Time-energy uncertainty does not create particles

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We criticise claims (found in many expositions) that the time-energy uncertainty principle allows both a violation of energy conservation, and particle creation, provided that this happens for a sufficiently short time. But we agree that there is a grain of truth in these claims: which we make precise and justify using perturbation theory.

1 Introduction

A popular dogma about quantum theory is that energy conservation can be violated, and that particles can 'pop in to existence' out of nowhere, thanks to a time-energy uncertainty principle. Thus:

> "a consequence of the Heisenberg Uncertainty Principle is that we can take seriously the possibility of the existence of energy non-conserving processes—provided the amount by which energy is not conserved, $E_{violation}$, exists for a time less than $t = \hbar/2E_{violation}$" Jones, 2002, p. 226

> "Even when there is not enough energy or pair creation, multiparticle states appear, for example, as intermediate states in second-order perturbation theory. We can think of such states as existing only for a very short time, according to the uncertainty principle $\Delta E \cdot \Delta t = \hbar$. As we go to higher orders in perturbation theory, arbitrarily many such 'virtual' particles can be created." Peskin and Schroeder 1995 p. 13.

We will criticise this. And indeed, not every textbook is so unwise! Thus Griffiths says

> "It is often said that the uncertainty principle means that energy is not strictly conserved in quantum mechanics-that you're allowed to 'borrow' energy ΔE, as long as you 'pay it back' in a time $\Delta t \sim \hbar/2\Delta E$; the greater the violation, the briefer the period over which it can occur. There are many legitimate readings of the energy-time uncertainty principle, but this is not one of them. Nowhere does quantum mechanics license violation of energy conservation". (1995, p. 115)

But we will also argue that the dogma contains a grain of truth: which we will make precise and justify using perturbation theory. We will focus on three ideas, which we will interpret using the tools of perturbation theory:

1. (*non-conservation*) There is some sense in which 'energy' associated with a perturbed system is not conserved; and
2. (*particle creation*) There is some sense in which that non-conservation allows the non-conservation of particle-number.
3. (*shorter times*) There is some sense in which more particle creation occurs during shorter times.

We will make these statements precise, and verify them. Our lesson will be that, in each case, it is not a time-energy uncertainty relation that provides the wiggle-room to create particles: hence our title, "time-energy uncertainty does not create particles." Instead, the particles are best viewed as artefacts of the shifted perspective one adopts when approximating a physical system using perturbation theory.

1 The perturbation view of virtual states

Perturbation theory uses a quantum system that is in some way tractable to approximate a quantum system that is not. We begin with a a quantum system $(\mathcal{H}, \mathcal{A}, t \mapsto U_t)$ and a set of operators in $\{A_\lambda\} \subset \mathcal{A}$ parametrised by a positive real number λ, one value of which gives the correct/physically real operator of interest:

$$A_\lambda = A_0 + V_\lambda. \tag{1}$$

This set is constructed in such a way that $A_\lambda \to A_0$ in the operator norm as $\lambda \to 0$. Then A_0 is called the 'unperturbed' operator, and the set A_λ is called a 'perturbation'. The hope is both that a physical system can be correctly described by A_λ for some value of λ, and also that its properties can be accurately approximated using known facts about the more tractable operator A_0.

We assume that the operator A_λ can be represented in a power series expansion around $\lambda = 0$. That is, one seeks an expression of A_λ for which one can write,

$$A_\lambda = A_0 + \lambda \left(\frac{d}{d\lambda} V_\lambda\right)\Big|_{\lambda=0} + \frac{\lambda^2}{2!}\left(\frac{d^2}{d\lambda^2} V_\lambda\right)\Big|_{\lambda=0} + \cdots \tag{2}$$

The nth-order approximation of A_λ is by definition the sum of the first n terms in this series. As $n \to \infty$, it follows that the series approaches A_λ in the operator norm. A wide class of problems can be solved by adopting the simple approximation where only the first two terms are calculated. Defining $V := \frac{d}{dt} V_\lambda\Big|_{\lambda=0}$ this gives an approximation known as a 'linear' perturbation:

$$A_\lambda \approx A_0 + \lambda V. \tag{3}$$

The eigenvalue problem for an operator expressed by as a linear perturbation can typically be given an approximate analysis, thanks to classic results in perturbation theory. (Kato 1995)

Virtual states arise out by applying perturbation theory to dynamical evolution. Let $(\mathcal{H}, \mathcal{A}, t \mapsto U_t)$ be a quantum system, and consider a second one-parameter unitary representation $t \mapsto U^0_t$. We write H and H_0 for their respective Hamiltonian generators, and refer to the former as the 'perturbed' (or 'interaction' or 'correct/physically real') Hamiltonian, while the the latter is the 'unperturbed' Hamiltonian. We define $V := H - H_0$ and refer to it as the 'perturbation'. Writing $U_t = U^0_t(U^0_{-t}U_t)$, we now can expand the term in parentheses as a power series around $t = 0$,

$$U_t = U^0_t \left(I + t\frac{d}{dt}(U^0_{-t}U_t)\Big|_{t=0} + \frac{t^2}{2!}\frac{d^2}{dt^2}(U^0_{-t}U_t)\Big|_{t=0} + \cdots\right). \tag{4}$$

To the extent that t is close to zero and $V = H - H_0$ is small (in the operator norm), the dynamics U_t is approximated by summing the first n terms in this series and cutting off the

remainder. Writing U_t^n to denote the sum of the first n terms in the series, the first few orders of approximation can be calculated by applying the Leibniz rule to the derivatives:

$$\begin{aligned} U_t^0 &= U_t^0 \\ U_t^1 &= (I - itV)U_t^0 \\ U_t^2 &= \left(I - itV + \tfrac{t^2}{2!}([V, H_0] + V^2)\right) U_t^0. \end{aligned} \tag{5}$$

The virtual state picture arises from thinking of each contribution as a separate 'event'. For example, suppose ψ_a and ψ_b are orthogonal eigenvectors of the unperturbed Hamiltonian H_0, and that we wish to approximate the amplitude $\langle \psi_b, U_t \psi_a \rangle$ associated with a transition during time t from ψ_a to ψ_b. In the first-order case, we can replace U_t with its approximate counterpart U_t^1 to get,

$$\begin{aligned} \langle \psi_b, U_t^1 \psi_a \rangle &= \langle \psi_b, (U_t^0 - itVU_t^0)\psi_a \rangle \\ &= e^{-ita} \underbrace{\langle \psi_b, \psi_a \rangle}_{0} - ite^{-ita} \langle \psi_b, V\psi_a \rangle \end{aligned} \tag{6}$$

where a is the ψ_a eigenvalue of H_0. The first term does not contribute to the amplitude because the two states are orthogonal. However, the second term does. So we say: first the state ψ_a evolves freely as a stationary state; then an interaction V is turned on that allows it to 'deflect' to ψ_b.

Virtual states arise as 'intermediate states' in this kind of analysis. In our example of the transition $\langle \psi_b, U_t \psi_a \rangle$, they begin to appear in the second-order case, U_t^2. To simplify discussion, consider just the final term in that series, $([V, H_0] + V^2)U_t^0$. Writing a and b for the respective eigenvalues of ψ_a and ψ_b, we find its contribution to the amplitude is given by,

$$(a - b)\langle \psi_b, V\psi_a \rangle + \langle \psi_b, V^2 \psi_a \rangle. \tag{7}$$

The first of these two terms is a contribution just like we saw in the first-order case. The second term can be written,

$$\langle \psi_b, V^2 \psi_a \rangle = \langle \psi_b, V\psi' \rangle \langle \psi', V\psi_a \rangle, \tag{8}$$

where $\psi' = \frac{1}{|V\psi_a|} V\psi_a$ is a normalised vector satisfying $\langle \psi', \psi' \rangle = 1$. So, instead of viewing the transition from ψ_a to ψ_b in the potential V^2, we can view it as consisting an intermediate transition from ψ_a to ψ' in V, followed by a transition from ψ' to ψ_b in V. The intermediate state ψ' is an example of a *virtual state*.

As expected, in third-order transitions, the presence of a V^3 term gives rise to a pair of virtual states, and so on up the series. There is a Feynman diagram associated with each contribution that illustrates the interaction in terms of these intermediate states; as a result, the nth-order amplitude can be calculated by the summing the contributions from each of the nth-order Feynman diagrams.

2 The appearance of energy non-conservation

To sum up: the perturbation view of a quantum system is one of shifting perspectives. We approximate a system from the perspective of some low-n-order dynamics U_t^n, recognising that

from this vantage point, the world will sometimes deviate from its 'true' behaviour according to the perturbed dynamics U_t, as well as from its 'idealised' behaviour according to U_t^0.

One example of this is energy conservation. Thus we can now interpret the claim from the introduction:

1. (*non-conservation*) There is some sense in which 'energy' associated with a perturbed system is not conserved

On the perturbation scheme above, the 'true' energy eigenstate of a perturbed dynamics U_t is not conserved under any n-order approximation; and neither is the 'idealised' energy eigenstate of the unperturbed dynamics U_t^0. This can be formulated as a simple proposition. Formally:

Proposition 1. *Let $(\mathcal{H}, \mathcal{A}, t \mapsto U_t)$ be a quantum system with Hamiltonian H. Let $t \mapsto U_t^0$ be a second unitary representation with Hamiltonian H_0, and let U_t^n be the nth-order approximation of U_t in terms of U_t^0 (i.e. U_t^n is the nth-order cutoff of $U_t = U_t^0(U_{-t}^0 U_t)$ with the factor $U_{-t}^0 U_t$ expanded around $t = 0$). Then neither H nor H_0 are conserved under U_t^n, provided that $[H, U_t^n] \neq 0$ and $[H_0, U_t^n] \neq 0$.*

However, there is a sense in which energy for the 'true' perturbed dynamics U_t becomes closer to being satisfied the higher the order n of approximation. As we add more terms, the resulting approximate dynamics U_t^n approximates better the true dynamics U_t; that is, $U_t^n \to U_t$ in the ooperator norm as n becomes arbitrarily large. And this implies,[1]

$$[U_t^n, H] \to 0 \text{ as } n \to +\infty. \tag{1}$$

This is ironic: Peskin and Schroeder (p. 13) write that, "As we go to higher orders in perturbation theory, arbitrarily many such 'virtual' particles can be created", suggesting that energy conservation gets worse with higher order terms'. Agreed: there is more room for virtual states in higher-order terms. But energy conservation does not get worse — it gets better! The 'true' perturbed energy just becomes more distant (in the operator norm) from the idealised, unperturbed description of the system.

3 Particle creation

We now turn to the statement of (ii), the particle-number claim. For this we need to add some notion of particle number to our description. This will consist in a representation of annihilation (a_i) and creation (a_i^*) operators on $\mathcal{H}$ for $i \in \mathbb{Z}^+$, which satisfy $[a_i, a_j^*] = \delta_{ij}$ and $[a_i, a_j] = [a_i^*, a_j^*] = 0$. Let $N = \sum_i a_i^* a_i$ be the 'particle number' operator. Interpreting H_0 now as the 'free' Hamiltonian associated with no interactions, we assume that $[N, H_0] = 0$, and hence that a free system is one in which particle-number is conserved. However, if $[N, H_\lambda] \neq 0$, then particle number will *not* be conserved along the 'true' particle dynamics, and as a consequence it will not be conserved under the dynamics generated by any of the n-order perturbation Hamiltonians $H_{\lambda,n}$. That is, we have,

2. (*particle-creation*) The unitary dynamics generated by the nth order approximation $H_{\lambda,n}$ of the Hamiltonian H_λ does not conserve particle number N.

[1] In general, $\|A_n - B\| \to 0$ implies $\|[A_n, B]\| \to 0$, since $\|[A_n, B]\| = \|(A_n - B)B + B(B - A_n)\| \leq \|A_n - B\|\|B\| + \|B\|\|B - A_n\| = 2\|B\|\|A_n - B\|$.

4 Shorter times

We finally turn to the shorter-time claim (iii). Here at last, we are in the realm of the time-energy uncertainty principle—or, rather, principles—that are invoked by the cavalier textbook tradition with which we began. These are surveyed by Busch (1990, 1990a, 2008). For us, there are two main points to make, corresponding to two broad understandings of time-energy uncertainty. The second will be more positive, in that it will vindicate the shorter-time claim (iii).

The first concerns what Busch (2008) suggests we call 'external time' (or in his 1990: 'pragmatic time'): namely, time as measured by clocks that are not coupled to the objects studied in the experiment. So in this role, time specifies a parameter or parameters of the experiment: e.g. an instant or duration of preparation or of measurement, or the time-interval between preparation and measurement. In this role, there seems to be no scope for uncertainty about time. And indeed, our first point here is a warning—following Busch (1990a).

For as Busch discusses, there is tradition (deriving from the founding fathers of quantum theory) of an uncertainty principle between:

(i) the duration of an energy measurement, and

(ii) either the range of an uncontrollable change of the measured system's energy or the resolution of the energy measurement or the statistical spread of the system's energy.

To give a little more detail: Busch (1990a, Sections 1 and 2) describes how various authorities (Landau and Peierls, Landau and Lifshitz, ...) endorse either of the following:

(P): An energy measurement of duration Δt leads to an uncontrollable and unpredictable change of the (previously sharply defined) energy by an amount of the order ΔE such that $\Delta E.\Delta t \geq \hbar$; so that there is no short-time reproducible (first kind) energy measurement.

(P'): An energy measurement of duration Δt must carry an inaccuracy ΔE such that the uncertainty relation $\Delta E.\Delta t \geq \hbar$ is satisfied.

Busch argues, and we agree, that Aharonov and Bohm (1961) refute this tradition; (Busch 1990a, especially Section 4; 2008 Section 3.1). They give a simple model of an arbitrarily accurate and arbitrarily rapid energy measurement. In short: two particles are confined to a line and are both free, except for an impulsive measurement of the momentum and so energy of the first by the second, with the momentum of the second being the pointer-quantity. (Note that Busch argues that a proper analysis and vindication of Aharonov and Bohm's refutation uses POVMs, i.e. the notion of physical quantity that generalizes PVMs.)

Our second, more positive, point concerns what Busch (2008) suggests we call 'intrinsic time' (or in his 1990: 'dynamical time'): namely, a dynamical variable of the *studied* system, that functions to measure the time. For example: the position of a clock's dial relative to the clock's face. Busch suggests that in principle every non-stationary quantity A defines for any quantum state ρ a characteristic time $\tau_\rho(A)$ in which $\langle A \rangle$ changes 'significantly'. For example: if $A = Q$, and ρ is a wave packet then $\tau_\rho(A)$ could be defined as the time for the bulk of the wave packet to shift by its width—in some sense of 'width'.

So we need some measure of temporal duration associated with the particle number operator N. Various are available; we choose what is sometimes called the 'characteristic time' associated with the dispersion of an operator-state pair. This obeys what is probably the best-known time-energy uncertainty principle for intrinsic times: the *Mandelstam-Tamm uncertainty principle*.

They combine (a) the Heisenberg equation of motion of an arbitrary quantity A

$$i\hbar\frac{dA}{dt} = [A, H] \tag{1}$$

with (b) the Heisenberg-Robertson uncertainty principle, that for any quantities A, B, and quantum state (density matrix) ρ:

$$\Delta_\rho A \Delta_\rho B \geq \frac{1}{2}|\langle [A, B]\rangle_\rho| \ ; \tag{2}$$

and (c) the definition of a characteristic time

$$\tau_\rho(A) \ := \ \Delta_\rho A \ / \ | \ d\langle A\rangle_\rho \ /dt \ | \ ; \tag{3}$$

i.e. as the time it takes for the expectation value of A to change by its standard deviation. They deduce

$$\tau_\rho(A)\Delta_\rho(H) \geq \frac{1}{2}\hbar. \tag{4}$$

To apply these ideas to our particle number operator N: let H be the Hamiltonian generator of a perturbed ('correct') unitary dynamics $t \mapsto U_t$, and let N be the 'free' number operator satisfying, as in Section 3, $[H_0, N] = 0$—but also $[H, N] \neq 0$. For any state $\psi \in \mathcal{H}$ evolving unitarily according to $\rho_t = U_t \rho U_t^*$, the *U_t-characteristic time* τ^H (with respect to ρ) of N is,

$$\tau^H(N)_\rho := \frac{\Delta_\rho N}{|\frac{d}{dt}\mathrm{Tr}(\rho N)|}. \tag{5}$$

Or one might speak of the *H-characteristic time* τ^H (with respect to ρ) of N.

We can replace U_t with an nth-order cut-off U_t^n and its generator H_n, giving us a sequence of U_t^n-characteristic times (a sequence of H_n-characteristic times) indexed by $n = 0, 1, 2, \ldots$. For $n \to 0$, we have that $N(n) \to N$, which is time-independent, and hence the characteristic time diverges to infinity. Conversely: *if* the degree of non-commutativity of N and U_t is monotonically increasing in n, *then* the characteristic time gets smaller as $n \to +\infty$. That is, we get a version of statement (iii) above:

3. (*shorter times*) If $||[H_n, N]||$ is monotonically increasing in n, then the U_t^n-characteristic time (the H_n-characteristic time) of N decreases monotonically as $n \to +\infty$.

That is, the characteristic time of the particle number operator decreases as one approaches the 'true' (perturbed) Hamiltonian H_λ by considering higher and higher order perturbative terms.

1 References

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