Dimensioned Algebra

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Abstract

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2.1 The formal structure of dimensional analysis

"**Question**: A water dispenser has two taps, one dispenses $36.7\,\mathrm{cm}^3/\mathrm{s}$ and the other $2.1\,\mathrm{L/min}$, how long will it take for an empty $300\,\mathrm{cm}^3$ cup to fill up when placed under both taps? **Answer**: The flow through a pipe measures volume of fluid traversing the pipe per unit time, so combining both taps means that we should add the flows. To this end we write both flows in the same units using the appropriate conversion factors to find a combined flow of $2.2\,\mathrm{L/min} + 2.1\,\mathrm{L/min} = 4.3\,\mathrm{L/min}$. Setting $F = V/T$ with $F = 4.3\,\mathrm{L/min}$ and $V = 300\,\mathrm{cm}^3 = 0.3\,\mathrm{L}$ we find $T = 0.06\,\mathrm{min}$ so the cup will fill up in approximately 4 seconds, again using the appropriate conversion factors."

This elementary hydraulics problem, familiar to pupils in the earliest stages of scientific education, will help us illustrate the main formal structure of physical quantities as commonly used in practical science and engineering.

We first observe how the relevant quantities are specified by a numerical value and a unit of measurement, which is arbitrary within the class of units *measuring the same type* of physical quantity. We thus see that concrete physical quantities, e.g. resulting from a measurement, are a sort of equivalence classes of numerical values in all possible appropriate units of measurement; indeed, they are far from being simply numerical values as sometimes conceptualised in theoretical discussions.

Another general observation is that algebraic operations correspond to specific physical phenomena, i.e. the addition of flows as physical quantities is correlated with the combination of tap streams and the multiplication by rates is correlated with the passage of time. Although we will not delve further on this topic here, we note that this is rarely considered beyond elementary applications and most theoretical treatments simply postulate algebraic operations as formal devices without explicit connection to physical phenomena.

The most relevant aspect to our discussion is the peculiar way in which algebraic operations behave: addition can only be performed between quantities specified by the same unit of measurement and only affects the numerical part, addition is otherwise undefined; multiplication can be performed between any two arbitrary physical quantities, it affects the numerical part and the units part; all the algebraic operations are compatible with conversion factors that allow to change between units of the same kind. These are, of course, the usual *rules of the game* in standard **dimensional analysis**. For a comprehensive review on the subject of dimensional analysis see [BI96].

Simplifying physical quantities as numerical values, i.e. elements of $\mathbb{Q}$ or $\mathbb{R}$, is ubiquitous in theoretical and mathematical models of science, however, in light of the comments above, this characterisation is manifestly insufficient. Given the long history of metrological science and dimensional analysis, it is perhaps surprising that it wasn't until the last few decades that the structure of physical quantities was given rigorous mathematical characterisations. For a review of the history of metrology see [Zap19, Ch. 3].

The main efforts at developing a general mathematical theory of physical quantities are due to Hart [Har12] and Janyška-Modugno-Vitolo [JMV07] [JMV10]. Particularly, the latter authors develop a rich theory of semi-vector spaces and positive spaces that captures all the standard features of dimensional analysis in a transparent and mathematically rigorous way. In a more abstract setting, Dolan and Baez found a characterisations of physical quantities as line objects in monoidal categories [BD09].

In the present paper we give a rigorous treatment of the general algebraic structures that appear in dimensional analysis with an emphasis on the partial nature of addition of physical quantities and the multiplicativity properties of units of measurement.

2.2 Dimensioned Sets and Binars

Following our discussion in Section 2.1, we are led to consider 'typed' or 'labelled' sets as the primary objects to study. Our approach is to formally define these 'labelled sets' and to identify the natural categories that emerge from considering different compatibility conditions between the 'labelling' structure and binary operations defined on the sets.

The notion that physical quantities have 'types' is captured simply by what we call a **dimensioned set** which is nothing but a surjection of sets $\delta : A \to D$. We call D the **set of dimensions**, δ the **dimensionality projection** and the preimages $A_d := \delta^{-1}(d) \subset A$ **dimension slices** of the set A. In the interest of brevity we may denote a dimensioned set $\delta : A \to D$ simply as A_D. A **morphism of dimensioned sets** or **dimensioned map** $\Phi_\varphi : A_D \to B_E$ is simply a morphism of surjections, that is, a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ D & \xrightarrow[\varphi]{} & E \end{array}$$

The **category of dimensioned sets** is denoted by DimSet. The **cartesian product** of two dimensioned sets $A_D \times B_E$ is defined in the obvious way:

$$\begin{array}{c} A \times B \\ \downarrow{\scriptstyle\delta\times\epsilon} \\ D \times E \end{array}$$

A distinguished singleton considered as a dimensioned set $\{\bullet\} \to \{\bullet\}$ clearly acts as a terminal object in DimSet and as a unit for $\times$. The unit object is denoted by 1 and will be called the **dimensionless set**. It is then easy to see that $(\mathsf{DimSet}, \times, 1)$ forms a cartesian monoidal category. Further assuming that 1 is an initial object in DimSet corresponds to assuming that all dimension sets have a distinguished element representing the 'dimensionless' dimension; we shall see in later sections that it is sometimes useful to make such assumption.

It will be useful to introduce a notation that reflects the dimensioned structure explicitly so that we can keep track of consistency of expressions. In what follows, unless redundant, elements of a dimensioned set will be denoted with a subscript indicating its dimension projection: for a dimensioned set $\delta : A \to D$ we denote

$$a_d \in A \;\text{ where }\; d = \delta(a) \in D.$$

With this convention, the action of a dimensioned map Φ_φ is notated $\Phi(a_d) = \Phi(a)_{\varphi(d)}$.

Let us now discuss binary operations on dimensioned sets. For the remainder of this section it will be useful to think of dimensioned sets as disjoint unions of their dimension slices:

$$A_D = \bigcup_{d\in D} A_d.$$

General, possibly partially-defined, binary operations defined on dimensioned sets appear, in principle, considerably more nuanced that ordinary binary operations. Natural compatibility conditions between binary operations and the dimensioned structure may be imposed in multiple ways. Without any further choices, however, there are two canonical types of binar-like structure whose composition domains intersect with the dimension slices in extremal ways: on the one hand binar structures whose composition domain equal to the dimension slices, these will be called dimensional binars; and on the other, binar structures whose composition domain is the entire set together with a transitivity condition between dimension slices, these will be called dimensioned binars. Intuitively, dimensional binars are operations *strictly within* the dimension slices and dimensioned binars are operations *strictly between* the dimension slices.

Let a dimensioned set $\delta : A \to D$, a **dimensional binar** structure $(A_D, *_D)$ is a partially-defined binary operation on A with

$$a * b \;\text{ defined only when }\; \delta(a) = \delta(a * b) = \delta(b).$$

This means that in the expression $a_d *_d b_d$ all subscripts must agree for the product to be defined. In other words, a dimensional binar is a collection of ordinary binars indexed by the set of dimensions $\{(A_d, *_d),\, d \in D\}$. A **morphism of dimensional binars** $\Phi_\varphi : (A_D, *_D) \to (B_E, \circ_E)$ is a dimensioned map Φ_φ such that

$$\forall\, d \in D,\, a, b \in A_d \qquad \Phi(a *_d b) = \Phi(a) \circ_{\varphi(d)} \Phi(b).$$

Again, a morphism of dimensional binars can be regarded as a collection of ordinary morphisms of binars between the dimension slices:

$$\{\Phi_d : (A_D, *_d) \to (B_{\varphi(d)}, \circ_{\varphi(d)}),\ d \in D\}.$$

The **category of dimensional binars** is denoted by DimBin. The cartesian product $\times$ on DimSet extends to the category dimensional binars via the obvious construction: given two dimensional binars $(A_D, *_D)$ and $(B_E, \circ_E)$ there is a dimensional binar structure on the product $(A_D \times B_E, \bullet_{D\times E})$ where:

$$(a_d, b_e) \bullet_{(d,e)} (a'_d, b'_e) := (a_d *_d a'_d, b_e \circ_e b'_e)$$

for all $a_d \in A_D$ and $b_e \in B_E$.

Interestingly, in direct analogy with ordinary sets and binars, the sets of morphisms of dimensional binars carry natural dimensional binar structure. A trivial observation is that morphisms of dimensioned sets, which are simply morphisms of surjections, have a natural surjection into maps of sets, i.e. $\pi : \Phi_\varphi \mapsto \varphi$. This means that for any two dimensioned sets A_D and B_E the set of morphisms together with the natural surjection $\pi : \mathsf{DimSet}(A_D, B_E) \to \mathsf{Set}(D, E)$ is a dimensioned set. Furthermore, when there are dimensional binar structures on the sets $(A_D, *_D)$ and $(B_E, \circ_E)$, a point-wise construction endows $\mathsf{DimSet}(A_D, B_E)$ with a dimensional binar structure: given two morphisms $\Phi_\varphi, \Psi_\psi \in \mathsf{DimSet}(A_D, B_E)$, for all $a_d \in A_d$ define

$$\Phi_\varphi \circ \Psi_\psi(a_d) := \Phi(a)_{\varphi(d)} \circ_e \Psi(a)_{\psi(d)}$$

which is indeed only possible when $\varphi(d) = e = \psi(d)$. Hence the point-wise operation so defined will endow the set of dimensioned maps with a dimensional binar structure:

$$(\mathsf{DimSet}(A_D, B_E)_{\mathsf{Set}(D,E)},\ \circ_{\mathsf{Set}(D,E)}).$$

Let a dimensioned set $\delta : A \to D$, a **dimensioned binar** structure $(A_D, *^D)$ is a totally-defined binary operation $*$ on A with

$$\forall\, d, d' \in D,\ \exists!\, d'' \in D : \qquad A_d * A_{d'} \subset A_{d''}.$$

This condition is equivalent to the set of dimensions carrying a binar structure $(D, \ )$ (denoted by juxtaposition) and the dimension projection being a morphism of binars $\delta : (A, *) \to (D, \ )$. We thus write $a_d * b_e = (a * b)_{de}$, where the juxtaposition de denotes de binar structure of the set of dimensions. A **morphism of dimensioned binars** between $(A_D, *^D)$ and $(B_E, \circ^E)$ is simply a dimensioned map $\Phi_\varphi : A_D \to B_E$ such that $\Phi : (A, *) \to (B, \circ)$ is a morphism of binars, since the binary operations are totally-defined. Note that this condition on Φ_φ makes $\varphi : (D, \ ) \to (E, \ )$ into a morphism of binars.

Let us now consider interactions between two (or more) binary operations on dimensioned sets. In the most general scenario, a set A has two independent dimensioned structures

$\delta : A \to D$ and $\epsilon : A \to E$ and two binary operations, one relative to the D dimensions and the other to the E dimensions. It is easy to see that a systematic analysis of compatibility conditions between two such binary operations becomes almost impossible to do in full generality due to the partial nature of the operations. The situation becomes much more tractable when only a single dimensioned structure is assumed on the set A_D and the binary operations are all defined relative to it. This case, which we shall take for the remainder of this work, is further justified by the motivating example of physical quantities, where there is only one notion of *physical dimension*.

Focusing now on the case of interest of a dimensioned set A_D with two binar structures, we find three possible cases depending on whether the operations are dimensional or dimensioned:

$$\text{i)}\,(A_D, *_D, \circ_D) \qquad \text{ii)}\,(A_D, *^D, \circ^D) \qquad \text{iii)}\,(A_D, *_D, \circ^D).$$

Cases i) and ii) in fact reduce to the ordinary theory of binars: collections of pairs of binar structures indexed by D in the case of i) and two binar structures on D in the case of ii). Case iii) seems to point at genuinely new possibilities for the interaction of two binar structures. This will indeed be confirmed by the results in the sections to follow.

3 Dimensioned Rings

Our goal is to replicate the standard theory of algebraic structures, i.e. groups, rings, modules, etc., while attempting to account for the characteristic structure of physical quantities discussed in Section 2.1. We shall see that developing such a theory is natural and straightforward when working with dimensioned sets and dimensioned binary operations as introduced in Section 2.2.

The theory of dimensional binars can be extended trivially to include familiar notions such as identity elements, associativity or invertibility. Consider a dimensional binar $(A_D, *_D)$ and some abstract property of binars P (e.g. commutativity, existence of identity, etc.), we will say that $(A_D, *_D)$ **satisfies property** P simply when the binar slices $(A_d, *_d)$ satisfy the property P for all $d \in D$.

Since we are aiming to identify ring-like structures where addition is partially-defined we begin by defining dimensional abelian groups. We say that $(A_D.+_D)$ is a **dimensional abelian group** when $(A_d, +_d)$ is an abelian group for all $d \in D$. Morphisms of dimensional abelian groups are simply morphisms of underlying dimensional binar structures. It follows from the general results for dimensional binars of Section 2.2 that the set of morphisms between two dimensional abelian groups $(A_D, +_D)$ and $(B_E, +_E)$ has the structure of a dimensional abelian group with dimension set given by the set of maps between the dimension sets D and E. This will be called the set of **dimensioned morphisms** or **dimensioned maps** and we will denote it by $(\mathrm{Dim}(A_D, B_E)_{\mathrm{Map}(D,E)}, +_{\mathrm{Map}(D,E)})$ or for the endomorphisms of a single dimensioned group $\mathrm{Dim}(A_D)_{\mathrm{Map}(D)} := \mathrm{Dim}(A_D, A_D)_{\mathrm{Map}(D,D)}$. Subscripts will be omitted whenever they can be inferred from context. The **category of dimensional abelian groups** will be denoted by DimAb.

Dimensional abelian groups display structures analogous to those of ordinary abelian groups. Firstly, subgroups, products and quotients can be naturally generalised to dimensional abelian groups. Let $(A_D, +_D)$ be a dimensional abelian group, then the subset $0_D := \{0_d \in (A_d, +_d), \quad d \in D\}$ is called the **zero** of A_D. A subset $S \subset A_D$ is called a **dimensional subgroup** when $S \cap A_d \subset (A_d, +_d)$ are subgroups for all $d \in D$. A dimensional subgroup $S \subset A_D$ is clearly a dimensional group with dimension set given by $\delta(S)$, where $\delta : A \to D$ is the dimension projection. We can define the **kernel** of a dimensional group morphism $\Phi : A_D \to B_E$ in the obvious way

$$\ker(\Phi) := \{a_d \in A_D | \quad \Phi(a_d) = 0_{\phi(d)}\}.$$

Clearly, the zero 0_D and kernels of dimensional morphisms $\ker(\Phi) \subset A_D$ are examples of dimensional subgroups. A dimensional subgroup $S \subset A_D$ whose slice intersections $S \cap A_d \subset (A_d, +_d)$ are normal subgroups also induces a natural notion of **quotient**:

$$A_D/S := \bigcup_{d \in \delta(S)} A_d/(S \cap A_d)$$

which has an obvious dimensioned group structure with dimension set $\delta(S)$. There is also a natural notion of **product** of two dimensional groups A_D, B_E given by the categorical product of dimensional binars $(A_D \times B_E, +_{D\times E})$. Furthermore, when we fix a dimension set D and we consider **dimension-preserving morphisms**, i.e. dimensional group morphisms $\Phi : A_D \to B_D$ for which the induced map on the dimension sets is the identity $\mathrm{id}_D : D \to D$, the dimensional abelian dimensional groups over D form a subcategory $\mathsf{DimAb}_D \subset \mathsf{DimAb}$ that, in addition to the notions of subgroup, kernel and quotient, also admits a **direct sum** defined as $A_D \oplus_D B_D := (A \times B)_D$ with partial multiplication given in the obvious way

$$(a_d, b_d) +_d (a'_d, b'_d) := (a_d +_d a'_d, b_d +_d b'_d).$$

It is easy to prove that this direct sum operation on DimAb_D acts as a product and coproduct for which the notions of kernel and quotient identified in the general category DimAb satisfy the axioms of an abelian category. We call DimAb_D the **category of D-dimensional abelian groups**. These constructions are indeed identical to those commonly defined within the categories of abelian group bundles.

Having identified the structure of a partially-defined additive operation as a dimensional abelian group, we are now in the position to attempt a definition of dimensioned ring by considering a multiplicative operation together with the additive operation. In accord with the guiding example of physical quantities, let us consider a dimensional abelian group $(A_D, +_D)$ with a total multiplication $\cdot$, i.e. assume $(A, \cdot)$ is a monoid. The key axiom that characterises rings across multiple conventions is distributivity of multiplication with addition, however the partial nature of addition may present an obstruction to demanding distributivity in general.

If one considers three elements $a_d, b_e, c_f \in A_D$ and attempts to write the (right) distributivity property:

$$(a_d + b_e) \cdot c_f = a_d \cdot c_f + b_e \cdot c_f,$$

it is clear that a few compatibility conditions in dimensions are required for distributivity to possibly hold in some generality in A. The fact that addition can only happen between elements of the same dimension means that in the above formula $d = e$ and that the dimension of $a_d \cdot c_f$ and $b_d \cdot c_f$ must be the same. Therefore, if we are to demand distributivity as generally as possible, the multiplicative operation must map transitively between dimension slices, in other words, the dimension of $a_d \cdot c_f$ only depends on d and f. This is precisely the notion of a dimensioned binary operation introduced in Section 2.2 and thus we are left with a clear motivation to consider case iii) of the possible ways in which two binary operations interact on a dimensioned set.

A **dimensioned ring** is a triple $(R_D, +_D, \cdot^D)$ where R_D is a dimensioned set, $(R_D, +_D)$ is a dimensional abelian group, $(R_D, \cdot^D)$ is a dimensioned monoid and the transitivity condition

$$(a + b) \cdot c = a \cdot c + b \cdot c \qquad c \cdot (a + b) = c \cdot a + c \cdot b$$

holds whenever it is defined for $a, b, c \in R$. Recall from the definition of dimensioned binar in Section 2.2 that the dimension projection $\delta : R \to D$ becomes a morphism of binars and so the dimension set D of a dimensioned ring $(R_D, +_D, \cdot^D)$ carries an associative, unital binary operation $(D, \ )$, denoted by juxtaposition, such that $\delta : (R, \cdot) \to (D, \ )$ is a monoid morphism. With a slight abuse of notation we denote by 1 the multiplicative identities of both $(R, \cdot)$ and $(D, \ )$, thus symbolically $1 = \delta(1)$. The slice containing $1 \in R$, or, equivalently, above $1 \in D$ is called the **dimensionless slice** of the dimensioned ring $R_1 \subset R$. A dimensioned ring is called **commutative** when the monoid structures are abelian. For the remainder of this text dimensioned rings and ordinary rings are assumed to be commutative unless otherwise stated.

Let $(R_D, +_D, \cdot^D)$ and $(P_E, +_E, \cdot^E)$ be two dimensioned rings, a dimensioned map $\Phi : R_D \to P_E$ is called a **morphism of dimensioned rings** when

$$\Phi(a \cdot b) = \Phi(a) \cdot \Phi(b), \qquad \Phi(1_R) = 1_P$$

for all $a, b \in R_D$. The map between the dimension monoids $\phi : D \to E$ is thus necessarily a monoid morphism. Dimensioned rings with these morphisms form the **category of dimensioned rings**, denoted by DimRing.

The **product** of two dimensioned rings $R_D \times P_E$ is defined as the obvious extension of the product of dimensional abelian groups above and the product of ordinary monoids. A dimensioned subgroup $S \subset (R_D, +_D)$ is called a **dimensioned subring** when $S \cdot S \subset S$ and $1 \in S$. A dimensioned subring $I \subset R_D$ is called a **dimensioned ideal** if for all elements $a_d \in R_D$ and $i_e \in I$ we have

$$a_d \cdot i_e \in I \cap R_{de}.$$

Note that the zero $0_D \subset R$ is an ideal since the dimensioned ring axioms imply that it acts as an absorbent set in the following sense:

$$0_d \cdot a_e = 0_{de}.$$

Proposition 3.1 (Quotient Dimensioned Ring)**.** *Let* $(R_D, +_D, \cdot^D)$ *be a dimensioned ring and and* $I \subset R_D$ *a dimensioned ideal, then the quotient dimensioned group* R/I *carries a canonical dimensioned ring structure such that the projection map:*

$$q : R \to R/I$$

is a morphism of dimensioned rings. This construction is called the ***quotient dimensioned ring****.*

Proof. Let us denote dimension slices of the ideal by $I_d := I \cap R_d$. From the construction of quotient dimensioned group we see that the projection map $q : R \to R/I$ is explicitly given by

$$a_d \mapsto a_d + I_d,$$

which makes R/I into a dimensioned abelian group with dimension set $\delta(I) \subset D$. The dimensioned ring multiplication on the quotient can be explicitly defined by:

$$(a_d +_d I_d) \cdot (b_e +_e I_e) = a_d \cdot b_e +_{de} a_d \cdot I_e +_{de} b_e \cdot I_d +_{de} I_d \cdot I_e = a_d \cdot b_e +_{de} I_{de}.$$

This is easily checked to be well-defined and to inherit all the dimensioned ring multiplication properties from R_D. The map q is then a morphism of dimensioned rings by construction. Note that the quotient ring has dimension projection $\delta' : R/I \to \delta(I)$ thus, in particular, $\delta(I) \subset D$ is a submonoid. □

A **unit** or **choice of units** u in a dimensioned ring R_D is a section of the dimension projection

$$\begin{array}{c} R \\ \delta\downarrow \ \big)^{\nwarrow} u \\ D \end{array} \quad \delta \circ u = \mathrm{id}_D, \qquad \text{such that} \quad u_{de} = u_d \cdot u_e \quad \text{and} \quad u_d \neq 0_d$$

for all $d, e \in D$. In other words, a unit is a splitting $u : (D, \ ) \to (R, \cdot)$ of the monoid surjection $\delta : (R, \cdot) \to (D, \ )$ with non-zero image. Units can be regarded as the dimensioned generalization of the notion of non-zero element of a ring with the caveat that they may not exist due to the non-vanishing condition being required for all of D. It was noted above that vector bundles are a form of extreme example of dimensioned rings; in this vein, considering the Moebius band as a dimensioned ring with dimension set the circle S^1 and the zero multiplication operation, we find an explicit example of a dimensioned ring that does not admit units, since they would correspond to global non-vanishing sections of a non-trivialisable vector bundle.

It turns out that all dimensioned rings carry ordinary ring structures within them.

Proposition 3.2 (Dimensionless Ring). *Let $(R_D, +_D, \cdot^D)$ be a dimensioned ring, then its dimensionless slice R_1 carries a natural ring structure induced from the partial addition defined on the slice $+_1$ and the restriction of the total multiplication $\cdot|_{R_1}$. This is called the* ***dimensionless ring*** *$(R_1, +_1, \cdot|_{R_1})$. Furthermore, morphisms, products and quotients of dimensioned rings induce their analogous counterparts for dimensionless rings.*

Proof. The fact that $(R_1, +_1, \cdot|_{R_1})$ is a ring stems simply from the fact that R_1 is closed under multiplication, since $1 \in D$ is the monoid identity. From a similar reasoning we see that for any dimensioned ideal $I \subset R_D$ the dimensionless slice $I_1 := I \cap R_1$ is an ideal of the dimensionless ring. Let two dimensioned rings $(R_D, +_D, \cdot^D)$ and $(P_E, +_E, \cdot^E)$. Since a morphism $\Phi : R_D \to P_E$ preserves the multiplicative identities, it is clear that the restriction

$$\Phi|_{R_1} : (R_1, +_1, \cdot|_{R_1}) \to (P_{\phi(1)}, +_{\phi(1)}, \cdot|_{P_{\phi(1)}})$$

is a morphism of rings. The dimensionless slice of the dimensioned product $R_D \times P_E$ is indeed $R_1 \times P_1$ with the direct product construction applying to rings in a straightforward manner. □

This shows that dimensioned rings are, in fact, a strict generalisation of ordinary rings since we recover them by considering trivial dimension monoids, i.e. singleton dimension sets. More precisely, the category of rings is a subcategory of the category of dimensioned rings $\mathsf{Ring} \subset \mathsf{DimRing}$.

Thus far we have seen that abelian group bundles and ordinary rings are extreme examples of dimensioned rings: the former by taking the zero multiplication and the latter by taking the dimension set to be a singleton. A natural example of dimensioned ring that is somewhat intermediate to the aforementioned two is what we call a **trivial dimensioned ring**: let $(R, +, \cdot)$ be a ring and $(D,)$ a monoid, then the cartesian product $R \times D$ carries a natural dimensioned ring structure defined in the obvious way

$$\mathrm{pr}_2 : R \times D \to D \qquad (a, d) +_d (b, d) := (a + b, d), \qquad (a, d) \cdot (b, e) := (a \cdot b, de).$$

The dimensionless ring of a trivial dimensioned ring is simply $(R \times D)_1 = R \times \{1\}$ and a unit in $R \times D$ is given by a monoid morphism $u : (D,) \to (R, \cdot)$ such that $u(d) \neq 0$ for all $d \in D$. Note that a trivial dimensioned ring always admits a unit given by the constant map $1 : D \to R$ such that $1(d) = 1$ for all $d \in D$.

The first non-trivial example of a dimensioned ring is the set of dimensioned maps of a dimensional abelian group.

Proposition 3.3 (Endomorphism Ring). *Let $(A_D, +_D)$ be a dimensional abelian group, then the set of dimensioned maps $\mathrm{Dim}(A_D)_{\mathrm{Map}(D)}$ carries a dimensioned (non-commutative) ring structure*

$$(\mathrm{Dim}(A_D)_{\mathrm{Map}(D)}, +_{\mathrm{Map}(D)}, \circ^{\mathrm{Map}(D)})$$

where $+$ denotes the dimensional abelian addition and $\circ$ is composition.

Proof. The set of dimensioned maps $\mathrm{Dim}(A_D)_{\mathrm{Map}(D)}$ has a canonical dimensional abelian group structure as shown for the category of dimensional binars in Section 2.2. Note that the dimension set, the maps from D into itself $\mathrm{Map}(D)$, carries a natural monoid structure given by composition of maps $\circ$. Let the dimensioned maps $\Phi_\phi, \Theta_\phi, \Psi_\psi : A_D \to A_D$ where $\phi, \psi : D \to D$ are the dimension maps. It follows by construction that

$$\Phi_\phi \circ \Psi_\psi = (\Phi \circ \Psi)_{\phi \circ \psi},$$

so we see that composition is indeed a dimensioned binary operation $\circ^{\mathrm{Map}(D)}$. It only remains to check the distributivity property:

$$(\Phi_\phi +_\phi \Theta_\phi) \circ \Psi_\psi = \Phi_\phi \circ \Psi_\psi +_{\phi\circ\psi} \Theta_\phi \circ \Psi_\psi \qquad \Psi_\psi \circ (\Phi_\phi +_\phi \Theta_\phi) = \Psi_\psi \circ \Phi_\phi +_{\psi\circ\phi} \Psi_\psi \circ \Theta_\phi$$

which follows from the point-wise definition of addition and the fact that the maps are dimensional abelian group morphisms. □

Division in dimensioned rings is formally analogous to division in ordinary rings since the multiplication operation is totally defined. An element of a dimensioned ring $a \in R_D$ is said to be **invertible** if there exists a (necessarily unique) element $1/a \in R_D$, called its **reciprocal**, such that $a \cdot 1/a = 1$. Subtleties of the dimensioned case appear, however, when considering the notion of **zero divisor** as an element $z \in R_D$ such that there exists a $z' \in R_D$ with $z \cdot z' \in 0_D$. We will not delve further into these questions here.

A dimensioned ring R_D is called a **dimensioned field** when all non-zero elements are invertible. Note that for this requirement to be consistent with the dimension projection $\delta : R \to D$, the monoid structure on D must be a group. A direct consequence of the defining condition of dimensioned field is that non-zero elements induce bijective maps between dimension slices. Indeed, for a non-zero element $0_d \neq a_d \in R_D$ we have the following induced maps called **slice-wise multiplications**:

$$\begin{aligned} a_d \cdot : R_e &\to R_{de} \\ b_e &\mapsto a_d \cdot b_e \end{aligned}$$

for all $e \in D$. The distributivity axiom implies that these are slice-wise abelian group isomorphisms with inverse given by $1/(a_d)\cdot$. These maps allow to prove a general result that confers a role to choices of unit on dimensioned fields similar to that of a trivialization of a fibre bundle.

Proposition 3.4 (Units in Dimensioned Fields)**.** *Let $(R_D, +_D, \cdot)$ be a dimensioned field, then a choice of units $u : D \to R$ induces an isomorphism with the trivial dimensioned field:*

$$R_D \cong R_1 \times D.$$

Proof. A choice of units induces the following map via slice-wise multiplication:

$$\begin{aligned}\Phi_u : R_1 \times D &\to R_D \\ (r, d) &\mapsto u_d \cdot r\end{aligned}$$

This is shown to be a bijection by explicitly constructing its inverse $\Phi_u^{-1}(a_d) := u_{d^{-1}} \cdot a_d$. It only remains to check that Φ_u is dimensioned ring morphism; this follows directly by construction and the fact that u is a morphism of monoids:

$$\begin{aligned}\Phi^u((r_1, d) \cdot (r_2, e)) = \Phi^u((r_1 \cdot r_2, de)) = u_{de} \cdot r_1 \cdot r_2 = u_d \cdot u_e \cdot r_1 \cdot r_2 = \\ = (u_d \cdot r_1) \cdot (u_e \cdot r_2) = \Phi^u(u_d \cdot r_1) \cdot \Phi^u(u_e \cdot r_2).\end{aligned}$$

□

This last proposition shows that the dimensioned fields for which choices of units exist are (non-canonically) isomorphic to the trivial dimensioned fields $F \times D$ with F an ordinary field and D an abelian group.

4 Dimensioned Modules

Ordinary modules are algebraic structures closely related to rings where addition is possible and multiplication is defined externally in such a way that properties, as formally close as possible to the ring axioms, are satisfied. We motivate the definition of dimensioned modules by investigating the structure present in natural constructions with dimensioned rings.

Considering the dimensional abelian group part of a dimensioned ring $(R_D, +_D, \cdot^D)$, we can form the product $R_D \times R_D$, which is a dimensional abelian group with dimension set $D \times D$, or the direct sum $R_D \oplus_D R_D$, which is a dimensional abelian group with dimension set D. In both cases we can form module-like maps by setting

$$a_d * (b_e, c_f) := (a_d \cdot b_e, a_d \cdot c_f), \qquad a_d * (b_e \oplus c_e) := a_d \cdot b_e \oplus a_d \cdot c_e.$$

These module-like actions are compatible with the dimensioned structure in the sense that, in the first case, D acts diagonally on $D \times D$ and, in the second case, D acts on itself by multiplication. Furthermore, from the defining axioms of dimensioned ring, these maps satisfy the usual linearity properties of the conventional notion of R-module with the only caveat that addition is partially defined.

Recall from our discussion in Section 3 that the dimensioned maps from R_D into itself form an abelian dimensioned group $(\mathrm{Dim}(R_D)_{\mathrm{Map}(D)}, +_{\mathrm{Map}(D)})$ where $\mathrm{Map}(D)$ denotes the set of maps from D onto itself. The presence of the dimensioned ring multiplication allows for the definition of the following module-like structure

$$* : R_D \times \mathrm{Dim}(R_D) \to \mathrm{Dim}(R_D)$$

defined via

$$(a_d * \Phi)(b_e) := a_d \cdot \Phi(b_e).$$

We note that $a_d * \Phi$ is a well-defined dimensioned morphism from the fact D acts naturally on $\mathrm{Map}(D)$ by composition with the monoid multiplication action of D on itself: indeed the if $\phi : D \to D$ is the dimension map of Φ, then $a_d * \Phi$ has dimension map $d \circ \phi : D \to D$. Once more, it follows directly from the axioms of dimensioned ring that that this operation satisfies the usual linearity properties of the conventional notion of R-module with the only caveat that addition is partially defined.

These examples motivate the following definition: let $(R_G, +_G, {}^G)$ be a dimensioned ring (in the interest of notational economy, ring multiplications will be denoted by juxtaposition hereafter) and $(A_D, +_D)$ a dimensioned abelian group. Note that G carries a monoid structure whereas D is simply a set. A_D is called a **dimensioned R_G-module** if there is a map

$$\cdot : R_G \times A_D \to A_D$$

that is compatible with the dimensioned structures via a monoid action $G \times D \to D$ (denoted by juxtaposition) in the following sense

$$r_g \cdot a_d = (r \cdot a)_{gd}$$

and that satisfies the following axioms

1) $r_g \cdot (a_d + b_d) = r_g \cdot a_d + r_g \cdot b_d$,
2) $(r_g + p_g) \cdot a_d = r_g \cdot a_d + p_g \cdot a_d$,
3) $(r_g p_h) \cdot a_d = r_g \cdot (p_h \cdot a_d)$,
4) $1 \cdot a_d = a_d$

for all $r_g, p_h \in R_G$ and $a_d, b_d \in A_D$. Note that these four axioms for a map $\cdot : R_G \times A_D \to A_D$ can only be demanded in consistency with the dimensioned structure in the presence of a monoid action $G \times D \to D$. With this definition at hand, we recover the motivating examples: the direct sum $R_G \oplus_G R_G$ is a dimensioned R_G-module with dimension set G and monoid action given by the multiplication action; the product $R_G \times R_G$ is a dimensioned R_G-module with dimension set $G \times G$ and monoid action given by the diagonal action; and the set of dimensioned maps of a dimensioned ring $\mathrm{Dim}(R_G)$ is a dimensioned R_G-module with dimension set $\mathrm{Map}(G)$ and monoid action given by composition with the multiplication action.

Let $(A_D, +_D)$ and $(B_E, +_E)$ be two dimensioned R_G-modules, a morphism of abelian dimensioned groups $\Phi : A_D \to B_E$ is called **R_G-linear** if

$$\Phi(r_g \cdot a_d) = r_g \cdot \Phi(a_d)$$

for all $r_g \in R_G$ and $a_d \in A_D$. Note that this condition forces the dimension map $\phi : D \to E$ to satisfy

$$\phi(gd) = g\phi(d)$$

for all $g \in G$ and $d \in D$, in other words, the dimension map ϕ must be G-equivariant with respect to the monoid actions of the dimension sets D and E. Let us denote the set of G-equivariant dimension maps as

$$\mathrm{Map}^G(D, E) := \{\phi : D \to E \,| \quad \phi \circ g = g \circ \phi \quad \forall\, g \in G\},$$

then it follows that the dimensioned group of morphisms $\mathrm{Dim}(A_D, B_E)_{\mathrm{Map}(D,E)}$ contains a dimensioned subgroup of morphisms covering G-equivariant dimension maps for which the following dimensioned module map can be defined

$$(r_g \cdot \Phi)(a_d) := r_g \cdot \Phi(a_d) = \Phi(r_g \cdot a_d).$$

The set of dimensioned maps $\mathrm{Dim}(A_D, B_E)_{\mathrm{Map}^G(D,E)} \subset \mathrm{Dim}(A_D, B_E)_{\mathrm{Map}(D,E)}$ that are R_G-linear is thus shown to carry a natural dimensioned R_G-module structure. We simply call these the **R_G-linear maps** between A_D and B_E and denote them by $\mathrm{Dim}_{R_G}(A_D, B_E)$.

Let $(A_D, +_D)$ be a dimensioned R_G-module, a dimensional abelian subgroup $S \subset A_D$ is called a **dimensioned submodule** if

$$r_g \cdot s_d \in S \cap A_{gd}$$

for all $r_g \in R_G$ and $s_d \in S$. Natural examples of dimensioned submodules are the **span** of a subset $X \subset A_D$, defined as all the possible R_G-linear combinations of elements in X, and the kernels and images of R_G-linear maps between modules. The dimensional abelian group quotient construction of Section 3 induces the notion of **quotient** of R_G-modules: let $\delta : A \to D$ be the dimension projection of the dimensioned R_G-module $(A_D, +_D)$ and $S \subset A_D$ a submodule, then by taking the quotient as dimensional abelian groups $A'_{\delta(S)} := A_D/S$ is a dimensioned R_G-module.

Let $(A_D, +_D)$ and $(B_D, +_D)$ be two dimensioned R_G-modules, the dimensional abelian group direct sum $A_D \oplus_D B_D$ carries a natural R_G-module structure:

$$r_g \cdot (a_d \oplus_d b_d) := r_g \cdot a_d \oplus_{gd} r_g \cdot b_d,$$

which gives the definition of **direct sum** of dimensioned R_G-modules. Our definitions so far allow for notions from ordinary module theory, such as **finitely generated**, **free**, **projective** or **injective**, to apply to dimensioned modules in an obvious way. By fixing a dimensioned ring R_G and a dimension set D, R_G-modules with dimensions in D together with D-preserving R_G-linear maps form an abelian category, essentially analogous to the category of D-dimensional abelian groups DimAb_D. This is called the **category of D-dimensional R_G-modules** denoted by $R_G\mathsf{DimMod}_D$.

Let $(A_D, +_D)$ and $(B_E, +_E)$ be two dimensioned R_G-modules, we define the **tensor product** from their product as dimensional abelian groups

$$A_D \otimes_{R_G} B_E := R_G \bullet (A_D \times B_E)/\sim$$

where $R_G \bullet (A_D \times B_E)$ denotes the free dimensional abelian group with coefficients in R_G and $\sim$ denotes taking a quotient with respect to the following relations:

$$(a_d + a'_d, b_e) \sim (a_d, b_e) + (a'_d, b_e), \qquad (a_d, b_e + b'_e) \sim (a_d, b_e) + (a_d, b'_e), \qquad (r_g \cdot a_d, b_e) \sim (a_d, r_g \cdot b_e).$$

This construction would make $A_D \otimes_{R_G} B_E$ into a dimensioned R_G-module with dimension set $D \times E$ as long as we can find a monoid action such that $g(d, e) = (gd, e) = (d, ge)$ so that the third relation holds at the level of dimensions. This is indeed achieved by considering the tensor product monoid action

$$G \otimes G \times D \times E \to D \times E$$

where $G \otimes G$ denotes the tensor product of abelian monoids so that the required dimension identities for the third relation are realised by:

$$(gd, e) = (g \otimes 1) \cdot (d, e) = g \cdot (d, e) = (1 \otimes g) \cdot (d, e) = (d, ge)$$

where we have used the natural inclusion $- \otimes 1 : G \hookrightarrow: G \otimes G$. Note that these are the same relations used to define the tensor product of ordinary modules with the added caveat that addition is partially defined. It follows from this that the tensor product construction can be characterised with the obvious universal property, defining in turn the tensor product of R_G-linear maps and establishing R_G-**bilinearity** of a map $\Phi : A_D \times B_E \to C_F$ as the fact that it factors through the tensor product via R_G-linear maps:

$$A_D \times B_E \xrightarrow{\otimes} A_D \otimes_{R_G} B_E \xrightarrow{\phi} C_F$$

where $\otimes$ denotes the natural element-wise tensor product as a dimensioned map covering the identity on $D \times E$.

The theory of dimensioned modules developed thus far is manifestly analogous to the theory of ordinary modules. Indeed, all the conventional notions appear essentially identical aside from all the *dimensioned technology* that is there to systematically account for the fact that additive operations are partially defined. The next proposition vindicates this view in which dimensioned algebra is formally analogous to ordinary algebra with the only caveat that addition is partially defined.

Proposition 4.1 (Dimensioned Distributive Symmetric Monoidal Category)**.** *Dimensioned R_G-modules together with R_G-linear maps form a category denoted by $R_G\mathsf{DimMod}$. Let $(\mathsf{Set}, \times)$ be the symmetric monoidal category of ordinary sets. The direct sum $\oplus$ on $R_G\mathsf{DimMod}$ is a dimensional binar with dimensions in* Set *and the tensor product $\otimes$ on $R_G\mathsf{DimMod}$ is a* *dimensioned binar compatible with the monoid product of* Set*. Furthermore,* $\otimes$ *is distributive with respect to* $\oplus$ *within the category* $R_G\mathsf{DimMod}$*, this makes*

$$(R_G\mathsf{DimMod}_{\mathsf{Set}}, \oplus_{\mathsf{Set}}, \otimes^{\mathsf{Set}})$$

into a ***dimensioned rig category****, the categorical counterpart of a dimensioned ring.*

Proof. ... □

So far we have only considered dimensioned modules over the same dimensioned ring. We can connect categories of dimensioned modules over different dimensioned rings via the **pullback** construction: let $(A_D, +_D)$ be a R_G-module and $\varphi : P_H \to R_G$ a dimensioned ring morphism, then A_D has a dimensioned P_H-module structure given by:

$$p_h \cdot a_d := \varphi(p_h) \cdot a_d$$

for all $p_h \in P_H$ and $a_d \in A_D$. This dimensioned module is denoted by $\varphi^* A_D$ since the base set of the module is unchanged and so is its dimension set; the monoid action $H \times D \to D$ is given by pullback with the dimension map $H \to G$ of the dimensioned ring morphism $\varphi : P_H \to R_G$. This construction motivates the extension of the notion of dimensioned module morphisms to account for maps between modules over different rings: let A_D be a R_G-module and B_E a P_H-module, the pair of maps Φ^φ is said to be a **twisted module morphism** if $\Phi_F : A_D \to B_E$ is a dimensioned map, $\varphi_f : R_G \to P_H$ is a dimensioned ring morphism and

$$\Phi(r_g \cdot a_d) = \varphi(r_g) \cdot \Phi(a_d)$$

for all $r_g \in R_G$ and $a_d \in A_D$. We also say that $\Phi : A_D \to B_E$ is a **φ-linear map**. Note that the φ-linearity condition implies that the dimension maps satisfy a sort of equivariance property with respect to the monoid actions:

$$F(gd) = f(g)F(d)$$

where $g \in G$, $d \in D$. For two twisted module morphisms Φ^ϕ and Ψ^ψ it is easily checked that:

$$\Phi^\phi \circ \Psi^\psi = (\Phi \circ \Psi)^{\phi \circ \psi}$$

and so the categories of dimensioned modules over a fixed dimensioned ring $R_G\mathsf{DimMod}$ can now be generalised to include all module morphisms twisted by the endomorphisms of R_G.

Proposition 4.2 (Pullback Functor)**.** *Let* $\varphi : P_H \to R_G$ *be dimensioned ring morphism, then the pullback construction is compatible with composition and categorical products. Furthermore, when* φ *is an isomorphism, the assignment*

$$\varphi^* : R_G\mathsf{DimMod} \to P_H\mathsf{DimMod}.$$

becomes a functor of dimensioned distributive monoidal categories, the categorical analogue of a dimensioned ring morphism.

Proof. Note that the pullback assignment φ^* is the identity functor at the level of dimensional abelian groups by construction. This means that we should only check compatibility with the pullback module multiplication structure. Compatibility with a ψ-linear map of R_G-modules $\Psi^\psi : A_D \to B_E$ is ensured by setting:

$$\varphi^*(\Psi^\psi) := \Psi^{\psi\circ\varphi}$$

which, by simple checks is shown to satisfy:

$$\varphi^*(\mathrm{id}_{A_D}^{\mathrm{id}_{R_G}}) = \mathrm{id}_{\varphi^* A_D}^{\mathrm{id}_{P_H}} \qquad \varphi^*(\Phi^\phi \circ \Psi^\psi) = \varphi^*\Phi^\phi \circ \varphi^*\Psi^\psi.$$

Further relying on the underlying dimensional abelian group structure of dimensioned modules, we can easily show that pullbacks preserve categorical products in the following sense:

$$\varphi^*(A_D \oplus_D B_D) \cong \varphi^* A_D \oplus_D \varphi^* B_D \qquad \varphi^*(A_D \otimes_{R_G} B_E) \cong \varphi^* A_D \otimes_{P_H} \varphi^* B_E$$

the isomorphisms are as dimensioned modules and we have crucially used additivity and multiplicativity of the dimensioned ring morphism φ. The pullback assignment defined in this way sends objects to objects between the categories of dimensioned modules over the fixed rings R_G and P_H, but it does not so for morphisms of those categories since $\varphi^*(\Psi^\psi) : \varphi^* A_D \to B_E$ is a map from a P_H-module to a R_G-module. When φ is an isomorphism, however, we can define the pullback assignment making use of the inverse:

$$\varphi^*(\Psi^\psi) := \Psi^{\varphi^{-1}\circ\psi\circ\varphi}$$

which then makes it into a well defined functor. □

An important example where twisted module morphisms appear naturally is the more general notion of quotient of modules induced by ideal submodules.

Proposition 4.3 (Quotient Dimensioned Module)**.** *Let $(A_D, +_D)$ be a dimensioned R_G-module and $I \subset R_G$ an ideal, then, if $S \subset A_D$ is a submodule such that $I \cdot A \subset S$, the quotient A_D/S inherits a dimensioned R_G/I-module structure such that the projection map*

$$Q : A_D \to A_D/S$$

is a q-linear map, where $q : R_G \to R_G/I$ is the quotient ring projection.

Proof. Since quotients of rings and modules are taken as dimensional abelian groups, this result follows from a simple computation showing the distributivity property of the ideal submodule S: let $r_g \in R_G$, $i_g \in I$, $a_d \in A_D$ and $s_d \in S$ then

$$(r_g + i_g) \cdot (a_d + s_d) = r_g \cdot a_d + r_g \cdot s_d + i_g \cdot a_d + i_g \cdot s_d.$$

The second and fourth terms are in S from the fact that S is a submodule and the third term is in S from the ideal submodule condition $I \cdot A \subset S$, then the above expression defines the R_G/I-module structure on A_D/S which, by construction, satisfies $Q(r_g \cdot a_d) = q(r_g) \cdot Q(a_d)$. □

5 Dimensioned Algebras

Let us motivate the dimensioned generalization of the notion of algebra by considering, once more, the guiding example of the set of dimensioned maps of a dimensioned ring R_G. Proposition 3.3 implies that the set of dimensioned maps of R_G, regarded as a dimensional abelian group, carries a dimensioned ring structure $(\mathrm{Dim}(R_G)_{\mathrm{Map}(G)}, +_{\mathrm{Map}(G)}, \circ^{\mathrm{Map}(G)})$; but $\mathrm{Dim}(R_G)_{\mathrm{Map}(G)}$ is also naturally a R_G-module structure. Depending on the further conditions imposed on the dimensioned maps, these two structures, the dimensioned ring and the R_G-module, may interact in different ways. A first obvious choice is to consider dimensioned ring homomorphisms, in which case the interaction manifests as the fact that composition acts as a twisted R_G-module morphism. Another direction is to consider differential operators. Although we will only focus on zeroth order operators and derivations, general differential operators are defined recursively from the R_G-linearity condition:

$$\Phi_\phi(r_g s_h) = r_g \Phi_\phi(s_h) = \Phi_\phi(r_g) s_h$$

for $r_g, s_h \in R_G$ and $\Phi_\phi : R_G \to R_G$ a dimensioned map. In the case at hand of commutative dimensioned rings, this condition can only be realised by multiplication by a ring element, which corresponds to the natural inclusion $R_G \hookrightarrow \mathrm{Dim}_{R_G}(R_G)$ via ring multiplication. Dimension maps of such R_G-linear operators correspond, in turn, to multiplication by monoid elements. It is then easy to see that R_G-linear operators satisfy

$$r_g \cdot (\Phi_\phi \circ \Psi_\psi) = (r_g \cdot \Phi_\phi) \circ \Psi_\psi = \Phi_\phi \circ (r_g \cdot \Psi_\psi)$$

for all $r_g \in R_G$ and $\Phi_\phi, \Psi_\psi \in \mathrm{Dim}_{R_G}(R_G)$. This shows that $\mathrm{Dim}_{R_G}(R_G)$ gives a prime example of a bilinear associative operation on a dimensioned module and prompts us to give the following general definition.

Let $(A_D, +_D)$ be a dimensioned R_G-module, a map $M : A_D \times A_D \to A_D$ is called a **dimensioned bilinear multiplication** if it satisfies

$$\begin{aligned} M(a_d +_d b_d, c_e) &= M(a_d, c_e) +_{\mu(d,e)} M(b_d, c_e) \\ M(a_d, b_e +_e c_e) &= M(a_d, b_e) +_{\mu(d,e)} M(a_d, c_e) \\ M(r_g \cdot a_d, s_h \cdot b_e) &= r_g \cdot s_h \cdot M(a_d, b_e) \end{aligned}$$

for all $a_d, b_d, b_e, c_e \in A_D$, $r_g, s_h \in R_G$ and for a **dimension map** $\mu : D \times D \to D$ which is G-equivariant in both entries, i.e.

$$\mu(gd, he) = gh\mu(d, e)$$

for all $g, h \in G$ and $d, e \in D$. When such a map M is present in a dimensioned R_G-module A_D, the pair (A_D, M) is called a **dimensioned R_G-algebra**. The notion of dimensioned tensor product given a the end of Section 4 allows to reformulate the definition of a dimensioned bilinear multiplication $M : A_D \times A_D \to A_D$ as a dimensioned R_G-linear morphism

$$M : A_D \otimes_{R_G} A_D \to A_D.$$

Note that the dimension set of the tensor product $A_D \otimes_{R_G} A_D$ is $D \times D$ with the diagonal G-action induced from the R_G-module structure, then we see that the double G-equivariant condition of μ is reinterpreted now as ordinary G-equivariance with respect to the natural monoid actions.

The natural notions of morphisms and subalgebras of ordinary algebras extend naturally to the dimensioned case. Let (A_D, M) and (B_E, N) be two dimensioned R_G-algebras, a R_G-linear morphism $\Phi : A_D \to B_E$ is called a **morphism of dimensioned algebras** if

$$\Phi(M(a, a')) = N(\Phi(a), \Phi(a')),$$

for all $a, a' \in A_D$. A submodule $S \subset A_D$ such that $M(S, S) \subset S$ is called a **dimensioned subalgebra**.

****Quotient and Product Algebras****

The dimension map μ of a dimensioned bilinear multiplication in a dimensioned R_G-algebra (A_D, M_μ) is naturally regarded as an binary operation on the set of dimensions D. In a general sense, dimension sets of dimensioned algebras carry the most basic algebraic structures, commonly known as binars. However, if one wishes to demand specific algebraic properties, such as commutativity or associativity, the algebraic structure present in the dimension binar becomes richer. Let (A_D, M_μ) be a dimensioned R_G-algebra, we say that it is **symmetric** or **antisymmetric** if

$$M(a_d, b_e) = M(b_e, a_d), \qquad M(a_d, b_e) = -M(b_e, a_d)$$

for all $a_d, b_e \in A_D$, respectively. The dimension binars of symmetric or antisymmetric dimensioned algebras are necessarily commutative, i.e. $\mu(d, e) = \mu(e, d)$ for all $d, e \in D$. The usual 3-element-product properties of ordinary algebras can be demanded for dimensioned algebras in an analogous way, in particular (A_D, M_μ) is called **associative** or **Jacobi** if

$$\mathrm{Ass}_M(a_d, b_e, c_f) = 0, \qquad \mathrm{Jac}_M(a_d, b_e, c_f) = 0$$

for all $a_d, b_e, c_f \in A_D$, respectively. The dimension binars of associative or Jacobi dimensioned algebras are necessarily associative, i.e. $\mu(\mu(d, e), f) = \mu(d, \mu(e, f))$ for all $d, e, f \in D$, making them into semigroups. Returning to the motivating example presented at the beginning of this section, we now see that the dimensioned morphisms of a dimensioned ring R_G give the prime example of dimensioned associative algebra $(\mathrm{Dim}(R_G), \circ)$.

In parallel with the definitions of ordinary algebras, we define **dimensioned commutative algebra** as a symmetric and associative dimensioned algebra and a **dimensioned Lie algebra** as an antisymmetric and Jacobi dimensioned algebra. Note that dimensioned commutative and dimensioned Lie algebras necessarily carry dimension sets that are commutative semigroups.

In keeping with the general philosophy to continue to scrutinize the natural algebraic structure present in the dimensioned module of dimensioned morphisms of a dimensioned ring R_G, let us attempt to find the appropriate dimensioned generalization of the notion of derivations of a ring. Working by analogy, a dimensioned derivation will be a dimensioned morphism $\Delta \in \mathrm{Dim}(R_G)$ covering a dimension map $\delta : G \to G$ satisfying a Leibniz identity with respect to the dimensioned ring multiplication

$$\Delta(r_g \cdot s_h) = \Delta(r_g) \cdot s_h + r_g \cdot \Delta(s_h),$$

for all $r_g, s_h \in R_G$, however, for the right-hand-side to be well-defined, both terms must be of homogeneous dimension, which means that the dimension map must satisfy

$$\delta(gh) = \delta(g)h = g\delta(h)$$

for all $g, h \in G$. Since G is a monoid, this condition is equivalent to the dimension map being given by left (or equivalently due to commutativity, right) multiplication with a monoid element, i.e. $\delta = L_d$ for some element $d \in G$. Following from this observation, we see that there is a natural dimensioned submodule of the dimensioned module of dimensioned morphisms $\mathrm{Dim}(R_G)_G \subset \mathrm{Dim}(R_G)_{\mathrm{Map(G)}}$ given by the dimensioned morphisms whose dimension maps are specified by multiplication with a monoid element. Recall that dimensioned rings are assumed to be commutative and, thus, the dimension monoid has commutative binary operation. This allows for the identification of the first natural example of dimensioned Lie algebra: consider the commutator of the associative dimensioned composition

$$[\Delta, \Delta'] := \Delta \circ \Delta' - \Delta' \circ \Delta,$$

it is easy to check that this bracket is indeed antisymmetric and Jacobi, thus making $(\mathrm{Dim}(R_G)_G, [,])$ into the **dimensioned Lie algebra of dimensioned morphisms** of a dimensioned ring R_G. Notice that this bracket can only be defined on the dimensioned submodule $\mathrm{Dim}(R_G)_G \subset \mathrm{Dim}(R_G)_{\mathrm{Map(G)}}$ since the two terms of the right-hand-side for general dimensioned morphisms will have dimensions given by the composition of maps from G into itself which is a non-commutative binary operation in general. It is then clear that the Leibniz condition proposed above can be demanded in consistency with the dimensioned structure of dimensioned morphisms within the Lie algebra of dimensioned morphisms, so we see the dimensioned Lie algebra of **derivations of a dimensioned ring** R_G as the natural dimensioned Lie subalgebra of the dimensioned morphisms

$$\mathrm{Der}(R_G) \subset (\mathrm{Dim}(R_G)_G, [,]).$$

Derivations covering the monoid identity, i.e. those with dimension map $\mathrm{id}_G : G \to G$, are called **dimensionless derivations** and it is clear by definition that they form an ordinary Lie algebra with the commutator bracket $(\mathrm{Der}(R_G)_1, [,])$. Restricting their action to elements of the dimensionless ring $R_1 \subset R_G$ we recover the ordinary Lie algebra of ring derivations, in other words, there is a surjective map of Lie algebras

$$(\mathrm{Der}(R_G)_1, [,]) \to (\mathrm{Der}(R_1), [,]).$$

The example of the dimensioned Lie algebra of derivations of a dimensioned ring illustrates the case of a dimensioned algebra whose space of dimensions is a (commutative) monoid and whose dimension map is simply given by the monoid multiplication. For the reminder of this chapter, the dimension sets of dimensioned modules will be assumed to carry a commutative monoid structure (with multiplication denoted by juxtaposition of elements) unless stated otherwise. Let A_G be a dimensioned module, a dimensioned algebra multiplication $M : A_G \times A_G \to A_G$ is said to be **homogeneous of dimension** m if the dimension map $\mu : G \times G \to G$ is given by monoid multiplication with the element $m \in G$, i.e. $\mu(g,h) = mgh$ for all $g, h \in G$. Assuming a monoid structure on the dimension set of a dimensioned module and considering dimensioned algebra multiplications of homogeneous dimension is particularly useful in order to study several algebra multiplications coexisting on the same set. Indeed, given two homogeneous dimensioned algebra multiplications (A_G, M_1) and (A_G, M_2) with dimensions $m_1 \in G$ and $m_2 \in G$, respectively, the fact that the monoid operation is assumed to be associative and commutative, allows for consistently demanding properties of the interaction of the two dimensioned multiplications involving expressions of the form $M_1(M_2(a,b),c)$ without any further requirements.

6 Dimensioned Poisson Algebras

Let A_G be a dimensioned R_H-module and let two dimensioned algebra multiplications $* : A_G \times A_G \to A_G$ and $\{,\} : A_G \times A_G \to A_G$ with homogeneous dimensions $p \in G$ and $b \in G$, respectively, the triple $(A_G, *_p, \{,\}_b)$ is called a **dimensioned Poisson algebra** if

1) $(A_G, *_p)$ is a dimensioned commutative algebra,

2) $(A_G, \{,\}_b)$ is a dimensioned Lie algebra,

3) the two multiplications interact via the Leibniz identity

$$\{a, b * c\} = \{a, b\} * c + b * \{a, c\},$$

for all $a, b, c \in A_G$.

Note that the Leibniz condition can be consistently demanded of the two dimensioned algebra multiplications since the dimension projections of each of the terms of the Leibniz identity for $\{a_g, b_h * c_k\}$ are:

$$bgphk, \quad pbghk, \quad phbgk,$$

but they are indeed all equal from the fact that the monoid binary operation is associative and commutative.

A morphism of dimensioned modules between dimensioned Poisson algebras $\Phi : (A_G, *_p, \{,\}_b) \to (B_H, *_r, \{,\}_c)$ is called a **morphism of dimensioned Poisson algebras** if $\Phi : (A_G, *_p) \to (B_H, *_r)$ is a morphism of dimensioned commutative algebras and also $\Phi : (A_G, \{,\}_b) \to (B_H, \{,\}_c)$ is a morphism of dimensioned Lie algebras. A submodule $I \subset A_G$

that is a dimensioned ideal in $(A_G, *_p)$ and that is a dimensioned Lie subalgebra in $(A_G, \{,\}_b)$ is called a **dimensioned coisotrope**.

Proposition 6.1 (Dimensioned Poisson Reduction)**.** *Let $(A_G, *_p, \{,\}_b)$ be a dimensioned Poisson algebra and $I \subset A_G$ be a coisotrope, then there is a dimensioned Poisson algebra structure induced in the subquotient*

$$(A'_G := N(I)/I, *'_p, \{,\}'_b)$$

where $N(I)$ denotes the dimensioned Lie idealizer of I regarded as a submodule of the dimensioned Lie algebra.

Proof. We assume without loss of generality that the dimension projection of I is the whole of G, the intersections with the dimension slices are denoted by $I_g := I \cap A_g$. The dimensioned Lie idealizer is defined in the obvious way

$$N(I) := \{n_g \in A_G | \quad \{n_g, i_h\} \in I_{bgh} \quad \forall i_h \in I\}.$$

We clearly see that $N(I)$ is the smallest dimensioned Lie subalgebra that contains I as a dimensioned Lie ideal. The Leibniz identity implies that $N(I)$, furthermore, is a dimensioned commutative subalgebra with respect to $*_p$ in which I sits as a dimensioned commutative ideal, since it is a commutative ideal in the whole A_G. It follows that we can form the dimensioned quotient commutative algebra $(N(I)/I, *')$ as described in PROPOSITION QUOTIENT ALGEBRA. The only difference with that case is that commutative multiplication covers a dimension map that is given by the monoid multiplication with a non-identity element $p \in G$, but this has no effect on the quotient construction. To obtain the desired quotient dimensioned Lie bracket we set:

$$\{n_g + I_g, m_h + I_h\}' := \{n_g, m_h\} + I_{bgh}$$

which is easily checked to be well-defined and that inherits the antisymmetry and Jacobi properties directly from dimensioned Lie bracket $\{,\}$ and the fact that $I \subset N(I)$ is a dimensioned Lie ideal. □

****Tensor Product of Dimensioned Poisson Algebras****

7 The Power Functor

In this final section we describe an important class of examples of dimensioned rings arising from ordinary 1-dimensional vector spaces. These examples are important for two reasons: on the one hand, mathematically, they constitute a large class of natural non-trivial examples of dimensioned rings, on the other, conceptually, they capture the standard structure of physical quantities described in Section 2.1 precisely.

We identify **the category of lines**, Line, as a subcategory of vector spaces over a field $\mathsf{Vect}_{\mathbb{F}}$. Objects are vector spaces of dimension 1, a useful way to think of these in the context of the present work is as sets of numbers without the choice of a unit. An object $L \in \mathsf{Line}$ will be appropriately called a **line**. A morphism in this category $b \in \mathrm{Hom}_{\mathsf{Line}}(L, L')$, usually simply denoted by $b : L \to L'$, is an invertible (equivalently non-zero) linear map. Composition in the category Line is simply the composition of maps. If we think of L and L' as numbers without a choice of a unit, a morphism b between them can be thought of as a unit-free conversion factor, for this reason we will often refer to a morphism of lines as a **factor**. We consider the field $\mathbb{F}$, trivially a line when regarded as a vector space, as a singled out object in the category of lines $\mathbb{F} \in \mathsf{Line}$.

It is a simple linear algebra fact that any two lines $L, L' \in \mathsf{Line}$ satisfy

$$\dim(L \oplus L') = \dim L + \dim L' = 2 > 1, \qquad \dim L^* = \dim L = 1, \qquad \dim(L \otimes L') = 1.$$

Then, we note that the direct sum $\oplus$, is no longer defined in Line, however, it is straightforward to check that $(\mathsf{Line}, \otimes, \mathbb{F})$ forms a symmetric monoidal category and that $* : \mathsf{Line} \to \mathsf{Line}$ is a duality contravariant autofunctor. Let us introduce the following notation:

$$\begin{cases} L^n := \otimes^n L & n > 0 \\ L^n := \mathbb{F} & n = 0 \\ L^n := \otimes^n L^* & n < 0 \end{cases}$$

which is such that given two integers $n, m \in \mathbb{Z}$ and any line $L \in \mathsf{Line}$ the following equations hold

$$(L^n)^* = L^{-n} \qquad L^n \otimes L^m = L^{n+m}.$$

Thus we see how one single line and its dual $L, L^* \in \mathsf{Line}$ generate an abelian group with the tensor product as group multiplication, the patron $\mathbb{F} \in \mathsf{Line}$ as group identity and the duality autofunctor as inversion. We define the **power** of a line $L \in \mathsf{Line}$ as the set of all tensor powers

$$L^{\odot} := \bigcup_{n \in \mathbb{Z}} L^n.$$

This set has than an obvious dimensioned set structure with dimension set $\mathbb{Z}$:

$$\pi : L^{\odot} \to \mathbb{Z}.$$

Since dimension slices are precisely the tensor powers L^n, they carry a natural $\mathbb{F}$-vector space structure, thus making the power of L into a dimensional abelian group $(L^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}})$. The next proposition shows that the ordinary $\mathbb{F}$-tensor product of vector spaces endows $L^{\odot}$ with a dimensioned field structure.

Proposition 7.1 (Dimensioned Ring Structure of the Power of a Line)**.** *Let $L \in \mathsf{Line}$ be a line and $(L^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}})$ its power, then the $\mathbb{F}$-tensor product of elements induces a dimensioned multiplication*

$$\odot : L^{\odot} \times L^{\odot} \to L^{\odot}$$

such that $(L^{\odot}_{\mathbb{Z}}, +_{\mathbb{Z}}, \odot)$ becomes a dimensioned field.

Proof. The construction of the dimensioned ring multiplication $\odot$ is done simply via the ordinary tensor product of ordinary vectors and taking advantage of the particular properties of 1-dimensional vector spaces. The two main facts that follow from the 1-dimensional nature of lines are: firstly, that linear endomorphisms are simply multiplications by field elements

$$\mathrm{End}(L) \cong L^* \otimes L \cong \mathbb{F}$$

which, at the level of elements, means that

$$\mathrm{End}(L) \ni \alpha \otimes a = \alpha(a) \cdot \mathrm{id}_L$$

as it can be easily shown by choosing a basis; and secondly, that the tensor product becomes canonically commutative, since, using the isomorphism above, we can directly check

$$a \otimes b(\alpha, \beta) = \alpha(a)\beta(b) = \alpha(b)\beta(a) = b \otimes a(\alpha, \beta),$$

thus showing

$$a \otimes b = b \otimes a \in L \otimes L = L^2.$$

The binary operation $\odot$ is then explicitly defined for elements $a, b \in L = L^1$, $\alpha, \beta \in L^* = L^{-1}$ and $r, s \in \mathbb{F} = L^0$ by

$$\begin{aligned} a \odot b &:= a \otimes b \\ \alpha \odot \beta &:= \beta \otimes \alpha \\ r \odot s &:= r \otimes s = rs \\ r \odot a &:= ra \\ r \odot \alpha &:= r\alpha \\ \alpha \odot a &:= \alpha(a) = a(\alpha) =: a \odot \alpha \end{aligned}$$

Products of two positive power tensors $a_1 \otimes \cdots \otimes a_q$, $b_1 \otimes \cdots \otimes b_p$ and negative powers $\alpha_1 \otimes \cdots \otimes \alpha_q$, $\beta_1 \otimes \cdots \otimes \beta_p$ are defined by

$$\begin{aligned} (a_1 \otimes \cdots \otimes a_q) \odot (b_1 \otimes \cdots \otimes b_p) &:= a_1 \otimes \cdots \otimes a_q \otimes b_1 \otimes \cdots \otimes b_p \\ (\alpha_1 \otimes \cdots \otimes \alpha_q) \odot (\beta_1 \otimes \cdots \otimes \beta_p) &:= \alpha_1 \otimes \cdots \otimes \alpha_n \otimes \beta_1 \otimes \cdots \otimes \beta_m \end{aligned}$$

and extending by $\mathbb{F}$-linearity. This clearly makes the dimensioned ring product satisfy, for $q, p > 0$,

$$\odot : L^q \times L^p \to L^{q+p}, \qquad \odot : L^{-q} \times L^{-p} \to L^{-q-p}, \qquad \odot : L^0 \times L^0 \to L^0.$$

For products combining positive power tensors $a_1 \otimes \cdots \otimes a_q$ and negative power tensors $\alpha_1 \otimes \cdots \otimes \alpha_p$ we critically make use of the isomorphism $L^* \otimes L \cong \mathbb{F}$ to define, without loss of generality for $p > q > 0$,

$$(a_1 \otimes \cdots \otimes a_q) \odot (\alpha_1 \otimes \cdots \otimes \alpha_p) := \alpha_1(a_1) \cdots \alpha_q(a_q)\alpha_{p-q} \otimes \cdots \otimes \alpha_p.$$

It is then clear that the multiplication $\odot$ satisfies, for all $m, n \in \mathbb{Z}$,

$$\odot : L^m \times L^n \to L^{m+n}$$

and so it is compatible with the dimensioned structure of $L^\odot_\mathbb{Z}$. The multiplication $\odot$ is clearly associative and bilinear with respect to addition on each dimension slice from the fact that the ordinary tensor product is associative and $\mathbb{F}$-bilinear. Then it follows that $(L^\odot_\mathbb{Z}, +_\mathbb{Z}, \odot)$ is a commutative dimensioned ring. It only remains to show that non-zero elements of $L^\odot$ have multiplicative inverses. Note that a non-zero element corresponds to some non-vanishing tensor $0 \neq h \in L^n$, but, since L^n is a 1-dimensional vector space for all $n \in \mathbb{Z}$, we can find a unique $\eta \in (L^n)^* = L^{-n}$ such that $\eta(h) = 1$. It follows from the above formula for products of positive and negative tensor powers that, in terms of the dimensioned ring multiplication, this becomes

$$h \odot \eta = 1,$$

thus showing that all non-zero elements have multiplicative inverses, making the dimensioned ring $(L^\odot_\mathbb{Z}, +_\mathbb{Z}, \odot)$ into a dimensioned field. □

We now prove that the construction of the power dimensioned field of a line is, in fact, functorial.

Theorem 7.1 (The Power Functor for Lines)**.** *The assignment of the power construction to a line is a functor*

$$\odot : \mathsf{Line} \to \mathsf{DimRing}.$$

Furthermore, a choice of unit in a line $L \in \mathsf{Line}$ induces a choice of units in the dimensioned field $(L^\odot_\mathbb{Z}, +_\mathbb{Z}, \odot)$ which, since $L^0 = \mathbb{F}$, then gives an isomorphism with the trivial dimensioned field

$$L^\odot \cong \mathbb{F} \times \mathbb{Z}.$$

Proof. To show functoriality we need to define the power of a factor of lines $B : L_1 \to L_2$

$$B^\odot : L_1^\odot \to L_2^\odot.$$

This can be done explicitly in the obvious way, for $q > 0$

$$\begin{aligned} B^\odot|_{L^q} &:= B \otimes \overset{q}{\cdots} \otimes B : L_1^q \to L_2^q \\ B^\odot|_{L^0} &:= \mathrm{id}_\mathbb{F} : L_1^0 \to L_2^0 \\ B^\odot|_{L^{-q}} &:= (B^{-1})^* \otimes \overset{q}{\cdots} \otimes (B^{-1})^* : L_1^{-q} \to L_2^{-q} \end{aligned}$$

where we have crucially used the invertibility of the factor B. By construction, $B^\odot$ is compatible with the $\mathbb{Z}$-dimensioned structure and since B is a linear map with linear inverse, all the tensor powers act as $\mathbb{F}$-linear maps on the dimension slices, thus making $B^\odot : L_1^\odot \to L_2^\odot$ into a morphism of abelian dimensioned groups. Showing that $B^\odot$ is a dimensioned ring morphism follows easily by the explicit construction of the dimensioned ring multiplication $\odot$ given in proposition 7.1

above. This is checked directly for products that do not mix positive and negative tensor powers and for mixed products it suffices to note that

$$B^{\odot}(\alpha)\odot B^{\odot}(a)=(B^{-1})^*(\alpha)\odot B(a)=\alpha(B^{-1}(B(a)))=\alpha(a)=\mathrm{id}_{\mathbb{F}}(\alpha(a))=B^{\odot}(\alpha\odot a).$$

It follows from the usual properties of tensor products in vector spaces that for another factor $C:L_2\to L_3$ we have

$$(C\circ B)^{\odot}=C^{\odot}\circ B^{\odot},\qquad (\mathrm{id}_L)^{\odot}=\mathrm{id}_{L^{\odot}},$$

thus making the power assignment into a functor. Recall that a choice of unit in a line $L\in\mathsf{Line}$ is simply a choice of non-vanishing element $u\in L^\times$. In proposition 7.1 we saw that $L^{\odot}$ is a dimensioned field, so multiplicative inverses exist, let us denote them by $u^{-1}\in(L^*)^\times$. Using the notation for $q>0$

$$\begin{aligned}u^q&:=u\odot\overset{q}{\cdots}\odot u\\ u^0&:=1\\ u^{-q}&:=u^{-1}\odot\overset{q}{\cdots}\odot u^{-1},\end{aligned}$$

it is clear that the map

$$\begin{aligned}u:\mathbb{Z}&\to L^{\odot}\\ n&\mapsto u^n\end{aligned}$$

satisfies

$$u^{n+m}=u^n\odot u^m.$$

By construction, all $u^n\in L^n$ are non-zero, so $u:\mathbb{Z}\to L^{\odot}$ is a choice of units in the dimensioned field $(L^{\odot}_{\mathbb{Z}},+_{\mathbb{Z}},\odot)$. The isomorphism of dimensioned fields $L^{\odot}\cong\mathbb{F}\times\mathbb{Z}$ follows from proposition 3.4 and the observation that, by definition, $(L^{\odot})_0=L^0=\mathbb{F}$. □

8 Comments and Further Research

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