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How to Be a Relativistic Spacetime State Realist

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ABSTRACT

According to spacetime state realism (SSR), the fundamental ontology of a quantum mechanical world consists of a state-valued field evolving in four-dimensional spacetime. One chief advantage it claims over rival wave-function realist views is its natural compatibility with relativistic quantum field theory (QFT). I argue that the original density operator formulation of SSR cannot be extended to QFTs where the local observables form type III von Neumann algebras. Instead, I propose a new formulation of SSR in terms of a presheaf of local state spaces dual to the net of local observables studied by algebraic QFT.

1 Introduction

There is growing belief that any viable solution to the measurement problem will require treating the quantum state as a genuine physical object rather than an indirect representational tool. This has prompted the development of numerous wave-function realist views, according to which, the fundamental ontology of a quantum mechanical world consists of a $\mathbb{C}$-valued field (the wave function) evolving in a high-dimensional space (usually the configuration space of a non-relativistic many-particle system). Ordinary four-

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dimensional spacetime and the objects bumping around inside of it are emergent structures that need to be explained in terms of this high-dimensional alien landscape.

Wallace and Timpson ([2010], p. 698) develop an altogether different kind of view that similarly aims to 'take the quantum state seriously'.[1] According to spacetime state realism (SSR), the fundamental ontology of a quantum mechanical world consists of a state-valued field evolving in four-dimensional spacetime. Each spacetime region is associated with a local Hilbert space whose density operators represent the possible values of the field in that region. Much as in classical field theories, these field values are interpreted as characterizing the intrinsic, local properties of the region.

Unlike wave-function realism, SSR treats spacetime as the arena in which physics unfolds. Objects like particles and cats are still emergent entities, patterns in the state-valued field, however the defender of SSR does not need to explain the emergence of spacetime itself. This might strike us as a decisive advantage, yet Wallace and Timpson demur. Instead, they think that the primary advantage of SSR is its extendability to relativistic quantum field theory (QFT). SSR, they contend, has an especially simple, elegant relativistic formulation, while there are a number of technical and conceptual pitfalls hampering relativistic extensions of wave-function realism.[2]

The reality is not so cut-and-dried. SSR faces its own foundational challenge in the relativistic domain that threatens to undermine this advantage. In standard formulations of QFT, the local states are pieces of derivative structure defined in terms of the global state and the local observable algebras. Self-adjoint elements of these algebras are usually interpreted as representing localized quantities like mass, spin, and charge. From the standpoint of SSR, however, the local observables are not part of the ontology at all. They are merely auxiliary mathematical tools. But if this is the case, we cannot use the concept of local observables to define the local states. Rather, we need a way to formulate QFT that treats the field of local states as fundamental and defines all other structures in terms of it.

In other work, Wallace ([2006], [2012]) suggests that such a state-first formulation of QFT is possible. In this article, I explore the viability of Wallace's proposal and find it wanting. His idea can only get off the ground if the local observable algebras are type I von Neumann algebras familiar from non-relativistic quantum theory. Unfortunately, results from constructive and algebraic QFT indicate that in order to consistently unify relativity and quantum mechanics, the local algebras need to be type III von Neumann algebras.

[1] See (Wallace [2012], Chapter 8) for further development of the view. (Arntzenius [2012], Chapter 3.13; Lewis [2013]; Baker [2016]; Ismael and Schaffer [2016]) represent early critical discussions.

[2] See (Wallace and Timpson [2010], Section 4, Section 8; Wallace [2012], pp. 304–5, 316–17) for further discussion.

Physically, this reflects the high degree of entanglement exhibited by the QFT vacuum and similar global states. Because these states are so highly entangled, the procedure that Wallace sketches for decomposing the global state into a field of local states falls apart at the hinges, leaving relativistic SSR without a coherent mathematical foundation.

Ultimately we can fix this problem, but it requires a much more radical overhaul of standard QFT than anything Wallace and Timpson suggest. It is well known that every operator algebra has a dual object of sorts, its state space, consisting of all positive, normalized, $\mathbb{C}$-valued linear functionals over the algebra. It is less well known how to make this duality precise—a deep reconstruction theorem due to Alfsen, Hanche-Olsen, and Shultz ([1980]; henceforth, AHS) proves that given a certain orientation structure on the state space, we can recover the algebra. Thus all of the physical information contained in the collection of local observables can in principle be encoded in the dual collection of local, oriented state spaces. To complete the picture, we need a set of axioms for the collection of local state spaces that mirror the usual axioms of algebraic or constructive QFT.[3]

The development of such axioms is an involved mathematical project that will be the subject of future work (Swanson and Halvorson [unpublished]). The aim of this article is to clearly articulate the foundational challenge SSR faces in the relativistic domain and to argue that the new state space formalism represents the best path forward for developing a QFT-compatible version of the view. In the next section, I review Wallace and Timpson's original density operator proposal for relativistic SSR. In Section 3, I argue that the type III character of local algebras in QFT poses a serious challenge for their view. A pair of no-go lemmas shows why easy fixes will not work. In Section 4, I outline my alternative proposal and sketch some ideas for state space axioms that mirror the Haag–Kastler axioms for algebraic QFT. Finally, in Section 5, I discuss how the new state space approach can help shed light on the metaphysics of SSR and respond to some recent criticisms of the view that turn on misunderstandings of the original density operator formulation. (Apart from this limited defence, the goal is not to provide a general argument in favour of SSR, nor is it to argue that the state space formalism is preferable to standard algebraic QFT.)

[3] The idea of studying QFT via a presheaf of state spaces can be traced back to unpublished work of John Roberts (Haag [1996], pp. 326–8, 131–2, 141–2). The major difference between Roberts's proposal and the one surveyed here is the use of the AHS-reconstruction theorem and the central notion of an orientation of a C^*-state space to capture dual versions of the usual algebraic axioms. Motivated by the problem of constructing particular models of QFT, Roberts did not seek dualized axioms. In addition, his characterization of the presheaf relies on special properties of von Neumann algebras and thus lacks the full generality of algebraic QFT.

2 Spacetime State Realism in Quantum Field Theory

In the original formulation of SSR, each spacetime region is associated with a local Hilbert space whose density operators correspond to possible field values encoding intrinsic properties of that region. Wallace and Timpson emphasize the parallels between this picture and classical field theories like electrodynamics. The tensor value of the electromagnetic field at a spacetime point represents certain intrinsic properties of that point. The physics are encoded in how these values vary across spacetime. Similarly, in SSR the field takes a particular quantum state as its value in a region. The physics are encoded in how these field values vary across spacetime, in the mosaic of local states.

In classical theories, specifying the field value at each point in a spacetime region suffices to fix the field configuration in that region. Due to quantum entanglement, this is not true in SSR. If the field in region O_1 is entangled with the field in region O_2, specifying the state of O_1 and O_2 will not fix the state of the joint region $O_1 \cup O_2$. Consequently, in order to specify the global field configuration one needs to provide a field value for every region, including the entirety of spacetime itself. This is known as the non-separability of the quantum state. According to SSR, non-separability reflects the presence of non-local relations between regions that do not supervene on the intrinsic properties of those regions.

Wallace ([2013]) observes that the mathematical signature of non-separability is the appearance of tensor product structure in quantum theory. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are the local Hilbert spaces associated with causally independent regions O_1 and O_2, the state space of the joint system $O_1 \cup O_2$ is the tensor product $\mathcal{H}_1 \bigotimes \mathcal{H}_2$. This Hilbert space includes density operators that cannot be factored into products of density operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, representing entangled states across O_1 and O_2. In contrast, in classical theories the state space of a joint system is given by a Cartesian product. As a result every state of the joint system is a product of subsystem states.

For non-relativistic particle theories we can combine these ideas into a simple mathematical picture. Galilean spacetime is naturally foliated into spatial hypersurfaces of constant universal time. Given an arbitrary partition of one of these hypersurfaces, each spatial region, Δ_i, can be assigned a Hilbert space, $\mathcal{H}_i$, whose density operators represent the possible particle states in that region.[4] The Hilbert space for the entire hypersurface is given by the tensor product of these local Hilbert spaces:

$$\mathcal{H} = \bigotimes_i \mathcal{H}_i. \tag{1}$$

If the state of the universe at time t is ρ, the local state in region Δ_i is given by

[4] Wallace and Timpson define $\mathcal{H}_i := \mathcal{F}(\mathcal{K}_i)$, where $\mathcal{F}(\mathcal{K}_i)$ is the Fock space built out of copies of the Hilbert space $\mathcal{K}_i$, consisting of single-particle wave functions with support in region Δ_i.

$$\rho(\Delta_i) = \mathrm{Tr}_{j \neq i}\, \rho, \tag{2}$$

the partial trace of ρ over all local Hilbert spaces except $\mathcal{H}_i$. The local state is a density operator acting on $\mathcal{H}_i$.

Extending this scheme to relativistic theories appears straightforward. Wallace and Timpson ([2010], pp. 711–12) observe:

> Things become simpler still when we move to full QFT. In the algebraic formulation of QFT, we associate to each spacetime region O a C^*-algebra $\mathfrak{A}(O)$ of operators, representing the dynamical variables associated to region O. A state $\rho(O)$ of such a region is a positive linear functional on $\mathfrak{A}(O)$ (often described in rather instrumentalist terms as giving the expectation value of each element of $\mathfrak{A}(O)$) and by the Gelfand–Naimark–Segal (GNS) construction we can associate $\rho(O)$ with a state in a Hilbert space $\mathcal{H}_O$, and represent $\mathfrak{A}(O)$ as an algebra of operators on $\mathcal{H}_O$ [...] $\mathcal{H}_O$ can then be taken as the Hilbert space of the field in region O. If preferred, one can even remain at the more abstract level, forego the representation theorems and just take the C^*-algebraic state itself as denoting the properties of a region.[5]

There is something rather odd, though, about the scattershot combination of ideas expressed in this paragraph. Algebraic QFT presents us with a global state plus a net of local observable algebras. By taking the restriction of the global state, ρ, on a local algebra,

$$\rho(O) := \rho|_{\mathfrak{A}(O)}, \tag{3}$$

we are able to define a unique local state associated with each region. The problem is that this concept of a 'local state' is defined in terms of another primitive concept, a 'local observable algebra'. But according to SSR, the observables are not part of the ontology at all! As Wallace and Timpson ([2010], p. 703) explain:

> To every different quantum state corresponds a different concrete way the world is. For Everett and for some readings of dynamical-collapse theories, the quantum state (perhaps together with some background space or spacetime) gives the only part of the ontology.

This viewpoint is reinforced by Wallace's ([2012], p. 320) critique of Deutsche and Hayden's alternative operator-valued field ontology:

> [...] different states of affairs are represented by different state vectors. From this perspective, the operators—time-dependent or not—do not directly give the state of the system; they are simply part of the mathematical machinery that breaks the symmetries of Hilbert space and allows different rays in Hilbert space to represent different states of affairs.

[5] The original notation has been modified for consistency with the notation of this article.

But if the local observables are just mathematical machinery, the various definitional procedures Wallace and Timpson discuss become physically opaque. We need to know what connects a particular state to a particular region. Why is $\rho(O)$ the local state for region O? We cannot point to the fact that $\rho(O)$ encodes the expectation values for the observables localized in O, since according to SSR the observables do not correspond to anything physical. We need a way to characterize $\rho(O)$ and its relationship to O that is independent of the mathematical crutch of local observables.

Although Wallace and Timpson never directly address this issue, Wallace's ([2012], p. 15) comments elsewhere point towards a possible resolution:

> In the so-called 'algebraic' description of QFT, an algebra of operators is associated to each spatial region, so that the operators associated with region O are intended to represent observables localized in O. At least formally, we can regard this as equivalent to decomposing the Hilbert space into quotient spaces, each one representing the quantum state of a different spatial region [...] When a QFT is specified in this way, there is no need to say *which* operator at O corresponds to which observable property of O—once the localization properties of all the operators are known, the quantum system is sufficiently structured to permit physics to be done, and in particular, to understand the emergence of particles.

Wallace ([2006]) gives a clearer idea of the formal equivalence alluded to here. In all standard formulations of QFT, we begin by assigning local operators (either unbounded field operators or bounded observable operators) to space-time regions. Choosing a space-like hypersurface and a partition as before, Wallace argues that there is a natural tensor product decomposition of the global Hilbert space, $\mathcal{H} = \bigotimes \mathcal{H}_i$, in which the operators localized in Δ_i act trivially on all $\mathcal{H}_j$ with $j \neq i$. In the usual picture, this justifies calling $\mathcal{H}_i$ the 'local state space' associated with region Δ_i. As Wallace emphasizes, however, nothing prevents us from switching the order of definition around. Given a tensor product decomposition, $\mathcal{H} = \bigotimes \mathcal{H}_i$, where $\mathcal{H}_i$ is the local Hilbert space associated with spatial region Δ_i, we can define operators localized in Δ_i as those linear transformations on $\mathcal{H}$ that act trivially on all $\mathcal{H}_j$ with $j \neq i$. This, he contends, provides a state-first formulation of QFT equivalent to the usual, operator-first perspective.

This equivalence thesis is needed in order to make sense of Wallace and Timpson's proposal for relativistic SSR. Despite the widespread use of local operators as part of the mathematical machinery of QFT, we can do without them in principle. All of the information encoded in the system of local field operators and observable algebras can be captured by tensor product decompositions of the global Hilbert space. According to SSR, these decompositions directly encode the physical relationship between local states and regions.

As it stands, the thesis is a bit rough around the edges. Given the arbitrariness of choosing a foliation, it would be far more elegant to have a formulation of SSR that utilizes spatiotemporal rather than spatial regions. Not only would this render SSR's starting point more similar to existing axiomatic approaches, but various no-go arguments suggest that local fields and observables in interacting QFTs are only well defined in four-dimensional regions, making such a generalization apparently necessary.[6] Another complication concerns the role of the tensor product structure, which on Wallace and Timpson's formulation effectively does double duty. It serves both to identify the local Hilbert spaces and to stitch them together into a single mathematical object whose sections represent field configurations. But the tensor product structure is ill suited for this second role. In general, we can only expect the joint Hilbert space associated with two regions to be given by a tensor product if the regions are space-like separated and thus causally independent.[7] Moreover, if there are an infinite number of local Hilbert spaces, and they each have dimension greater than two, their infinite tensor product will be non-separable (that is, the global Hilbert space will lack a countable basis). But all standard frameworks for QFT employ separable global Hilbert spaces, so if we are looking for a similar formulation of SSR, we can only use the tensor product structure to talk about finite collections of space-like separated regions.[8] In order to do physics, we will need additional information about the relations between different tensor product decompositions.

We can readily modify Wallace's proposal to take these complications into account while preserving the idea that we rely on tensor products to identify and isolate subsystems. Let O' denote the space-like complement of region O (that is, the set of points space-like separated from all points in O). We can view O' as the causally independent environment associated with

[6] These arguments typically rely on the singular nature of the renormalized perturbative expansion of the dressed fields in interacting QFTs (for example, Haag [1996], pp. 55–7) and are therefore not entirely rigorous. There are some rigorous results that suggest such smearing is necessary, but they are not conclusive. For a survey, see (Halvorson and Müger [2006], Section 6).

[7] On Wallace's picture, given a tensor product, $\mathcal{H}_1 \otimes \mathcal{H}_2$, associated with regions O_1 and O_2, the corresponding local observables will be elements of the algebras $B(\mathcal{H}_1) \otimes I$ and $I \otimes B(\mathcal{H}_2)$, which mutually commute. In generic models of QFT, however, only the algebras of space-like separated regions are guaranteed to commute.

[8] A formalization of QFT using non-separable Hilbert spaces and infinite tensor products might be possible, but such a framework would be unlike anything currently on the table (at least from a mathematical perspective). In contrast, the state space view I develop in Section 4 hews closely to standard algebraic QFT. Halvorson and Müger ([2006], Section 6.2.2) discuss some possible physical applications of non-separable Hilbert spaces.

spatiotemporal system O. The key idea will be to focus on the collection of all such system-environment decompositions.

2.1 Equivalence thesis

Given a system-environment decomposition of the global Hilbert space,

$$\mathcal{H} = \mathcal{H}_O \bigotimes \mathcal{H}_{O'},$$

for each region O, define the local algebra of observables associated with O as the algebra of bounded linear operators on $\mathcal{H}_O$:

$$\mathfrak{A}(O) := B(\mathcal{H}_O).$$

Given this collection of system-environment decompositions, along with some suitable family of mappings between them, we can define a corresponding net of local observable algebras acting on $\mathcal{H}$ that satisfy the Haag–Kastler axioms for algebraic QFT. Moreover, any net satisfying the axioms arises in such a way.

To make this precise, we need to specify the relevant family of mappings encoding relations between different system-environment decompositions. (At a minimum, if O_1 is a subregion of O_2, the local Hilbert space $\mathcal{H}_{O_1}$ should be identified with a subspace of $\mathcal{H}_{O_2}$.) In addition, we might also hope to show that the reconstructed net of observables is either unique or natural in some sense. In the end it does not matter. The thesis cannot possibly be true in its current formulation. To see why, we will have to delve deeper into the algebraic structure of QFT.

3 Entanglement and the Type III Property

A C^*-algebra is an abstract collection of operators isomorphic to some subalgebra of $B(\mathcal{H})$. In algebraic QFT, a theory is given by a net of local C^*-algebras, $\{\mathfrak{A}(O)\}$, satisfying the Haag–Kastler axioms, along with a collection of physically possible global states, $\{\rho\}$, on the quasi-local algebra, $\mathfrak{A}$ (the norm-closed algebra generated by all local algebras). The self-adjoint elements of the net are taken to represent localized physical quantities, while the Haag–Kastler axioms ensure that the assignment of quantities to regions obeys the joint requirements of relativity and quantum mechanics.[9] Via the GNS theorem, each global state determines a unique representation of the net as bounded operators on a separable global Hilbert space, which in turn determines an expanded net of local von Neumann algebras, $\{\mathfrak{R}(O)\}$, and

[9] The standard six axioms are isotony, microcausality, covariance, vacuum, spectrum, and weak additivity. (Halvorson and Müger [2006], Section 2) contains a precise formulation of these axioms and a discussion of basic structural results including the Reeh–Schlieder theorem.

the global algebra, $\mathfrak{R}$, defined as the double commutant (or equivalently the weak-closure) of $\{\mathfrak{A}(O)\}$ and $\mathfrak{A}$, respectively, in the given representation. This allows for the definition of additional representation-dependent observables including the Hamiltonian and superselected charges.

Von Neumann algebras are special C^*-algebras that contain a complete set of projection operators. They can be divided into three types based on the structure of their lattice of projections. Algebraically speaking, a projection is an element $E \in \mathfrak{R}$ such that $E^2 = E$. Geometrically speaking, projection operators correspond to orthogonal projections onto closed subspaces, $E\mathcal{H} \subseteq \mathcal{H}$. For our purposes we will only need a few basic facts. Type I algebras contain minimal projections called atoms that correspond to one-dimensional subspaces of $\mathcal{H}$ and act as generators of the algebra. It is typically assumed that the local von Neumann algebras in QFT are factors (that is, they do not contain any central elements, which commute with the entire algebra).[10] A type I factor is always isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If $\mathcal{H}$ is finite-dimensional, then the factor is type I_n. If $\mathcal{H}$ is infinite-dimensional, the factor is type I_∞. Non-relativistic quantum mechanics employs type I_n algebras almost exclusively.

In contrast, the local von Neumann algebras in QFT are generically type III factors. These algebras are always isomorphic to some proper subalgebra of $B(\mathcal{H})$, where $\mathcal{H}$ is infinite-dimensional. Type III factors lack atoms entirely and have no pure, normal states.[11] In all known rigorous models of QFT, the local algebras assigned to compact double-cone regions are type III. Moreover, general theorems (Fredenhagen [1985]; Buchholz and Verch [1995]) show that for QFTs with a well-defined ultraviolet scaling limit, double-cone algebras must be type III.[12]

From a physical standpoint, the type III property reflects the high degree of entanglement displayed by the QFT vacuum and similar global states. As a consequence of the Reeh–Schlieder theorem, Halvorson and Clifton ([2000])

[10] While the physical motivation for this assumption is still unclear, nothing in the present discussion turns on it.

[11] Normal states are weak*-continuous and give rise to countably additive probability measures. They correspond to states that can be represented by density operators on the relevant GNS Hilbert space.

Type III factors are also characterized by lack of atoms and pure, normal states, but they are in many ways less pathological than type III algebras (for example, it is possible to define a semi-finite trace). While these algebras have interesting applications in quantum statistical mechanics, they are of limited use in QFT.

[12] A double cone is formed by the intersection of a past and future light cone. Double cones are compact and causally complete (that is, $O = O''$). The algebras assigned to such regions play a central role in defining many important models of QFT. The scaling limit assumption requires the renormalization group flow to approach a conformal fixed point in the short-distance/ high-energy limit. Fredenhagen secures this using the compact localization properties of Wightman field operators, while Buchholz and Verch employ the more general framework of scaling algebras. For an overview of these results, see (Halvorson and Müger [2006], Section 2.5; Haag [1996], Chapter 5.6).

prove that any global state that is analytic for the energy is entangled across arbitrary space-like separated regions. Such states encompass many (if not all) physically reasonable global states including the vacuum state as well as charged states described by Doplicher–Haag–Roberts and Buchholz–Fredenhagen superselection theory. (Informally, the analyticity requirement means that the energy cannot grow too rapidly as a function of field strength.) The Reeh–Schlieder theorem only depends on the presence of long-range, infrared correlations. In contrast, the type III property is thought to be an ultraviolet effect. Extant proofs ruling out type I and II algebras all rely on short-distance scaling assumptions that appear to be ineliminable.[13] While we expect widespread entanglement based on the Reeh–Schlieder theorem, the type III property indicates just how bad this entanglement is. Summers and Werner ([1988]) use it to show that every normal state maximally violates Bell's inequalities across space-like tangent double cones. Clifton and Halvorson ([2001]) deploy it to prove that no local operation can disentangle the local vacuum state from its environment. Both results mark a significant departure from the non-relativistic norm.

The type III property poses a severe problem for the equivalence thesis. Since the local algebras are not type I, they cannot be defined as $B(\mathcal{H}_O)$ for some local Hilbert space $\mathcal{H}_O$; they can only be concretely represented as a proper subalgebra. By itself, however, the tensor product decomposition $\mathcal{H} = \mathcal{H}_O \bigotimes \mathcal{H}_{O'}$ does not suffice to pick out a privileged type III subalgebra of $\mathcal{H}_O$. At a minimum, some additional input besides the collection of system-environment decompositions is needed to recover the full structure of algebraic QFT.

In fact, the problem is much worse. An immediate corollary of Summers and Werner's results is that in any QFT with a well-defined ultraviolet scaling limit, the required system-environment decompositions do not exist.

3.1 No-Go lemma 1

In any model of the Haag–Kastler axioms with a well-defined ultraviolet scaling limit, if the local double-cone algebras, $\mathfrak{R}(O)$, are not type I,

[13] The local algebras of unbounded space-like wedges must be type III regardless of any ultraviolet scaling assumptions. This is usually proven using the technique of half-sided modular translations (Driessler [1977]; Longo [1979]). The proof that double-cone algebras are also type III, proceeds by showing that in the short distance scaling limit, the modular structure of the double-cone algebras, $\mathfrak{R}(O)$ coincides with the modular structure of the wedge algebras, $\mathfrak{R}(W)$, and thus $\mathfrak{R}(O)$ must also be type III. Since thermodynamically well-behaved QFTs are thought to satisfy the split property, however, there will always be a type I factor, $\mathfrak{R}$, such that $\mathfrak{R}(O) \subset \mathfrak{R} \subset \mathfrak{R}(W)$. Hence it appears that some assumption about the ultraviolet scaling properties of the double-cone algebras is essential.

then then there is no pair of Hilbert spaces, $(\mathcal{K}_1, \mathcal{K}_2)$, and an isomorphism, $V: \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2$, such that $V^*\mathfrak{R}(O)V \subset B(\mathcal{K}_1) \otimes I$, with $V^*\mathfrak{R}(O')V \subset I \otimes B(\mathcal{K}_2)$.[14]

Like proofs of the type III property, this no-go lemma relies on assumptions about the ultraviolet regime of QFT. This should give us pause. Wallace ([2006], [2011]) forcefully argues that when interpreting QFT we should be cautious of drawing metaphysical conclusions from the short-distance/high-energy aspects of the theory. He contends that QFTs like the standard model are best viewed as effective field theories, low-energy approximations of some unknown, underlying theory of quantum gravity. Given results in modern renormalization analysis, we have good reason to believe these approximations are largely insensitive to the details of the exact short-distance theory. Nonetheless, we should not trust effective theories in the ultraviolet limit where we suspect that they break down. Consequently, Wallace and Timpson urge that for all practical purposes we can ignore the complications created by non-type I algebras.[15]

Insofar as we are interested in the metaphysics of the actual world, I agree. Any physical results that hinge on the ultraviolet structure of QFT should be viewed as provisional guide, at best. But SSR has the potential to be a highly general ontology for quantum theories of all shapes and sizes. We have many reasons to explore the metaphysics of exact, non-approximate QFTs. For example, we might be interested in high-level metaphysical debates about things like quantities, laws, and causation that turn on possible physics, not just actual physics. More narrowly, we might be interested in conceptual questions about the compatibility of relativity and quantum mechanics, which will require the detailed study of both exact and effective QFTs to

[14] Proof: This is a corollary of (Summers and Werner [1988], Theorem 6.1), which shows that there are no normal product states across tangent double cones O_1 and O_2 in the situation described by the lemma. Since $\mathfrak{R}(O_2) \subset \mathfrak{R}(O_1{}')$, there can be no normal product states across O_1 and $O_1{}'$ either. Two commuting von Neumann algebras acting on the same Hilbert space can be split if and only if there exist normal product states across them.

If a theory satisfies Haag duality, then $\mathfrak{R}(O') = \mathfrak{R}(O)'$, and a more elementary algebraic argument argument can be given. For purposes of reductio, assume such an isomorphism exists. Since $V^*\mathfrak{R}(O)V \subset B(\mathcal{K}_1) \otimes I$, it follows that $(B(\mathcal{K}_1) \otimes I)' \subset (V^*\mathfrak{R}(O)V)'$. But $(B(\mathcal{K}_1) \otimes I)' = I \otimes B(\mathcal{K}_2)$ and $(V^*\mathfrak{R}(O)V)' = V^*\mathfrak{R}(O)'V$, so $I \otimes B(\mathcal{K}_2) \subset V^*\mathfrak{R}(O)'V$. But by hypothesis $V^*\mathfrak{R}(O')V \subset I \otimes B(\mathcal{K}_2)$, and so by Haag duality, $V^*\mathfrak{R}(O)'V = I \otimes B(\mathcal{K}_2)$. But this is impossible. By the Tomita-Takesaki theorem $\mathfrak{R}(O)$ and $\mathfrak{R}(O)'$ must have the same type, so $V^*\mathfrak{R}(O)'V$ cannot be type I, whereas $I \otimes B(\mathcal{K}_2)$ is type I. □

[15] See (Wallace [2006], p. 711, Footnote 14, [2012], p. 301, Footnote 13). Note that in order for the local algebras to be finite as they suggest, we would have to treat both the ultraviolet and infrared cutoffs literally. If we only view the former as a real, physical cutoff (as Wallace [2011] proposes) then the local algebras must be infinite as a consequence of the Reeh–Schlieder theorem. This situation is of course compatible with type I_∞ local algebras. The problem with extending SSR to exact QFTs without an ultraviolet cutoff is not that the local algebras are infinite, it is that they are not type I.

fully answer. Even if the QFTs describing our world are effective, some theorists (for example, Rivasseau [1991]) argue that exact QFTs can provide a unifying, explanatory idealization akin to the thermodynamic limit in statistical mechanics. In addition, both the string world-sheet formalism and the AdS-CFT correspondence suggest that studying exact conformally invariant QFTs may help us to better understand the metaphysics of string theory. If it turns out that SSR is only compatible with effective QFTs, this would be an important and interesting caveat. Moreover, the status of the type III property in effective theories is still unresolved. We can wager that the details do not matter in the end, but we cannot be sure at this stage. All things considered, it would be far better to have a version of SSR that can be applied both to effective and exact models of QFT regardless of what type the local algebras turn out to be.

If we drop the standing idea that localization information is encoded by tensor products we might make some progress in this direction. Given a local state $\rho(O)$, the GNS construction generates a canonical representation of the local algebra $\mathfrak{A}(O)$ as bounded linear operators acting on the GNS Hilbert space $\mathcal{H}_{\rho(O)}$. Wallace and Timpson suggest that this GNS Hilbert space can serve as the local Hilbert space of the field in region O. If we could somehow identify $\mathcal{H}_{\rho(O)}$ with a proper subspace of the global Hilbert space $\mathcal{H}_\rho$, we might hope to eventually recover the net of observable algebras from the collection of local GNS subspaces.

As it turns out though, the so-called local GNS Hilbert space $\mathcal{H}_{\rho(O)}$ does not really deserve the title. It is actually the global Hilbert space in disguise.

3.2 No-go lemma 2

Given a net of C^*-algebras satisfying the Haag–Kastler axioms, let ρ denote a global state (analytic in energy) on the quasi-local algebra, $\mathfrak{A}$, and let $\rho(O) = \rho|_{\mathfrak{A}(O)}$. Up to unitary equivalence, the local GNS Hilbert space $\mathcal{H}_{\rho(O)}$ is naturally isomorphic to the global GNS Hilbert space $\mathcal{H}_\rho$, that is, there is a natural family of isomorphisms mapping $\mathcal{H}_{\rho(O)}$ onto $\mathcal{H}_\rho$, intertwining the actions of $\pi_{\rho(O)}(\mathfrak{A}(O))$ and $\pi_\rho(\mathfrak{A}(O))$.[16]

[16] Proof: Let $\hat{\rho}$, $\hat{\rho}(O)$ denote the cyclic vectors representing ρ and $\rho(O)$ in the GNS representations $(\pi_\rho, \mathcal{H}_\rho)$ and $(\pi_{\rho(O)}, \mathcal{H}_{\rho(O)})$, respectively. First we show that $\mathcal{H}_{\rho(O)}$ can be naturally identified with a $\mathbb{C}$-closed subspace of $\mathcal{H}_\rho$. Taking $\pi_\rho(\mathfrak{A}(O))$ as a subalgebra of $B(\mathcal{H}_\rho)$, we consider the compression map $\kappa : \pi_\rho(A) \to \pi_\rho(A)|_{\mathcal{K}}$, for all $A \in \pi_\rho(\mathfrak{A}(O))$, where $\mathcal{K} := \overline{\pi_\rho(\mathfrak{A}(O))\hat{\rho}}$ (the overline denotes the norm closure). This map is natural in the following sense: the GNS representation is fixed (up to unitary equivalence) by ρ and $\mathfrak{A}$, and $\mathcal{K}$ is a $\pi_\rho(\mathfrak{A}(O))$-invariant subspace of $\mathcal{H}_\rho$, thus κ is fixed (up to unitary equivalence) by ρ, $\mathfrak{A}$, and $\mathfrak{A}(O)$. Since $\mathcal{K}$ is $\pi_\rho(\mathfrak{A}(O))$-invariant, it follows that $\kappa \circ \pi_\rho$ is a representation of $\mathfrak{A}(O)$ on $\mathcal{K}$. Moreover, $\hat{\rho}$ a cyclic vector for $(\kappa \circ \pi_\rho, \mathcal{K})$ and $\rho(O) = \rho \circ (\kappa \circ \pi_\rho)$. These two conditions define the local GNS representation $(\pi_{\rho(O)}, \mathcal{H}_{\rho(O)})$ up to unitary equivalence (Kadison and Ringrose [1997], Proposition 4.5.3). Thus there exists an isomorphism U from $\mathcal{H}_{\rho(O)}$ onto $\mathcal{K}$ intertwining the actions of $\pi_{\rho(O)}(\mathfrak{A}(O))$ and $\pi_\rho(\mathfrak{A}(O))$.

In general, given a C^*-subalgebra $\mathfrak{B} \subset \mathfrak{A}$ and a restricted state $\rho|_{\mathfrak{B}}$ as inputs, the GNS construction yields a Hilbert space $\mathcal{H}_{\rho|\mathfrak{B}}$ that can be naturally identified with a $\mathbb{C}$-closed subspace of $\mathcal{H}_\rho$. The hitch is that if the vector implementing ρ is cyclic for $\mathfrak{B}$, then this subspace is identical to $\mathcal{H}_\rho$. The Reeh–Schlieder theorem guarantees that this will be the case for any global state analytic for the energy. Thus there is no formal sense in which applying the GNS theorem to local inputs yields a subsector of the global Hilbert space.[17]

If we take the message of this second lemma to heart, we should dispense not only with tensor products, but the very idea of a local Hilbert space. There is no obvious structure in the standard algebraic formalism that can be coherently identified as the local Hilbert space associated with a given region of spacetime. Perhaps there is some other clever way to work the standard formalism into alignment with SSR, but taken together these results strongly suggest that a different approach is needed.

4 State Space Axioms for Quantum Field Theory

In order to formalize a version of SSR compatible with type III local algebras, we have to rethink the relationship between states and observables familiar to us from non-relativistic quantum mechanics. Fortunately, algebraic QFT offers such a perspective. Rather than rays in a local Hilbert space, in algebraic QFT local states are viewed as functionals over the local algebras. Towards the end of their discussion, Wallace and Timpson ([2010], p. 712) suggest that we can adopt this algebraic perspective as a foundation for SSR: 'one can even remain at the more abstract level, forego the representation theorems and just take the C^*-algebraic state itself as denoting the properties of a region'. The idea is never fully developed, however, and given that the local algebraic state is defined in terms of the observables it looks like a non-starter. What can the algebraic state possibly be besides a list of expectation values for the local observables?

In spite of this, thinking about the local state in abstract algebraic terms actually gives us a way to reframe SSR to accommodate the type III property.

Next, we show that in fact $\mathcal{K} = \mathcal{H}_\rho$. By the Reeh–Schlieder theorem, we obtain $\pi_\rho(\mathfrak{A}(O))''\hat{\rho} = \mathcal{H}_\rho$. But if $\hat{\rho}$ is cyclic for $\pi_\rho(\mathfrak{A}(O))''$, it is separating for $\pi_\rho(\mathfrak{A}(O))'' = \pi_\rho(\mathfrak{A}(O))'$, and hence must also be cyclic for $\pi_\rho(\mathfrak{A}(O))$. Therefore $\mathcal{K} = \overline{\pi_\rho(\mathfrak{A}(O))\hat{\rho}} = \mathcal{H}_\rho$. It follows that the compression κ is the identity map, and the local GNS representation $(\pi_{\rho(O)}(\mathfrak{A}(O)), \mathcal{H}_{\rho(O)})$ is unitarily equivalent to $(\pi_\rho(\mathfrak{A}(O)), \mathcal{H}_\rho)$. Although the isomorphism U implementing this equivalence may not be unique, it is part of a family determined by the natural compression κ, and each such isomorphism maps $\mathcal{H}_{\rho(O)}$ onto $\mathcal{H}_\rho$. □

[17] Results from the modular localization program (Brunetti *et al.* [2002]) indicate that it is possible to naturally assign a $\mathbb{R}$-closed proper subspace of the global Hilbert space to each spacetime region. Since these subspaces are not $\mathbb{C}$-closed, however, they are not large enough to be interpreted as local state spaces.

The key is to shift attention from the local state to the collection of possible local states. The state space, $\mathcal{S}(\mathfrak{A})$, of a C^*-algebra $\mathfrak{A}$, is defined as the collection of all positive, normalized, $\mathbb{C}$-valued linear functionals on $\mathfrak{A}$. While the state space is not a Hilbert space, it has a rich geometry; it is not just a bare set of functionals.

First, $\mathcal{S}(\mathfrak{A})$ is a convex set. Pure states are extremal points, while each mixed state can be written as a convex linear combination of distinct pure states,

$$\rho = \lambda \rho_1 + (1 - \lambda)\rho_2, \tag{4}$$

where $\lambda \in (0, 1)$. Second, $\mathcal{S}(\mathfrak{A})$ also has an order structure inherited from the order structure on $\mathfrak{A}$: $\rho_1 \geq \rho_2$ if and only if $\rho_1(A) - \rho_2(A) \geq 0$ for all self-adjoint $A \in \mathfrak{A}$ with positive spectrum. In addition, $\mathcal{S}(\mathfrak{A})$ is compact in the weak*-topology.[18] For each self-adjoint operator $A \in \mathfrak{A}$, there is a weak*-continuous affine function, $\hat{A} : \mathcal{S}(\mathfrak{A}) \to \mathbb{R}$, defined by setting

$$\hat{A}(\rho) = \rho(A) \tag{5}$$

for all $\rho \in \mathcal{S}(\mathfrak{A})$. Kadison ([1951]) proves that the mapping sending $A \mapsto \hat{A}$ is an isometric isomorphism. This fact is the foundation for the spectral calculus on $\mathfrak{A}$ and shows that the self-adjoint part of $\mathfrak{A}$ is determined by the affine, order, and topological structure of $\mathcal{S}(\mathfrak{A})$. If $\mathfrak{A}$ is a von Neumann algebra, we may also choose to restrict attention its normal state space, the set of all weak*-continuous elements of $\mathcal{S}(\mathfrak{A})$. The normal state space has similar convexity and spectral properties but different topological features.[19]

An instructive, easily visualizable example is given by a two-level quantum system. The relevant von Neumann algebra is $M_2(\mathbb{C})$, the algebra of 2×2 matrices with complex entries. This is a type I_2 factor isomorphic to $B(\mathcal{H}_2)$. In this case, every state is normal, so the state space and the normal state space coincide. It is affinely isomorphic to a three-dimensional Euclidean sphere with unit radius (that is, a Euclidean three-ball). Each state can be written as a positive trace-one matrix,

$$\frac{1}{2}\begin{pmatrix} 1 + x & y + iz \\ y - iz & 1 - x \end{pmatrix}, \tag{6}$$

where (x, y, z) are the Cartesian coordinates of the corresponding point in

[18] This is defined as the coarsest topology such that every element of $\mathcal{S}(\mathfrak{A})$ corresponds to a continuous map on $\mathfrak{A}$.

[19] Typically, the normal state space will not be compact in any useful topology, and the norm topology plays a more prominent role than the weak*-topology in many structural theorems. Interestingly, the state space of an arbitrary C^*-algebra, $\mathfrak{A}$, is affinely isomorphic to the normal state space of its universal enveloping von Neumann algebra, $\mathfrak{A}^{**}$ (Alfsen and Shultz [2001], Corollary 2.126). Viewed as such, it has additional properties (for example, weak*-compactness) that generic normal state spaces lack. Hence, while a von Neumann algebra is a special case of a C^*-algebra, a C^*-state space is more properly viewed as a special case of a normal state space.

sphere.[20] Boundary points represent pure states and interior points represent mixed states. If ρ is a statistical mixture of pure states ρ_1 and ρ_2, then ρ lies on the line segment connecting boundary points ρ_1 and ρ_2. Using Kadison's mapping, self-adjoint elements of $M_2(\mathbb{C})$ correspond to bounded $\mathbb{R}$-valued affine maps on the ball that attain their maximum and minimum on antipodal points.

These observations raise the following question: if we take local states rather than observables as primitive, is it possible to reconstruct the local observables as an algebra of functionals on the local state space? A reconstruction theorem due to Alfsen, Hanche-Olsen, and Shultz ([1980]) shows that the answer is yes.[21] Geometrically, the spectral information encoded by Kadison's isomorphism appears in the orthogonality relations between different faces of the convex set $\mathcal{S}(\mathfrak{A})$. A face is a convex subset, $F \subseteq \mathcal{S}(\mathfrak{A})$, that contains any line segment in $\mathcal{S}(\mathfrak{A})$ with interior points in F. (This generalizes the concept of a face of a polygon to an arbitrary convex set.) A face is exposed relative to some topology if the face can be isolated by intersecting $\mathcal{S}(\mathfrak{A})$ with a supporting hyperplane closed in that topology.

There is a tight structural analogy between exposed faces of $\mathcal{S}(\mathfrak{A})$ and projection operators in $\mathfrak{A}$. In the special case of a von Neumann algebra, this analogy is perfect: the norm-exposed faces of the normal state space generate an orthomodular lattice that is naturally isomorphic to the lattice of projection operators. Thus the quantum logic of projections is mirrored in the facial structure of state space.[22] In our two-level example, the norm-exposed faces are just the boundary points of the three-ball. Each such point is a pure state associated with a projection operator with expectation value one in that state. Orthogonal projection operators are associated with antipodal points.

In order to recover the full structure of the corresponding algebra, one additional piece of geometric information is needed, a state space orientation. The smallest face of $\mathcal{S}(\mathfrak{A})$ containing a pair of distinct pure states is affinely isomorphic to either a straight line or a Euclidean three-ball. (The face is a line if and only if the GNS representations of the associated states are quasiequivalent and a three-ball if and only if the representations are disjoint.) The non-commutative operator product on $\mathfrak{A}$ determines an orientation for each facial three-ball given by an equivalence class of affine automorphisms where two automorphisms are equivalent just in case they can be related by a rotation.[23] Alfsen *et al.* ([1980]) prove the converse, showing that a suitably

[20] See (Alfsen and Shultz [2001], Theorem 4.4).

[21] Alfsen and Shultz ([2001], [2003]) give a comprehensive treatment of these results.

[22] The general case is a bit more complicated since C^*-algebras do not always possess a complete set of projections, but the basic idea is the same. In full generality, the lattice of weak*-semi-exposed faces in $\mathcal{S}(\mathfrak{A})$ is isomorphic to the lattice of upper semi-continuous projections in the universal enveloping algebra $\mathfrak{A}^{**}$ (Alfsen and Shultz [2001], Theorem 3.61).

continuous choice of orientation for each facial three-ball suffices to determine the operator product of the associated algebra.

Geometrically, choosing an orientation determines a unique correspondence between $\mathbb{R}$-valued affine functions on $\mathcal{S}(\mathfrak{A})$ (that is, self-adjoint observables) and one-parameter groups of affine automorphisms of $\mathcal{S}(\mathfrak{A})$. From a physical standpoint, this correspondence captures the role that observables play as infinitesimal generators of particular symmetries. The two-level example elegantly illustrates this idea. Each self-adjoint $A \in M_2(\mathbb{C})$ acts as a $\mathbb{R}$-valued functional on the Euclidean three-ball, attaining a maximum and minimum value on a pair of antipodal points. The operators iA and $-iA$ generate infinitesimal state space symmetries, rotations of the three-ball around the diameter between these points. The choice of an orientation structure determines which of these generate clockwise and counterclockwise rotations.

The AHS-reconstruction theorem shows that it is possible to recover a local observable algebra from its oriented state space. This is not enough to capture the full physics of QFT, however, since in most models the local algebras are isomorphic. Consequently the physical differences between various models of QFT must be encoded in the relations between local algebras rather than their internal algebraic structure. We must find a way to recover the structure of the complete net of observable algebras given suitable relations between their state spaces.

The eventual goal is to formulate extensionally equivalent state space analogues of the standard Haag–Kastler axioms. The first axiom, isotony, requires that if O_1 is a subregion of O_2, the local algebra $\mathfrak{A}(O_1)$ must be a subalgebra of $\mathfrak{A}(O_2)$. This gives the collection of local algebras the structure of a net, and allows the quasi-local algebra to be defined as the colimit of this structure. Its dual, $\mathcal{S}$-isotony requires the existence of a privileged restriction mapping, $\psi : \mathcal{S}(O_2) \to \mathcal{S}(O_1)$, between the corresponding oriented local state spaces. Formally, ψ must be an $\mathcal{S}$-homomorphism, a weak*-continuous affine surjection that preserves complementary faces and whose inverse preserves orientation. This gives the collection of local state spaces the structure of a presheaf, dual to the net of observable algebras, and allows us to define the quasi-local state space as its limit.

The translation of the remaining axioms is somewhat mathematically involved, but the basic motivation is straightforward. $\mathcal{S}$-microcausality says that the exposed faces of space-like separated local state spaces must be antipodal (except possibly on their intersection) when embedded within the state space of any enveloping region. This mirrors the idea that two algebras

[23] For each facial three-ball, this effectively corresponds to the usual notion of orientation for a manifold. In particular, a three-ball always admits two possible orientations, 'right-' and 'left-handed'. This is not true across the entire state space. In general there will be infinitely many suitably continuous ways to choose an orientation for each facial three-ball corresponding to the infinite number of Lie products compatible with the Jordan product on $\mathfrak{A}$.

commute if and only if their associated spectral projections do. $\mathcal{S}$-covariance requires the existence of a suitably continuous representation of the Poincaré group (or the translation subgroup) in the group of automorphisms of the presheaf of local state spaces. This encodes the dynamics and allows us to identify the vacuum state as a fixed point of the group action. The $\mathcal{S}$-analogue of the spectrum condition requires that the vacuum state lie in the (exposed, split) face generated by the positive energy states (that is, those states whose Fourier transformations have particular spectral support properties relative to the momentum-space forward light cone).

Drawing upon these ideas, we can frame a revised version of Wallace's equivalence thesis.

4.1 Revised equivalence thesis

Given an assignment, $\{\mathcal{S}(O)\}$, of oriented state spaces to spacetime regions satisfying the $\mathcal{S}$-axioms, we can define a dual net of observables satisfying the Haag–Kastler axioms. Moreover, any net satisfying the usual axioms arises in this way.

This gives us sharpened mathematical conjecture that has a fighting chance of actually being true. The $\mathcal{S}$-axioms characterize the family of mappings between local state spaces missing from Wallace's original version. Moreover, because of the 1–1 correspondence between oriented state spaces and C^*-algebras given by the AHS-reconstruction theorem, the revised thesis is compatible with local algebras of any type. It stands to provide a state-first formulation of QFT that has the same flexibility and power as algebraic QFT.

A follow-up paper (Swanson and Halvorson [unpublished]) will provide a mathematically precise formulation of the $\mathcal{S}$-axioms and endeavour to prove the revised equivalence thesis. The primary goal here has been to motivate the kind of formal framework needed to circumvent the type III problem. Our conceptual understanding of QFT has been greatly enhanced by shifting to an abstract algebraic picture on the observable side. It is only natural to suspect that the parallel move on the state side will provide similar insight into the foundations of relativistic SSR. To assuage the sceptical reader, we note that Shultz ([1981]) proves that $\mathcal{S}$-homomorphisms are dual to *-homomorphisms, establishing that the AHS-reconstruction result is categorical. Thus from a purely mathematical angle, anything we can do with C^*-algebras, we can do with oriented state spaces. It remains to be seen if the revised equivalence thesis is similarly categorical and what independent physical motivation can be given for the $\mathcal{S}$-axioms.[24]

[24] In addition, we might eventually hope to develop state space analogues of standard quantization techniques. As long as our methods for constructing QFTs rely on operators as tools, there will be the worry that the state space framework is not sufficiently independent. The relationship between the new framework and constructive QFT is an important issue, but one beyond the

The defender of SSR thus has two options on the table. She can restrict the scope of her thesis to effective QFTs and use Wallace and Timpson's original proposal, or she can broaden her sights and pursue the state space approach advocated here. The flexibility to treat non-type I local systems is a marked advantage of the latter. In addition, the dual state space language facilitates connections with the extensive literature on algebraic and constructive QFT. In many instances we can appeal to the duality between algebras and state spaces to show that a certain construction must be possible on the state space side, even if we do not yet know all of the details. In the final section, we will use this technique to diffuse a number of preliminary objections that have been levied against SSR.

5 Discussion

SSR claims that the local state encodes intrinsic properties of a region, but what exactly are these properties? Wallace and Timpson ([2010], pp. 699–700) recognize that they must be 'admittedly somewhat alien', but urge that the situation is in principle no different from classical field theories: 'it is not as if we really have an intuitive grasp of what an electric or a magnetic field is, other than indirectly and by means of instrumental considerations ("A test charge would be accelerated thus", for example)'. They argue that everything must ultimately be analysed in terms of patterns of fundamental properties of spacetime regions, and that our epistemic grip on the properties themselves comes from two sources: (i) the structure of the mathematical entities used to represent them, and (ii) instrumental connections between these entities and observation. As critics have noted, however, there are serious concerns about both (i) and (ii).

One prominent objection to SSR is that we do not adequately understand the relevant instrumental connections (ii). Both Lewis ([2013]) and Baker ([2016]) worry that the view makes the relationship between fundamental quantities and experimental predictions hopelessly obscure, if not incoherent. Lewis observes that density operators are represented by complex matrices, 'but the matrix elements themselves determine a probability distribution over outcomes. How do the actual outcomes give us insight into probabilistic properties'? As Baker puts it:

> We derive statistical predictions from the quantum state via expectation values, but what is the meaning of an expectation value of energy, for

scope of this article. Landsman ([1998], Chapter 1.2–3) develops a geometric quantization procedure linking the pure state space of a type I quantum mechanical system to a classical Poisson manifold where both are viewed as transition probability spaces. These techniques could provide a bridge to more general state space quantization procedures.

> example, if energy is not among the fundamental quantities (which are exhausted by the local states)? Moreover, how are we to make sense of a deterministic theory whose fundamental quantities appear to be specified indirectly by way of statistical expectation values?

Baker goes on to point out that Wallace and Timpson criticize rival views on the grounds that they posit a brute connection between operators and expectation values, rather than providing an explanation of how expectation values arise through an analysis of the physics of measurement. But according to Baker, essentially the same criticism applies to SSR: the fundamental quantities, the local states, directly encode expectation values by fiat.

While these objections target (ii), at their core lies a host of confusions surrounding (i), most conspicuously, the idea that the local state is nothing more than a list of expectation values. The defender of SSR should reject this idea (as Wallace and Timpson do), but it is incredibly difficult to see how this is possible from the perspective of the original density operator formalism.

The difficulty is brought out by a pair of puzzles about the symmetries of Hilbert space that cast doubt on whether the field of local density operators is sufficient to encode all of the relevant physics. Wallace ([2012]) observes that we usually think of the quantum state as represented by a ray rotating in Hilbert space. But Hilbert spaces are highly symmetrical (up to isomorphism they are only distinguished by dimension). Each ray is just like every other, so it is not clear how a ray can represent a structured entity like the quantum state in a region. A second, related puzzle, originally put to me by Baker (personal communication), turns on essentially the same technical point. All existing models of QFT employ infinite-dimensional, separable Hilbert spaces as the global state space, and all such Hilbert spaces are isomorphic. Therefore by Gleason's theorem, all models of QFT have the same set of global density operators. These in turn determine the local density operators. But if this is all there is to the ontology of QFT, as SSR suggests, then it seems that SSR cannot distinguish between physically inequivalent models of QFT, for instance, a free Klein–Gordon model and an interacting ϕ^4-model.

In response to the first puzzle, Wallace ([2012], p. 296) concludes that in addition to density operators we need to specify a privileged set of observables acting on the Hilbert space. This breaks the symmetry, allowing different rays to represent distinct physical possibilities: 'Differences between states correspond to differing patterns of assignments of numbers to operators, and *those* patterns can be highly structured'. A similar response can be given to the second puzzle: although the Klein–Gordon model and the ϕ^4-model are implemented on isomorphic Hilbert spaces, they carry unitarily inequivalent representations of the canonical commutation relations that breaks the symmetry. The patterned assignment of numbers to operators is different in the free and interacting theory. Yet even if we treat the operators as merely formal tools as Wallace and Timpson urge, it remains unclear how we should interpret the numbers assigned to them by the states. If they represent expectation values, then the Lewis–Baker objection seems on point—differences between states are differences in fundamentally probabilistic properties. If they represent something else, then what, and why are these entities directly correlated with expectation values?

The equivalence thesis suggests a possible escape route. All of this algebraic information can instead be encoded in the localization structure, in the tensor product decomposition of the global Hilbert space into subsystems. A state is not just a ray or a density operator, but rather a section of this tensor product structure. Differences between theories boil down to differences in their subsystem decompositions. Although patterns of expectation values can reveal the complexity of this geometry, these patterns emerge from the localization structure, not vice versa.

Unfortunately, in its original formulation the equivalence thesis is far too vague to be of much practical help. As we have seen, most of the physical content is actually contained in mappings between different tensor product decompositions, however the thesis is largely silent on what this network of mappings should look like. It is not clear at this stage how to identify SSR field configurations as sections of some suitable bundle-like structure, nor is it clear how differences between unitarily inequivalent QFTs can be grounded in different tensor product decompositions. Here, the new state space formulation of SSR can be of great service. It offers a rich geometry capable in principle of encoding everything that can be found in algebraic QFT. Three basic observations about this geometry serve to dissolve the two symmetry puzzles and counter the Lewis–Baker objection.

First, the $\mathcal{S}$-isotony and $\mathcal{S}$-microcausality axioms provide a more precise characterization of the localization structure in QFT than Wallace's vague suggestions about tensor product decompositions. The antipodality constraints imposed by $\mathcal{S}$-microcausality ensure that space-like separated systems are suitably independent, while the network of restriction mappings characterized by $\mathcal{S}$-isotony provides information about the relations between regions, weaving the collection of local state spaces together into a single mathematical object whose sections represent SSR field configurations. It also ensures that each point in the state space associated with region O has a canonical restriction to a unique point in the state space of each subregion. Thus we can speak coherently of the local presheaf associated with each region. The local state is not just a point in the convex set $\mathcal{S}(O)$, it is a section of the local presheaf. Points that are intrinsically alike with respect to the geometry of $\mathcal{S}(O)$ can correspond to geometrically distinct sections of the local presheaf.

Second, by itself, this presheaf structure is insufficient to do physics. Unitarily inequivalent QFTs like the Klein–Gordon model and ϕ^4-model or Klein–Gordon models with different masses can be constructed by starting with isomorphic nets of C^*-algebras. They are differentiated by choosing inequivalent representations of the Poincaré group acting as automorphisms of the net. In suitable positive energy GNS representations, privileged sets of operators are singled out as generators of the corresponding spacetime symmetries, justifying their role as energy-momentum and angular-momentum observables. In Wallace's picture, these observables serve a critical function, essentially labelling sectors of Hilbert space so that otherwise identical rays can encode states with different physical content. In our revised state space picture, the $\mathcal{S}$-covariance axiom specifies a representation of the Poincaré transformations acting as one-parameter groups of presheaf automorphisms. The orbits of these groups, rather than a set of energy-momentum observables, serve to label portions of state space so that otherwise identical sections can represent different configurations of the state-valued field.

Third, given a net of C^*-algebras equipped with a representation of the Poincaré group, it remains possible to choose different global states and, via the GNS construction, arrive at unitarily inequivalent representations of the net. Such representations can be distinguished by different global boundary conditions characterized by different global von Neumann algebras. This scenario arises, for instance, when describing different charge sectors in theories characterized by DHR/BF superselection theory. The global algebras in each sector contain non-trivial central elements that are the spectral projections of associated charge observables. These structural differences are mirrored in the convex geometry of the dual global state spaces (where central projections correspond to split faces). Like algebraic QFT, the state space version of SSR thus gives us the flexibility to handle reducible representations with global algebras other than $B(\mathcal{H})$.

To sum up: A state is not just a ray or density operator, nor is just a point in a convex, oriented set. It is a section of a presheaf of such sets, carrying a privileged representation of the Poincaré group, and thus a highly structured entity. This geometry is dual to a net of C^*-algebras, and provided that the revised equivalence thesis is correct, it gives us all of the necessary tools to individuate unitarily inequivalent models of QFT. Differences between models boil down to differences in the presheaf structure, the Poincaré representation, or the structure of the global state space in particular representations. These observations also serve to clarify Wallace's conjecture that the localization structure is sufficient to determine the physics. If 'localization structure' is interpreted to mean a presheaf of oriented C^*-state spaces satisfying $\mathcal{S}$-isotony and $\mathcal{S}$-microcausality, then the conjecture is false. In addition to knowing which states are localized in which regions, we need to know which symmetries of the presheaf correspond to which spacetime symmetries and which global states are physically possible.[25] But SSR does not sink with this conjecture (whose primary purpose was to help make the equivalence thesis plausible). Rather, we have simply gained a better understanding of the convex geometry that characterizes the field of local states.[26]

By clearly separating this geometry from talk about 'assignments of numbers to operators', the view also provides more breathing room to begin responding to Lewis and Baker. At the fundamental level, the field dynamics are completely deterministic. There are no primitive probabilistic properties (unless we modify the theory in the service of a collapse interpretation). We can introduce observables as bounded affine functions on the presheaf of local state spaces, but although their mathematical structure is fixed once the state space structure is specified, their physical interpretation is not. Which functions encode the expectation values for which kinds of measurements must be explained. One possible strategy would analyse measurement procedures in terms of functional patterns in the state-valued field configuration. (The details would depend on how we choose to solve the measurement problem.) Particular affine functions would then be identified with equivalence classes of measurement procedures. If successful, this strategy would in some sense vindicate the early operationalist

[25] Is this enough? That remains to be seen. If unitary equivalence is sufficient for physical equivalence, then it should be. Much of the algebraic QFT literature assumes that physical differences between models supervene on the net of C^*-algebras, the Poincaré representation, and the set of physically possible global states. Once these structures are fixed, we do not need to be told which operators represent which physical quantities. This can be inferred from the kinematical and dynamical properties of the model. SSR adopts the same mantra on the state space side. If it turns out that quantization rules or some other physical correspondence principles are essential not just for constructing models but for endowing them with their physical content, then more might be needed. The aim here is not the settle this question decisively, but the put SSR on a formally coherent foundation, so that its merits and flaws can be better assessed.

A further point of clarification: if 'localization structure' is instead interpreted to mean a presheaf of state spaces dual to the net of von Neumann algebras in a particular representation, then the status of Wallace's localization conjecture is less clear. The global von Neumann algebra now appears explicitly as part of the presheaf structure, and algebraic proofs of the PCT and spin-statistics theorems indicate that we can naturally reconstruct a representation of the Poincaré group from a net of concrete von Neumann algebras satisfying microcausality and a further technical condition, modular covariance (Guido and Longo [1995]).

It is doubtful that this move plausibly captures Wallace's original intuition, however. Modular covariance requires that certain algebraic invariants (the modular automorphism groups associated with wedge-localized von Neumann algebras) generate certain spacetime symmetries. Thus we still need information relating symmetries of the net of to symmetries of spacetime, only now this information is encoded in the algebraic properties of an enriched net of operators. It seems unlikely that this constraint can be plausibly interpreted as an assumption about the localization of physical quantities.

[26] It might be objected that the current proposal, which posits geometric relations between local states, is less ontologically parsimonious than the original version of SSR, which simply posits an assignment of states to regions. It should be noted, however, that the original proposal already posits substantial geometric relations between local states encoded in the Hilbert space and tensor-product structure (for example, convexity, extendability, and orthogonality relations). It is true that additional geometric relations are needed, but without them SSR will not be capable of adequately differentiating models of QFT.

interpretation of observables in algebraic QFT.[27] The crucial difference is that it would do so by providing a realist explication of concepts like 'measurement device' and 'preparation procedure', concepts taken as primitive by the operationalist view. Differences between states ultimately give rise to different patterns of expectation values, but only after a long, complex physical explanation. The states do not encode expectation values by fiat. In fact, it is the observables that emerge as lists of expectation values on this view.[28]

Obviously, much work remains to be done in order to spell out the details of such a story. There may be other viable strategies as well. Baker ([2016]) is right to point out that 'a perspicuous explanation for the expectation values is a major missing piece in extant field interpretations' like SSR. But by recasting the quantum state as a section of a highly structured presheaf, we can start to see how such an explanation might be possible, as well as some of the broad contours it could take. (For this reason, the present proposal should be of interest not just to philosophers concerned with the technical foundations of QFT, but to those interested in non-relativistic quantum theory as well.) The ultimate viability of relativistic SSR, including its relationship to more traditional interpretations of algebraic QFT, hinges on a proof of the revised equivalence thesis and a detailed study of the $\mathcal{S}$-axioms. Only this will reveal whether or not the state-valued field is sufficiently structured to ground the kind of physical emergence story that SSR wants to tell. By framing the thesis as a precise mathematical conjecture and providing the tools to solve the type III problem, the preceding investigation gives us the roadmap necessary to begin this project in earnest.

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[27] See (Araki [1999], Chapter 1) for a summary of the central motivating ideas.

[28] To be clear, the view is not that physical quantities, such as energy, are equivalence classes of measurement procedures, only that the corresponding observables are. The state-valued SSR field is a physical field, and just like the tensor-valued electromagnetic field, facts about its energy content will supervene on facts about the underlying field configuration. If this is right, we should be able to identify some structural pattern in the SSR field as its energy (similar to the definition of the electromagnetic stress energy-tensor in terms of the electromagnetic field tensor). The energy observable, an $\mathbb{R}$-valued functional over the SSR field, is interpreted operationally as a list of expectation values associated with a suitable equivalence class of measurement procedures that reliably track information about the energy content of the SSR-field. In short, observables no longer directly represent physical quantities as they do on the usual interpretation of algebraic QFT.

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