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Symmetry, Reduction and Gauge; an Introduction

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1. Introduction :
We will survey the constrained theories that arise from setting to zero the conserved quantities that arise from the symmetries of a Hamiltonian system. Sections 2 and 3 will discuss mechanics and symmetries; Section 4 constraints, and hence "gauge" (from a Hamiltonian perspective). Section 5 will discuss reduction. Section 6 will discuss relational mechanics as an example. Section 7 will discuss vacuum Maxwell theory, and so the Aharonov-Bohm effect. Finally, we will turn to quantization.

2. Hamiltonian mechanics :
A phase space $M \ni (q,p) = (q^1, ...; p_1, ...)$; $H : M \to \mathbb{R}$; $\dot{q}^i = \partial H/\partial p_i$ and $\dot{p}^i = -\partial H/\partial q_i$. A closed non-degenerate 2-form (a *symplectic form*, that makes M a *symplectic manifold*), ω, on M fixes the dynamical vector field X_H on M in terms of the gradient dH, namely by solving for X_H in

$$\omega(X_H, \cdot) = dH. \tag{1}$$

ω defines the Poisson bracket of any two dynamical variables $f, g : M \to \mathbb{R}$ by $\{f, g\} = \omega(X_f, X_g)$. The rate of change of f under the evolution determined by H is: $\dot{f} = \{f, H\}$. So Noether's theorem can be expressed as: if f generates a symmetry of the Hamiltonian in the sense that $\{H, f\} = 0$, then $\dot{f} = 0$.

Alternatively, and a bit more generally: equip $C^\infty(M)$ with a primitive Poisson bracket (i.e. Lie algebra with a Leibniz rule; and so define a *Poisson manifold*. This is a generalization of symplectic manifold: any Poisson manifold can be shown to be foliated by symplectic manifolds [not necessarily all of the same dimension].

A *simple mechanical system* is given by a configuration space Q with metric g, together with a smooth potential function $V : Q \to \mathbb{R}$. So using the cotangent bundle $T^*Q \ni (q,p)$, it is given by (T^*Q, g, V).

Here g defines the kinetic energy T on T^*Q via $T : (q,p) \mapsto g_q(p,p)$, and we take the Hamiltonian $H : T^*Q \to \mathbb{R}$ to be defined by $H(q,p) := T(q,p) + V(q)$.

3. Symmetries and mechanics :
Recall: Let a finite-dimensional Lie group $G \ni g$ act on a manifold $M \ni x$: the action can be: free, proper (a topological condition), fair (that the isotropy groups

of any two points in M are conjugate to each other). Fairness implies that the dimension $\dim(G_x)$ of the isotropy group G_x of $x \in M$ is independent of x.

Given a simple mechanical system (T^*Q, g, V): suppose G acts on Q by isometries that leave V invariant; and that the action is proper and fair. Then (T^*Q, g, V, G) is called a *simple mechanical G-system.*

Example: N gravitating Newtonian point-particles. The familiar ("absolutist") configuration space $\mathbb{R}^{3N}$ carries the Euclidean metric g and V. So we have a simple mechanical system $(T^*\mathbb{R}^{3N}, g, V)$. The Euclidean group $E(3)$ of rotations, reflections and translations acts on $\mathbb{R}^{3N}$ properly—but not fairly. Alhough a generic point $q \in \mathbb{R}^{3N}$ is fixed only by $e \in E(3)$ so that $\dim(E(3)_q) = 0$: a suitably symmetric configuration (e.g. the particles all collinear in space) has an isotropy group of larger dimension, i.e. larger than 0. So we excise these points and call the remaining space $Q := \mathbb{R}^{3N} - \{q \in \mathbb{R}^{3N} \mid \dim(E(3))_x \neq 0\}$. The excised points are closed under the action of $E(3)$, and so is Q. $E(3)$ acts freely on Q, and g and V are still well-defined on Q. So we conclude that $(T^*Q, g, V, E(3))$ is a simple mechanical $E(3)$-system.

Given any simple mechanical G-system (T^*Q, g, V, G):

(i): G's action on Q lifts to a proper and fair action on T^*Q

(iii) This lifted action leaves invariant: H and less obviously, the symplectic structure of T^*Q, in particular the symplectic form ω.

(iii): So if $x, y \in T^*Q$ with $x = g \cdot y$ for some $g \in G$, then x and y are "qualitatively identical" from the perspective of dynamics: e.g. coordinate expressions for dynamical trajectories around x, y assume the same form for coordinate systems related by g. Moreover, the orbit through x, $O_x := G \cdot x := \{y \in T^*Q \mid y = g \cdot x \text{ some } g \in G\}$ is a regular sub-manifold of T^*Q. That is: around any $x \in T^*Q$, there is a neighbourhood $U \subset T^*Q, U \ni x$ and local coordinates $\{x_1, ... : z_1, ...\}$ on U such that:

$$O_x \cap U = \{y \in T^*Q \mid z_1(y) = 0, z_2(y) = 0, z_3(y) = 0, ...\} \tag{2}$$

(iv): Since varying the coordinates x_i carries one along the orbit, i.e. between "dynamically qualitatively identical" states, one can here envisage *reduction*: i.e. setting up a *relationist* dynamics in which each orbit O_x is treated as a basic dynamical state. That is: one works with the set of orbits T^*Q/G, endowed with the "projected Hamiltonian structure".

(v): But we will not do that here! For NB:

[a]: The quotient T^*Q/G is in general *not* a symplectic manifold, but "at best" a Poisson manifold. (Recall: a generalization of symplectic manifold: foliated by symplectic manifolds [not necessarily all of the same dimension]).

[b]: The paradigm mechanical examples are the rigid body and the ideal

fluid. In these examples, the configuration space is isomorphic to the Lie group G: think of each configuration being fixed by a motion from an arbitrary reference configuration. So the quotient is T^*G/G.

[c]: The paradigm mathematical example of a Poisson manifold is $\mathfrak{g}^*$, the dual of the Lie algebra of any Lie group, carrying the co-adjoint representation of the group: this example was known to Lie in 1990–but forgotten and rediscovered in the 1970s ... yielding the theorem:

[d]: For any Lie group G, there is an isomorphism of Poisson manifolds $(T^*G/G) \cong \mathfrak{g}^*$.

We will restrict attention to a submanifold of T^*Q *before* identifying G-related points ...

4. Symmetry and constraint :

A: Conserved quantities, momentum maps and Noether's theorem:—

Recall: When a finite-dimensional Lie group $G \ni g$ acts on a manifold $M \ni x$: $\xi \in \mathfrak{g} \equiv L(G)$, the Lie algebra of G, defines a vector field ξ_M on M, by: $\xi_M(x) := (d/dt)(\exp(t\xi) \cdot x)|_{t=0}$.

For a simple mechanical G-system, ξ_{T^*Q} is the Hamiltonian vector field of the scalar $J(\xi) \equiv J^\xi \in C^\infty(T^*Q)$, i.e. $\xi_{T^*Q} = X_{J^\xi}$ where

$$J^\xi(q,p) \;:=\; \langle p,\, \xi_Q(q)\rangle_{q\in Q} \,. \tag{3}$$

Since H is invariant under the G-action, the scalar J^ξ is a constant of the motion: $\dot{J}^\xi = 0$. In fact, the map $J : \xi \in \mathfrak{g} \mapsto J^\xi \in C^\infty(T^*Q)$ is a Lie algebra homomorphism.

Side-remark: J^ξ defines the dual *momentum map* $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ via

$$\langle \mathbf{J}(q,p),\, \xi\rangle_{\mathfrak{g}\sim\mathfrak{g}^*} \;:=\; J^\xi(q,p) \,. \tag{4}$$

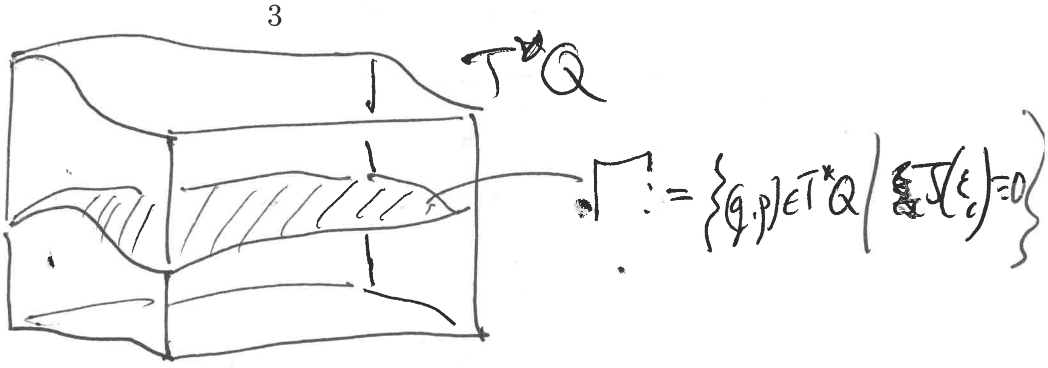
Recall that since the action of G on Q, and so also on T^*Q, is fair: the dimension of the isotropy group $G_{(q,p)}$, $\dim(G_{(q,p)})$, is independent of $(q,p) \in T^*Q$.

So $J : \mathfrak{g} \to C^\infty(T^*Q)$ maps $\mathfrak{g}$ onto a subspace of dimension $\dim_{act}G := \dim(G) - \dim(G_{(q,p)})$.

That is: a basis $\{\xi_i\}$ in $\mathfrak{g}$ (so i runs from 1 to $\dim(G)$) gives $\dim_{act}G$ independent conserved quantities, which generate a Poisson algebra that is a homomorphic image of $\mathfrak{g}$. Besides: if the G-action is free (as in Sections 6 to 8), then this is an isomorphism and there are $\dim G$ conserved quantities. This is (an abstract version of) *Noether's theorem.*

B. Setting the conserved quantities to zero:— Take a basis $\{\xi_i\}$ in $\mathfrak{g}$, and define:

$$\Gamma := \{(q,p) \in T^*Q \mid 0 = J(\xi_1)(q,p) = J(\xi_2)(q,p) = \dots\} \tag{5}$$



So $\dim(\Gamma) = 2\dim Q - \dim_{act} G$.

There are two ways to think of Γ. Considered *extrinsically*, it is a sheet in the extended phase space T^*Q, with a unique dynamical trajectory through each $(q,p) \in \Gamma$. But let us consider Γ *intrinsically*. We replace H by $H|_\Gamma : \Gamma \to \mathbb{R}$. Then we want $X_{H|_\Gamma}(q,p) \in T_{(q,p)}\Gamma$. We will write X_H as short for $X_{H|_\Gamma}$. We can get this by using the identity injection map $i : \Gamma \to T^*Q$ to pullback T^*Q's symplectic form ω to give a form $\omega|_\Gamma := i^*\omega$ on Γ; and then defining $X_H \equiv X_{H|_\Gamma}$ as solving

$$\omega|_\Gamma(X_H, \cdot) = d(H|_\Gamma). \tag{6}$$

But $\omega|_\Gamma$ is a *degenerate*, albeit closed, 2-form. It has a set of null vectors

$$\mathrm{Ker}(\omega|_\Gamma) := \{v \in T_{(q,p)}\Gamma \mid \omega|_\Gamma(v,w) = 0 \ \forall w \in T_{(q,p)}\Gamma\} \ ; \tag{7}$$

so that if X_H is a vector field on Γ that solves the intrinsic dynamical problem, eq. **??**, and N is any null vector field on Γ, then $X_H + N$ also solves the intrinsic dynamical problem.

In fact: $\omega|_\Gamma$'s null vectors are the infinitesimal generators of the G-action at that point, i.e.

$$\mathrm{Ker}(\omega|_\Gamma) = \{\xi_{T^*Q}(q,p) \ \mid \ \xi \in \mathfrak{g}\ \}. \tag{8}$$

That is: the null vectors are the tangent vectors to the orbits of the action of G. So Γ is foliated by orbits of G.

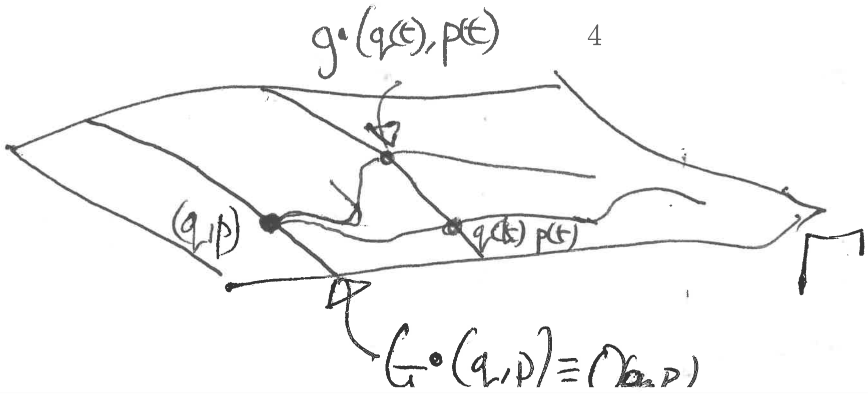
Combining the last two paragraphs: for all $(q,p) \in \Gamma$, the (tangent to the) intrinsic dynamical trajectory is fixed only up to an arbitrary infinitesimal generator of the action of G on T^*Q (and so by restriction on Γ). In other words (integral version): the intrinsic dynamical trajectories are of the form $g(t)\cdot(q(t),p(t))$ where $(q(t),p(t))$ solves the extrinsic problem, and $g : \mathbb{R} \to G_0$ is a smooth function from $\mathbb{R}$ to G's identity-connected component.

5. Symmetry and reduction :

A. Taking the quotient:— We shorten $(q,p) \in T^*Q$ to $x \in T^*Q$; and we define the equivalence relation $x \sim y$ for $x, y \in \Gamma$ by: $y = g \cdot x$. And we define $\Gamma/G := \{[x] \mid x \in \Gamma\}$. So we have

$$\dim(\Gamma/G) = \dim(\Gamma) - \dim_{act}(G) = 2(\dim(Q) - \dim_{act}(G)) \ . \tag{9}$$

The action of G on T^*Q maps dynamical trajectories on to dynamical trajectories. So if $x \sim y$, with $x(t), y(t)$ the extrinsic dynamical trajectories through x and y respectively, then $x(t) \sim y(t)$ (indeed, with the same $g \in G$). So the extrinsic dynamics defines a unique curve through any point $[x]$ of Γ/G. Similarly, the



pencil of intrinsic dynamical trajectories through $x \in \Gamma$ projects to a single curve in Γ/G. Summing up: through any point $[x]$ of Γ/G, there is a unque curve that is the image of all the intrinsic and extrinsic dynamical trajectories through all the points $y \in [x] \subset \Gamma$.

These curves are generated by a Poisson structure and Hamiltonian on Γ/G that Γ/G inherits from the identity embedding $i; \Gamma \rightarrow T^*Q$. For example:

(i): our original Hamiltonian on T^*Q was G-invariant; so it projects to a well-defined Hamiltonian $\tilde{H} : \Gamma/G \rightarrow \mathbb{R}$.

(i): We take "$C^\infty(\Gamma/G)$" to be the restrictions to Γ of G-invariant smooth functions on T^*Q.

B. Another characterization of the quotient dynamics:— This equivalent characterization is available because of the cotangent bundle structure, and the original action of G on Q being proper and fair.

We define the *reduced configuration space* Q/G. (Philosophy of point-particle mechanics! Leibniz, Mach, Barbour envisage a reduced configuration space whose elements are the relative configurations of the particles; cf. Section 6 below.) The dimensions of the orbits of G on Q and on T^*Q are equal: viz. $\dim_{act}(G)$. So

$$\dim(Q/G) = \dim(Q) - \dim_{act}(G) \ . \tag{10}$$

The metric g and the potential V on Q are G-invariant. So they define by projection $\tilde{g}$ and $\tilde{V}$ on Q/G. So *now*, we take the cotangent bundle of Q/G and get a simple mechanical system $(T^*(Q/G), \tilde{g}, \tilde{V})$. This is called a *reduced phase space*. Eq. ?? implies that its dimension is: $2(\dim(Q) - \dim_{act}(G))$.

We see that this ("relationist") theory and the theory in Subsection A (i.e. defined on Γ/G) both use a phase space whose dimension is: $2(\dim(Q) - \dim_{act}(G))$. In fact: *they are the same theory*: $\Gamma/G \cong T^*(Q/G)$.

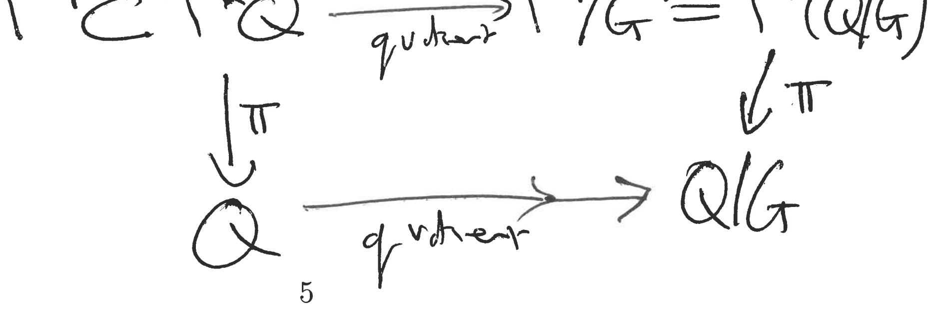
To summarize: We have three main spaces carrying a dynamics:—

(a): T^*Q: $\dim = 2\dim(Q)$;

(b): $\Gamma \subset T^*Q$. $\dim(\Gamma) = 2\dim Q - \dim_{act} G$.

(c): Reduced phase space: $\Gamma/G \cong T^*(Q/G)$. $\dim(\Gamma/G) = 2(\dim Q - \dim_{act} G)$.

And we have a diagram showing the commutation of taking a cotangent bundle with quotienting:—



Finally, a remark on *gauge-fixing* and *bundles*:—

(1): A gauge-fixing is a smooth choice of a single representative of eac guage-equivalence class. So in our notation: a choice in each G-orbit $\subset \Gamma$. We of course aim that the choice yields a symplectic manifold that is isomorphic to $\Gamma/G \cong T^*(Q/G)$.

(2): If G's action is free, then $\Gamma \to \Gamma/G$ (which, by the isomorphism $\Gamma/G \cong T^*(Q/G)$, can be identified with $\Gamma \to T^*(Q/G)$) and $Q \to Q/G$ are principal G-bundles. So a gauge-fixing is a choice of a section of the principal G-bundle $\Gamma \to T^*(Q/G)$.

(3): Gauge-fixing is always possible locally, but can be globally impossible (Gribov 1977, Singer 1978): even when $\Gamma \to T^*(Q/G)$ and $T^*(Q/G)$ is simply connected. (Singer showed that under mild conditions, it is globally impossible for non-abelian classical Yang Mills theory; cf. Healey 2007, p. 76-77.)

6. Relational mechanics :
Consider N gravitating Newtonian point-particles. In Section 3, we started from the familiar ("absolutist") configuration space $\mathbb{R}^{3N}$ carrying the Euclidean metric g and V, and so the simple mechanical system $(T^*\mathbb{R}^{3N}, g, V)$. Considering the Euclidean group $E(3)$ and excising the configurations that were unduly symmetric (i.e. that have a non-trivial isotropy group), and calling the remaining space $Q := \mathbb{R}^{3N} - \{q \in \mathbb{R}^{3N} \mid \dim(E(3))_x \neq 0\ \}$, we had: $E(3)$ acts properly and freely on Q, and g and V are well-defined on Q—and we concluded that $(T^*Q, g, V, E(3))$ is a simple mechanical $E(3)$-system.

The relationist idea is:

to start from the reduced configuration space whose elements are the relative configurations of the particles; this will be $Q/E(3)$.

to have a velocity phase space (i.e. tangent bundle $T(Q/E(3))$ on this preferred configuration space) consisting of (i) the possible relative configurations, naturally coordinatized by the set of relative distances, and (ii) the tangent vectors, naturally coordinatized by the set of relative velocities.

to have as the momentum phase space, the cotangent bundle $T^*(Q/E(3))$; (though Barbour et al. in fact pursue the Lagrangian framework).

Starting from the ("absolutist") simple mechanical $E(3)$-system $(T^*Q, g, V, E(3))$: the conserved quantities are the (components of) linear momentum and the (centre of mass) angular momentum. So $\Gamma =$ states with vanishing linear and angular momentum . So the intrinsic dynamics is determined only upto a time-dependent translation and rotation; (here $SE(3) = E(3)_0$ contains no reflections).

On the alternative approach, starting from the reduced configuration space

$Q/E(3)$ (cf. B of Section 5), we get a vividly relationist picture.

NB: the number of relative distances is $N(N-1)/2$ while $\dim(Q/E(3)) = 3N-6$. For $N >> 4, N(N-1)/2 \; >> 3N-6$. So though relative distances are natural coordinates on $Q/E(3)$, they are a vastly *over-complete* set of coordinates on it. That is: There are many constraints on the relative distances (not just the triangle inequality!)

Similarly "upstairs": though relative distances and relative momenta are natural coordinates on $T^*(Q/E(3))$, they are vastly over-complete. (And similarly, of course for the Lagrangian approach using $T(Q/E(3))$.)

Here there is an analogy with the Aharonov-Bohm effect: there, holonomies will be a natural but vastly over-complete set of coordinates. Cf the end of Section 7.

There will also be a *disanalogy*. In mechanics, everyone except a relationist will treat the extrinsic dynamics as fundamental; while we will see that in electromagnetism, it is normal to treat the intrinsic dynamics as fundamental (and the same applies in classical Yang-Mills theory). Why? The answer is presumably our different attitudes to whether the states *off* the constraints surface are genuine physical possibilities. In mechanics, for everyone except a relationist, they are possible. But as we will see: in electromagnetism, they are not—or better, they are only possible by changing the theory concerned. Again: cf end of Section 7.

7. Vacuum Maxwell theory :

The idea: This is a close (though infinite-dimensional) analogue of relational mechanics, with A (or B) configurational and E momentum-like, and gauge transformations the analogue of Euclidean motions, and holonomies the analogue of inter-particle distances (viz. as an over-complete set of coordinates on a reduced configuration space).

A. Basics: defining the simple mechanical G-system:

Let S be a flat Riemannian 3-manifold representing space. Maxwell's equations in terms of E, B are:

$$\dot{B} = -\text{curl}E \; ; \; \dot{E} = \text{curl}B \; ; \; \text{div}B = 0 \; ; \; \text{div}E = 0 \; . \tag{11}$$

Now take as the configuration space, the space of vector potentials in "temporal gauge", i.e.

$$\mathcal{A} := \{A : S \to \mathbb{R}^3 \;\mid\; A \text{ smooth } \} \; . \tag{12}$$

$\mathcal{A}$ is a vector space, so $T_A\mathcal{A}$ can be identified with $\mathcal{A}$. The phase space (cotangent bundle) is

$$T^*\mathcal{A} := \{(A, E) \;\mid\; A : S \to \mathbb{R}^3, E : S \to \mathbb{R}^3 \text{ both smooth } \} \tag{13}$$

with the pairing between vector and covector given by integrating the naive scalar product

$$\langle E,\ A\rangle_{A\in\mathcal{A}} := \int_S A\cdot E\ d^3x\ ; \tag{14}$$

the canonical symplectic form given by

$$\omega((A_1,E_1),(A_2,E_2)) := \int_S (E_1\cdot A_2 - E_2\cdot A_1)\ d^3x\ ; \tag{15}$$

and the Poisson bracket given by

$$\{F,\ G\} := \int_S (\frac{\delta F}{\delta A}\frac{\delta G}{\delta E} - \frac{\delta F}{\delta E}\frac{\delta G}{\delta A})\ d^3x\ . \tag{16}$$

Then the Hamiltonian given by a sum of kinetic and potential terms, viz.

$$H(A,E) := \frac{1}{2}\int_S |E|^2 + |\text{curl}A|^2\ d^3x\ . \tag{17}$$

yields as Hamilton's equations, $\dot{A} = -E, \dot{E} = \text{curl curl}A$. If we then define $B = \text{curl}A$, these give the first three equations of our Maxwell's equations eq. ??. For the fourth equation, div $E = 0$, cf. the definition below of the constraint surface *Gamma*.

Let $\mathcal{G}$ be the additive group of smooth functions $f : S \to \mathbb{R}$. (This will be our "gauge group".) $\mathcal{G}$ acts on $\mathcal{A}$ by setting $f\cdot A$ to be $A+\nabla f$. That is: $\Phi_f : A \mapsto A+\nabla f$. Then the lifted action on $T^*\mathcal{A}$ is: $\tilde{\Phi}_f : (A,E)\mapsto(A+\nabla f, E)$. This action leaves E, and so the kinetic energy, invariant. It also leaves the potential energy $\frac{1}{2}\int_S |\text{curl}A|^2\ d^3x$ invariant.

But $\mathcal{G}$ does not act fairly (since symmetric fields have larger-than generic isotropy groups). So we restrict to the pointed gauge transformations $\mathcal{G}_* := \{f \in \mathcal{G} \mid f(x_0) = 0\}$ for some fixed $x_0 \in S$. Then $\mathcal{G}_*$ acts fairly, indeed freely, on $\mathcal{A}$ and $T^*\mathcal{A}$. (And using $\mathcal{G}_*$ loses very little, in that $\mathcal{G}_*$ is a normal subgroup of $\mathcal{G}$, and $\mathcal{G}/\mathcal{G}_* = U(1)$.) To sum up: we have a simple mechanical $\mathcal{G}_*$-system.

B. Defining Γ *and the quotient dynamics*:
$\dim(\mathcal{G}_*) = \infty$. So there are infinitely many conserved quantities: namely the value of div E at each $x \in S$! Physically: Nothing in our framework determines the charge distribution ρ, nor that charge should respond to the field (i.e. the Lorentz force-law). Thus we now secure the fourth Maxwell equation, div$E = 0$, by *fiat*; i.e. we restrict attention to

$$\Gamma := \{(A,E) \in T^*\mathcal{A}\ \mid\ \text{div}E = 0\} \tag{18}$$

The null vectors of the presymplectic form $\omega|_\Gamma$ at the point $(A, E) \in \Gamma$ are the infinitesimal generators of the $\mathcal{G}_*$ action at that point, i.e. $\{\xi_{T^*\mathcal{A}}(A, E) \mid \xi \in L(\mathcal{G}_*)\ \}$. So the intrinsic dynamical trajectories in Γ are defined only up to a small gauge transformation. That is: if $A(t), E(t)$ is a solution, so is $(A + \nabla g(t), E(t))$ for any smooth map $g : \mathbb{R} \to \mathcal{G}_{*0}$.

We now quotient to consider Γ/G; or *isomorphically* (by the general equivalence discussed in Section 5.B), $T^*(\mathcal{A}/\mathcal{G}_*)$, i.e. the cotangent bundle of the reduced configuration space consisting of vector potential modulo pointed gauge transformations.

For $S = \mathbb{R}^3$, $\mathcal{A}/\mathcal{G}_* = \{\text{divergence} - \text{free}B : S \to \mathbb{R}^3\ \}$, and the reduced phase space can be taken as $\{(B, E) \mid \text{div}B = \text{div}E = 0\ \}$, with equations of motion given by the first two of Maxwell's equations eq. ??. But for other topological structures for S...

C. The Aharonov-Bohm effect: and comparison with relational mechanics:
If S is not simply connected, then: there are gauge-*inequivalent* A, A' with $\text{curl}A = \text{curl}A'$. That is: specifying a magnetic field does not determine a gauge-equivalence class of vector potentials.

But note that for any closed curve γ starting and ending at our base-point $x_0 \in S$, the *holonomy of A around γ*

$$H_\gamma(A) := \exp i \oint_\gamma A \cdot dx \tag{19}$$

is gauge-invariant: $H_\gamma(A) \equiv H_\gamma(A + \nabla f), \forall f \in \mathcal{G}_*$.

Moreover, $A\tilde{A}'$ (i.e. A, A' are gauge-equivalent) iff for all $\gamma, H_\gamma(A) = H_\gamma(A')$. Thus the set of holonomies are good coordinates on $\mathcal{A}/\mathcal{G}_*$—but over-complete!

To sum up: The Aharonov-Bohm effect brings out that "understanding electromagnetism" needs $\mathcal{A}/\mathcal{G}_*$: you cannot manage with the coarse-grained information in E and B. In any case, you need holonomies for vacuum Yang-Mills theory...[whose details we skip]

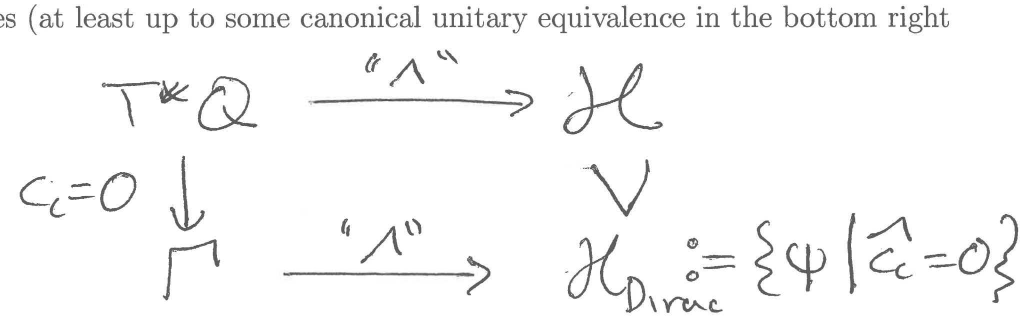
9 Quantization :
The reduced theories in Sections 6, 7 and 8 are Hamiltonian; and they are defined on a cotangent bundle, viz. the cotangent bundle of the reduced configuration space $\mathbb{Q}$ say—thanks to the general equivalence discussed in Section 5.B. So in principle, these theories can be subject to "usual canonical quantization", mapping a Poisson sub-algebra, $\mathcal{A}$ say, of $C^\infty(T^*\mathbb{Q})$, the C^∞ functions on the cotangent bundle, to a subalgebra of self-adjoint operators defined on $L^2(\mathbb{Q})$, the L^2 space of functions on the reduced configuration space.

But in practice, there is trouble: due to the over-completeness of the natural coordinates, and the non-locality of holonomies. This prompts the idea of *Dirac (aka: constrained) quantization.* The idea is to proceed in two stages:

(i) first: quantize the original extended phase space theory, *without* constraints $c_i = 0$ (for us: without setting to zero the values of conserved quantities $J(\xi_i) = 0$; getting a Hilbert space $\mathcal{H}$

(ii) second: restrict your attention to the subspace of $\mathcal{H}$ consisting of vectors annihilated by the quantum version of the constraints, i.e. to $\mathcal{H}_{\rm Dirac} := \{\psi \in \mathcal{H} \mid \hat{c}_i = 0\}$.

The hope is that this two-stage procedure gives the same quantum theory as directly quantizing the reduced theory: in other words, that the following diagram commutes (at least up to some canonical unitary equivalence in the bottom right corner!):



This hope turns out true for many finite-dimensional systems. We fix on a procedure, *geometric quantization*, applied to a simple mechanical G-system (T^*Q, g, V, G): with some "extra meshing" conditions that imply a unique quantization that makes the diagram commute up to canonical unitary equivalence.

Some details: The extra meshing implies that:

(i): G also acts on $L^2(Q)$ as: for $g \in G, q \in Q : (\psi \cdot g)(q) := \psi(g \cdot q)$. (This implies that $(\psi \cdot (g_1 g_2)(q) = ((\psi \cdot g_1) \cdot g_2)(q)$; so G's action on $L^2(Q)$ is a right-action.)

(ii): the infinitesimal generators of G's action on $L^2(Q)$ are the operators $\hat{J}^{\xi_i}$ corresponding to the J^{ξ_i} and the $\xi_i \in \mathfrak{g}$ via $\xi_{iT^*Q} = X_{J^{\xi_i}}$. So corresponding to the classical restriction to the constraint surface Γ, i.e. $J^{\xi_i} = 0$, we have the *Dirac condition*, $\hat{J}^{\xi_i}(\psi) = 0$.

(iii): Since (ii) means that G's action on $L^2(Q)$ is given by $\exp i\hat{J}^{\xi_i}$; and if $\hat{J}^{\xi_i}(\psi) = 0$, then $\exp i\hat{J}^{\xi_i}(\psi) = \psi$, we have: the Dirac Hilbert space, $\mathcal{H}_{\rm Dirac} \equiv \{\psi \in \mathcal{H} \mid \hat{J}^{\xi_i}(\psi) = 0\}$, is $\{\psi \in \mathcal{H} \mid \psi \cdot g = \psi, \forall g \in G\}$.

(iv): As (iii) suggests: $\mathcal{H}_{\rm Dirac}$ carries a representation (as self-adjoint operators) of some G-invariant quantities. This suggests it can be written as, or is unitarily equivalent to, $L^2(Q/G)$ equipped with such quantities: i.e. a quantization of the reduced theory—so that the diagram commutes.