open, $U = \bigcup_{a \in I} C_a$, for some subset I of B . Since U is closed in the compact space $G(B)$, it follows that U is compact. Thus, there is a finite subset F of B such that $U = \bigcup_{a \in F} C_a$. And since $a \mapsto C_a$ is a Boolean homomorphism, $\bigcup_{a \in F} C_a = C_b$, where $b = \bigvee_{a \in F} a$. Therefore, η_B is surjective. \square

LEMMA 3.7.4 *The family of maps $\{\eta_A : A \rightarrow KS(A)\}$ is natural in A .*

Proof Suppose that A and B are Boolean algebras and that $f : A \rightarrow B$ is a Boolean homomorphism. Consider the following diagram:

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \eta_A & & \downarrow \eta_B \\
KS(A) & \xrightarrow{KS(f)} & KS(B)
\end{array}$$

For $a \in A$, we have $\eta_B(f(a)) = C_{f(a)}$, and $\eta_A(a) = C_a$. Furthermore,

$$KS(f)(C_a) = S(f)^{-1}(C_a) = C_{f(a)},$$

by Eqn. 3.1. Therefore, the diagram commutes, and η is a natural transformation. \square

Now we define a natural isomorphism $\theta : \mathbf{1}_S \Rightarrow SK$. For a Stone space X , $K(X)$ is the Boolean algebra of clopen subsets of X , and $SK(X)$ is the Stone space of $K(X)$. For each point $\phi \in X$, let $\hat{\phi} : K(X) \rightarrow 2$ be defined by

$$\hat{\phi}(C) = \begin{cases} 1 & \phi \in C, \\ 0 & \phi \notin C. \end{cases}$$

It's straightforward to verify that $\hat{\phi}$ is a Boolean homomorphism. We define $\theta_X : X \rightarrow SK(X)$ by $\theta_X(\phi) = \hat{\phi}$.

LEMMA 3.7.5 *The map $\theta_X : X \rightarrow SK(X)$ is a homeomorphism of Stone spaces.*

Proof It will suffice to show that θ_X is bijective and continuous. (Do you remember why? Hint: Stone spaces are compact Hausdorff.) To see that θ_X is injective, suppose that ϕ and ψ are distinct elements of X . Since X is a Stone space, there is a clopen set U of X such that $\phi \in U$ and $\psi \notin U$. But then $\hat{\phi} \neq \hat{\psi}$. Thus, θ_X is injective.

To see that θ_X is surjective, let $h : K(X) \rightarrow 2$ be a Boolean homomorphism. Let

$$\mathcal{C} = \{C \in K(X) \mid h(C) = 1\}.$$

In particular $X \in \mathcal{C}$; and since h is a homomorphism, \mathcal{C} has the finite intersection property. Since X is compact, $\bigcap \mathcal{C}$ is nonempty. Let ϕ be a point in $\bigcap \mathcal{C}$. Then for any $C \in K(X)$, if $h(C) = 1$, then $C \in \mathcal{C}$ and $\phi \in C$, from which it follows that $\hat{\phi}(C) = 1$. Similarly, if $h(C) = 0$ then $X \setminus C \in \mathcal{C}$, and $\hat{\phi}(C) = 0$. Thus, $\theta_X(\phi) = \hat{\phi} = h$, and θ_X is surjective.

To see that θ_X is continuous, note that each basic open subset of $SK(X)$ is of the form

$$\hat{C} = \{h : K(X) \rightarrow 2 \mid h(C) = 1\},$$

for some $C \in K(X)$. Moreover, for any $\phi \in X$, we have $\hat{\phi} \in \hat{C}$ iff $\hat{\phi}(C) = 1$ iff $\phi \in C$. Therefore,

$$\theta_X^{-1}(\hat{C}) = \{\phi \in X \mid \hat{\phi}(C) = 1\} = C.$$

Therefore, θ_X is continuous. □

LEMMA 3.7.6 *The family of maps $\{\theta_X : X \rightarrow SK(X)\}$ is natural in X .*

Proof Let X, Y be Stone spaces, and let $f : X \rightarrow Y$ be continuous. Consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \theta_X & & \downarrow \theta_Y \\ SK(X) & \xrightarrow{SK(f)} & SK(Y) \end{array}$$

For arbitrary $\phi \in X$, we have $(\theta_Y \circ f)(\phi) = \widehat{f(\phi)}$. Furthermore,

$$SK(f) = \text{hom}(K(f), 2) = \text{hom}(f^{-1}, 2),$$

In other words, for a homomorphism $h : K(X) \rightarrow 2$, we have

$$SK(f)(h) = h \circ f^{-1}.$$

In particular, $SK(f)(\hat{\phi}) = \hat{\phi} \circ f^{-1}$. For any $C \in K(Y)$, we have

$$(\hat{\phi} \circ f^{-1})(C) = \begin{cases} 1 & f(\phi) \in C, \\ 0 & f(\phi) \notin C. \end{cases}$$

That is, $\hat{\phi} \circ f^{-1} = \widehat{f(\phi)}$. Therefore, the diagram commutes, and θ is a natural isomorphism. □

This completes the proof that K and S are quasi-inverse, and yields the famous theorem:

Stone Duality Theorem

The categories **Stone** and **Bool** are dual to each other. In particular, any Boolean algebra B is isomorphic to the field of clopen subsets of its state space $S(B)$.

PROPOSITION 3.7.7 *Let $A \subseteq B$, and let $a \in B$. Then the following are equivalent:*

1. *For any states f and g of B , if $f|_A = g|_A$ then $f(a) = g(a)$.*
2. *If h is a state of A , then any two extensions of h to B agree on a .*
3. *$a \in A$.*

Proof Since every state of A can be extended to a state of B , (1) and (2) are obviously equivalent. Furthermore, (3) obviously implies (1). Thus, we only need to show that (1) implies (3).

Let $m : A \rightarrow B$ be the inclusion of A in B , and let $s : S(B) \rightarrow S(A)$ be the corresponding surjection of states. We need to show that $C_a = s^{-1}(U)$ for some clopen subset U of $S(A)$.

By (1), for any $x \in S(A)$, either $s^{-1}\{x\} \subseteq C_a$ or $s^{-1}\{x\} \subseteq C_{-a}$. By Proposition 3.6.29, s is a closed map. Since C_a is open, Proposition 3.6.28 entails that the sets,

$$U = \{x \in S(B) \mid s^{-1}\{x\} \subseteq C_a\}, \quad \text{and} \quad V = \{x \in S(B) \mid s^{-1}\{x\} \subseteq C_{-a}\},$$

are open. Since $U = S(A) \setminus V$, it follows that U is clopen. Finally, it's clear that $s^{-1}(U) = C_a$. \square

PROPOSITION 3.7.8 *In **Bool**, epimorphisms are surjective.*

Proof Suppose that $f : A \rightarrow B$ is not surjective. Then $f(A)$ is a proper subalgebra of B . By Proposition 3.7.7, there are states g, h of B such that $g \neq h$, but $g|_{f(A)} = h|_{f(A)}$. In other words, $g \circ f = h \circ f$, and f is not an epimorphism. \square

Combining the previous two theorems, we have the following equivalences:

$$\mathbf{Th} \cong \mathbf{Bool} \cong \mathbf{Stone}^{op}.$$

We will now exploit these equivalences to explore the structure of the category of theories.

PROPOSITION 3.7.9 *Let T be a propositional theory in a countable signature. Then there is a conservative translation $f : T \rightarrow T_0$, where T_0 is an empty theory – i.e., a theory with no axioms.*

Proof After proving the above equivalences, we have several ways of seeing why this result is true. In terms of Boolean algebras, the proposition says that every countable Boolean algebra is embeddable into the free Boolean algebra on a countable number of generators (i.e., the Boolean algebra of clopen subsets of the Cantor space). That well-known result follows from the fact that Boolean algebras are always generated by their finite subalgebras. (In categorical terms, every Boolean algebra is a filtered colimit of finite Boolean algebras.)

In terms of Stone spaces, the proposition says that for every separable Stone space Y , there is a continuous surjection $p : X \rightarrow Y$, where X is the Cantor space. That fact is well known to topologists. One interesting proof uses the fact that a Stone space Y is *profinite* – i.e., Y is a limit of finite Hausdorff (hence discrete) spaces. One then shows that the Cantor space X has enough surjections onto discrete spaces and lifts

these up to a surjection $p : X \rightarrow Y$. See, for example, Ribes and Zalesskii (2000). Or, for a more direct argument: each clopen subset U of Y corresponds to a continuous map $p_U : Y \rightarrow \{0, 1\}$. There are countably many such clopen subsets of Y . Since $X \simeq \prod_{i \in \mathbb{N}} \{0, 1\}$, these p_U induce a continuous function $p : X \rightarrow Y$. Moreover, since every point $y \in Y$ has a neighborhood basis of clopen sets, p is surjective. \square

DISCUSSION 3.7.10 (Quine on eliminating posulates) It's no surprise that one can be charitable to a fault. Suppose that I am a theist, and you are an extremely charitable atheist. You are so charitable that you want to affirm the things I say. Here's how you can do it: when I say "God," assume that I really mean "kittens." Then when I say "God exists," you can interpret me to be saying "kittens exist." Then you can smile and say "I completely agree!"

Proposition 3.7.9 provides a general recipe for charitable interpretation. Imagine that I accept a theory T , which might be controversial. Imagine that you, on the other hand, like to play it safe: you only accept tautologies, viz. empty theory T_0 . The previous proposition shows that there is a conservative translation $f : T \rightarrow T_0$. In other words, you can reinterpret my sentences in such a way that everything I say comes out as true by your lights – i.e., true by logic alone.

Since we're dealing merely with propositional logic, this result might not seem very provocative. However, a directly analogous result – proven by Quine and Goodman (1940); Quine (1964) – was thought to refute the analytic–synthetic distinction that was central to the logical positivist program. Quine's argument runs as follows: suppose that T is intended to represent a contingently true theory, such as (presumably) quantum mechanics or evolutionary biology. By making a series of clever definitions, the sentences of T can be reconstrued as tautologies. That is, any contingently true theory T can be reconstrued so that all of its claims come out as true by definition.

What we see here is an early instance of a strategy that Quine was to use again and again throughout his philosophical career. There is a supposedly important distinction in a theory T . Quine shows that this distinction doesn't survive translation of T into some other theory T_0 . This result, Quine claims, shows that the distinction must be rejected.

Whether or not Quine's strategy is generally good, we should be a bit suspicious in the present case. The translation $f : T \rightarrow T_0$ is not an *equivalence* of theories – i.e., it does not show that T is equivalent to T_0 . Since f is conservative, it does show a sense in which T is *embeddable in* or *reducible to* T_0 . But we are left wondering: why should the existence of a formal relation $f : T \rightarrow T_0$ undercut the importance of the distinctions that are made within T ?

If Proposition 3.7.9 was surprising, then the following result is even more surprising:

PROPOSITION 3.7.11 *Let T be a consistent propositional theory in a countably infinite signature. If T has a finite number of axioms, then T is equivalent to the empty theory T_0 .*

Sketch of proof Suppose that T has a finite number of axioms. Without loss of generality, we assume that T has a single axiom ϕ . Let X be the Cantor space – i.e., the Stone space of the empty theory T_0 . Let $U_\phi \subseteq X$ be the clopen subset of all models in which

ϕ is true. Then U_ϕ is homeomorphic to the Stone space of T . Assume for the moment any nonempty clopen subset of the Cantor space is homeomorphic to the Cantor space. In that case, U_ϕ is homeomorphic to the Cantor space X ; and by Stone duality, T is equivalent to T_0 .

We now argue that nonempty clopen subset of the Cantor space is homeomorphic to the Cantor space. (This result admits of several proofs, some more topologically illuminating than the one we give here.) We begin by arguing that if ϕ is a conjunction of literals (atomic or negated atomic sentences), then U_ϕ is homeomorphic to the Cantor space. Indeed, there is a direct proof that the theory $\{\phi\}$ is equivalent to the empty theory; hence, by Stone duality, U_ϕ is homeomorphic to X . Now, an arbitrary clopen subset U of X has the form U_ϕ for some sentence ϕ . We may rewrite ϕ in disjunctive normal form – i.e., as a finite disjunction of conjunctions of literals. Thus, U_ϕ is a disjoint union of $U_{\phi_1}, U_{\phi_2}, \dots, U_{\phi_n}$. By the previous argument, each U_{ϕ_i} is homeomorphic to X , and a disjoint union of copies of X is also homeomorphic to X . \square

The previous proposition might suggest that the notion of equivalence we have adopted (Definition 1.4.6) is too *liberal* – i.e., that it counts too many theories as equivalent. If you think that's the case, we enjoin you to propose another criterion and explore its consequences.

DISCUSSION 3.7.12 The Stone duality theorem suggests that accepting a theory T involves accepting some claims about nearness/similarity relations among possible worlds. One theory T leads to a particular topological structure on the set of possible worlds, and another theory T' leads to a different topological structure on the set of possible worlds. That fact applies not just to propositional theories, but also to real-life scientific theories. For example, when one accepts the general theory of relativity, one doesn't simply believe that our universe is isomorphic to one of its models. Rather, one believes that the situation we find ourselves in is one among many other situations that obey the laws of this theory. Moreover, some such situations are more similar than others. See Fletcher (2016) for an extended discussion of this example.

3.8 Notes

We have given only the most cursory introduction to the rich mathematical fields of Boolean algebras, topology, and the interactions between them. There is much more to be learned and many good books on these topics. Some of our favorites are the following:

- For more on Boolean algebras, see Sikorski (1969); Dwinger (1971); Koppelberg (1989); Givant and Halmos (2008); Monk (2014),
- There are many good books on topology. We learned originally from Munkres (2000), and our favorites include Engelking (1989) and Willard (1970). The latter is notable for its presentation of the ultrafilter approach to convergence.

- Stone spaces, being a particular kind of topological space, are sometimes mentioned in books about topology. But for a more systematic treatment of Stone spaces, you'll need to consult other resources. For a fully general and categorical treatment of Stone duality, see Johnstone (1986). For briefer and more pedestrian treatments, see Bell and Machover (1977); Halmos and Givant (1998); Cori and Lascar (2000). For a proof that Stone spaces are profinite, see Ribes and Zalesskii (2000).