Jordan Lin Operators for Qm Mechanics Dover

p.1

Ch 1 Lin Spaces & Lin Fnals

Notes J Butterfield

1.1 Vector spaces

1.2 Inner products. : Schwarz inequality.

1.3 Hilbert space

a) converge of vectors $\psi_n \to \psi$:= $\|\psi - \psi_n\| \to 0$ as $n \to \infty$.

b) defn of infinite lin combinations on analogy with $w = \Sigma z_k$

c) Idea of completeness: every Cauchy seqce converges

Hbt space := complete I.P. space.

separable

lin manifold vs. subspace.

d) example $l^2 = \{(x_1, x_2 \ldots) \; x_c \in \mathbb{C}$ (or $\mathbb{R}$!) $| \; \Sigma^\infty |x_k|^2 < \infty \}$.

orthobasis $(1,0,0,0\ldots), (0,1,0,0,0\ldots) = \{\phi_k\}$

So each vector ψ is $\Sigma^\infty x_k \phi_k$

The partial sums are $(x_1, x_2 \ldots x_N, 0,0,0)$ and these converge to $(x_1, x_2 \ldots x_N, x_{N+1}, \ldots x_m \ldots)$

e) in sep Hbt space general ψ has a unique expression in terms of an orthonormal basis $\{\phi_k\}$

$\psi = \Sigma^\infty (\phi_k, \psi) \phi_k$

(Use Schwarz to show this with partial sums)

f) Any 2 Hbt space (over $\mathbb{C}$ / $\mathbb{R}$) are isomorphic by basis $\to$ basis

Cor Any inf-dimnl sep Hbt sp can be identified with l^2

g) A space L^2 of sq-Sable fnctns $\psi : X \to \mathbb{C} \ni \int_X |\psi(x)|^2 d\mu < \infty$

is a Hbt space.

$L^2(0,1)$ — fourier series give an orthonormal basis of continuous functions

h) NB Lebesgue vs Riemann integration!

i) orthocomplements & projections: $\mathcal{M}$ a subspace of $\mathcal{H}$.

(closed lin mfld)

If $\mathcal{H}$ is separable, so is $\mathcal{M}$

And $\mathcal{M}^\perp$ as usual with

$\forall \psi \in \mathcal{H}$, has ! expression $\varphi + \varphi^\perp$

$\varphi \in \mathcal{M}$ and $\varphi^\perp \in \mathcal{M}^\perp$

1.4 Lin Fnals

a) In an inner product space, each ψ defines a lin fnal F_ψ by $F_\psi(\phi) := (\psi, \phi)$

b) For n-dimensional space: $\{\phi_k\} = \{\phi_1 \ldots \phi_n\}$ an o.basis.

To each F on this space, assign the vector $\psi_F := \Sigma^n F(\phi_k)^* \phi_k$

Indeed the defn in a), applied to this ψ_F, yields F again.

That is for any vector $\phi = \Sigma(\phi_k, \phi)\phi_k$, we have

$F(\phi) = \Sigma(\phi_k, \phi) F(\phi_k) = (\psi_F, \phi)$.

c) For ∞-dimnl space, use defn of continuity

d) (Riesz) every contns lin fnal on sep ble Hbt space F, $\exists! \, \psi_F \in \mathcal{H}$ with $F(\phi) = (\psi_F, \phi)$

Ch 2 Lin Operators   Jordan p2

2.5 (1st f) Operators & matrices
For general $AB \neq BA$! $(A\psi)(x) = \frac{d}{dx}\psi(x)$ $(A\psi)(x) = \int a(x,y)\psi(y)dy$

2.6 Bdd opers.
Defn of A is continuous, and of bdd with norm $\|A\|$
Thm 6.1   A lin oper is conts iff bdd

from norm for vectors: $\left\{ \begin{array}{l} \|A+B\| \leq \|A\| + \|B\|, \\ \|aA\| = |a|\,\|A\| \quad \|A\| = 0 \text{ iff } A = 0 \\ \|AB\| \leq \|A\|\|B\| \end{array} \right.$ } so $\|\,\|$ is a "norm"

A bdd lin oper on an ∞-dim'l but sep Hilb sp can be repr'ted by an inf matrix.

2.7 Inverses
A has an inverse if there is ~~an~~ a lin oper B s.t. $AB = 1 = BA$
iff $\forall \psi \exists! \varphi$ with $\psi = A\phi$.

Thm 7.2   fin dim'l vec space. $\{\phi_i\}$ any basis   NSCs for A to have inverse
(i) there is no nonzero vector ϕ s.t. $A\phi = 0$
(ii) the set $\{A\phi_1 \ldots A\phi_n\}$ is lin indpdt
(iii) there is a lin oper B, s.t. $BA = 1$
(iv) the matrix corresp'dg to A has non-zero detmt.

NB for inf-dim'l space (i) (ii) (iii) are NOT sufficient.
Let $A: \ell^2 \to \ell^2$: $\phi = (x_1 x_2 \ldots) \xrightarrow{A} (0, x_1 x_2 \ldots)$ right shift
Let B be "drop 1st component & left shift" $(x_1 x_2 \ldots) \xrightarrow{B} (x_2, x_3 x_4 \ldots)$

Then (i) (ii) (iii) all hold
But A has no inverse. If $\psi = (x_1 x_2 \ldots)$ with $x_1 \neq 0$
then there is no vector ϕ with $\psi = A\phi$.

2.8 Unitaries.
U is unitary if it has an inverse and $\|U\psi\| = \|\psi\|$ for all ψ
every unitary is bdd $\|U\| = 1$.

Thm 8.1 $(U\psi, U\varphi) = (\psi, \varphi)$ $\forall \psi, \varphi$.   Cor 8.2   U(obasis) is an obasis
Thm 8.3   "Conversely" to Cor: If U is bdd and ~~U~~ U(some obasis) is an obasis, then U is unitary

2.9 Adjoints, Hermit^{n} Opers

Let A be bdd (and so continuous). Then for each $\psi \in \mathcal{H}$, the lin functional $F[\psi]$ defined by $F^{[\psi]}(\phi) := (\psi, A\phi)$ is continuous. So by Riesz, $\exists$ a ! vector ~~x~~, call it $A^{\dagger}\psi$, with $F^{[\psi]}(\phi) = (A^{\dagger}\psi, \phi)$

$A^{\dagger}$ is trivially linear

Thm 9.1 A is bdd $\Rightarrow A^{\dagger}$ is bdd and $\|A^{\dagger}\| = \|A\|$

Usual stuff: $A^{\dagger\dagger} = A$ $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ $(aA)^{\dagger} = a^{*}A^{\dagger}$ etc.
: adjoint/Herm conjugate of the repr^{ng} matrix.

A bdd lin oper is self-adjoint or Hermit^{n} if $A^{\dagger} = A$.
NB It is impossible to define an unbdd Herm^{n} oper for all vectors
This is Thm 11.1 below

i.e. $(\phi, A\psi) = (A\phi, \psi)$ i.e. $(\phi, A\psi) = (\psi, A\phi)^{*}$ (Herm)
or $(\psi, A\psi) \in \mathbb{R} \ \forall \psi$

Example On $L^2(0,1)$ A with $(A\psi)(x) := x\psi(x)$
is bdd: $\|A\psi\|^2 \le \|\psi\|^2$ and so $\|A\| = 1$.
and (Herm). So A is Hermitian.
But "comparable defin" $L^2(\mathbb{R})$ is NOT bdd; need to damp eg $(V\psi)(x) := e^{-|x|}\psi(x)$

Thm 9.2 If A is bdd with a bdd inverse A^{-1}, then $(A^{\dagger})^{-1}$ exists and $(A^{\dagger})^{-1} = (A^{-1})^{\dagger}$

Thm 9.3 A lin oper is unitary iff $U^{\dagger}U = 1 = UU^{\dagger}$.

For bdd lin opers A, B, we can use the adjoints to show that the repr^{ng} matrix (c_{jk}) of the $C := AB$ is the matrix product
$$c_{jk} = \sum_{i=1}^{\infty} a_{ji} b_{ik}$$

2.10 Project^{n} operators

Given subspace $\mathcal{M} < \mathcal{H}$, define ~~projection~~ $E_{\mathcal{M}}$ by $E_{\mathcal{M}} : \mathcal{H} \to \mathcal{H}$ with $\psi \in \mathcal{M} \Rightarrow E_{\mathcal{M}}\psi = \psi$.
Then

Thm 10.1 A bdd lin op E is a projection iff $E^2 = E = E^{\dagger}$
(nontrivial ~~for E~~)

Thm 10.2 If $E_1E_2 = E_2E_1$, then E_1E_2 is the project^{n} onto $\operatorname{ran}(E_1) \cap \operatorname{ran} E_2$
If $E_1 \perp E_2$ $E_1 + E_2$ projects on $\operatorname{ran} E_1 + \operatorname{ran} E_2$
If $E_1 \le E_2$, $E_2 - E_1$ projects on rel o complement

2.11 Unbdd Operators

Jordan p4

Example. Posn in Schrod repsentn for $L^2(\mathbb{R})$

i.e. $(Q\psi)(x) := x\psi(x)$.

$\|Q\psi\|^2 \equiv \int_{-\infty}^{\infty} |x\psi(x)|^2 dx$ can be arbitly larger than $\|\psi\|^2 = \int |\psi(x)|^2 dx$

Thm 11.1 If a lin oper is defined for all vectors and if $(\phi, A\psi) = (A\phi, \psi)$ for all ψ, φ, then A is bded.

a) QM Thy needs unbdd opers with the Hermitian property. So it needs ~~dom~~ opers with domains less than all of $\mathcal{H}$. Hence jargon of domains, dense domains, and extensions

b) If A has a dense domain, we can define A^{+} viz:–

Dom$(A^+) = \{ \psi$ s.t. there is a vector, $\tilde{\psi}$, with: $\forall \phi \in \text{dom} A$ $(\phi, \tilde{\psi}) = (A\phi, \psi) \}$

Then define $A^+ : \psi \underset{A^+}{\mapsto} \tilde{\psi}$

This defines A^+ as linear and dom A^+ a lin mfd. because dom A is dense.

c) An operator is symmetric if it has dense domain and $(\phi, A\psi) = (A\phi, \psi)$ $\forall \phi, \psi \in \text{dom}(A)$

Then $A^+\psi$ is defined and $A^+\psi = A\psi$, for all $\psi \in$ dom A

IE: A^+ is an extension of A.

If $A^+ = A$, we say A is self-adjoint or Hermitian.

d) Example. Define Q on $L^2(\mathbb{R})$ to have dom$(Q) := \{ \psi \mid \int |x\psi(x)|^2 < \infty \}$

This is trivially dense. So Q is symmetric.

And so Q^+ is defined & extends Q

One argues by "chopping" (p.31) that the extension is "vacuous"

That is: $Q^+ = Q$ and Q is self-adjoint.

e) A symmetric oper that cannot be extdd to a larger domain is maximal symmetric

Thm 11.2 Every s-a operator is a maximal symmetric operator.

NB not conversely

f) Closed operators Defn of "closed" ("2nd best to continuity")

Thm 11.3 If dom(A) is dense, then A^+ is closed

So every s-adjoint operator is closed

3.12 Eigenvalues & eigenvectors

...usual...

Let A be Hermit or unitary
Let $a_1 \ldots a_k \ldots$ be its evalues, with corrspdg espaces M_k
Then $\oplus_k M_k$ is the subspace spanned by eigenvectors of A. "Eig(A)"

Define M reduces A if both M and $M^\perp$ are invariant under A.

Thm 12.7 M reduces A iff $EA = AE$ iff $(1-E)A = A(1-E)$

Thm 12.8 A Hermit or unitary. Then Eig(A) reduces A.

3.13 Eigenvalue decomposit.

Thm 13.1 For findim, eigenvect of Hermit span whole space.

Write, with evalues $a_1 < a_2 .. < a_k < a_{m-1} < a_m$ with esps M_k.
So m evalues in all and $k = 1, 2, \ldots m$.

$$A = \sum_{k=1}^m a_k I_k \quad \text{with } I_k \text{ projectors onto } M_k$$

For each real number x, $E_x := \sum_{a_k < x} I_k$

Then $E_x = 0$ for $x < a_1$ and $E_x = 1$ for $x > a_m$.
If $x < y$, $E_x E_y = E_x = E_y E_x$. i.e. $E_x \le E_y$

Define for each x let $dE_x = E_x - E_{x-\varepsilon}$
with ε so small that there is no a_j with $x - \varepsilon \le a_j < x$.

So dE_x is not zero only when x is an evalue a_k
— in which case $dE_x = I_k$

For $\sum_{k=1}^m I_k = 1$, we write $\int_{-\infty}^{\infty} dE_x = 1$

And for $A = \sum a_k I_k$ we write $A = \int_{-\infty}^{\infty} x \, dE_x$.

Now, $(\phi, E_x \psi)$ is a complex fn of x that jumps in value by $(\phi, I_k \psi)$ at $x = a_k$. So

Contin from the right

ordinary Riemann integrals

$$(\phi, \psi) = \int d(\phi, E_x \psi)$$

$$(\phi, A\psi) = \int x \, d(\phi, E_x \psi)$$

Similarly unitaries. eigenvalues $u_k = e^{i\theta_k}$ $\quad 0 < \theta_1 < \theta_2 .. \theta_k < .. < \theta_m \le 2\pi$

$$E_x := \sum_{\theta_k < x} I_k \qquad U = \int_0^{2\pi} e^{i\theta} dE_x$$

$$(\phi, U\psi) = \int_0^{2\pi} e^{i\theta} d(\phi, E_x \psi)$$

3.14 Spectral decomposition

Define A family $\{E_x\}_{x\in\mathbb{R}}$ is spectral family iff
(i) $x \le y \Rightarrow E_x \le E_y$ i.e. $E_xE_y = E_x = E_yE_x$
(ii) If $\varepsilon > 0$, then $E_{x+\varepsilon}\psi \to E_x\psi$ as $\varepsilon \to 0$: $\forall \psi$, and $\forall x$ "continuous from right"
(iii) $E_x\psi \to 0$ as $x \to -\infty$; $E_x\psi \to \psi$ as $x \to \infty$: $\forall\psi$

Thm 14.1 For each selfadjt opr A, $\exists!$ spectral family E_x s.t.
$\forall \varphi, \psi$ $\quad (\phi, A\psi) = \int_{-\infty}^{\infty} x \, d(\phi, E_x\psi)$ $\quad$ we write $A = \int x \, dE_x$

Thm 14.2 Similarly for unitary ~~U~~ $E_x = 0$ for $x \le 0$, $E_x = 1$ for $x > 2\pi$
and $(\phi, U\psi) = \int_0^{2\pi} e^{ix} d(\phi, E_x\psi)$
So we write $U = \int_0^{2\pi} e^{ix} dE_x$.

Examples "position" on $L^2(0,1)$ and $L^2(\mathbb{R})$
On $L^2(0,1)$, define E_x as "chopping" $(E_x\psi)(y) \begin{cases} = \psi(y) & \text{if } y \le x \\ = 0 & \text{if } y > x \end{cases}$
Then
$\|E_{x+\varepsilon}\psi - E_x\psi\|^2 = \int_x^{x+\varepsilon} |\psi(y)|^2 dy \to 0$
and $\{E_x\}$ is a spectral family.
Define A on $L^2(0,1)$ by $(A\psi)(x) = x\psi(x)$ bdd s-adjoint.
Then for any ψ, and ϕ
$$\int_{-\infty}^{\infty} x \, d(\varphi, E_x\psi) = \int_{-\infty}^{\infty} x \, d\int_0^1 \varphi(y)^* (E_x\psi)(y) dy =$$
$$= \int x \, d\int_0^x \varphi(y)^* \psi(y) dy = \int_0^1 \varphi(x)^* x \psi(x) \, dx \equiv (\phi, A\psi)$$
So we have spectral decomp. of A.
$(\phi, E_x\psi)$ never jumps in value $\quad (\phi, E_x\psi) - (\phi, E_{x-\varepsilon}\psi) = \int_{x-\varepsilon}^{x} \varphi(y)^* \psi(y) dy \to 0$ as $\varepsilon \to 0$

~~Thm~~ We say E_x jumps in value at x iff for some ψ $(E_x - E_{x-\varepsilon})\psi$ does not converge to 0 as $\varepsilon \to 0$ (ε positive of course). Otherwise continuous at x
So in this example ~~A~~ E_x is continuous at all x, since A has no eigenvalues/eigenvectors.

Thm 14.3 A selfadjoint with $A = \int x \, dE_x$
Then E_x jumps in value at a iff a is an evalue of A.
Let I_a be the projection onto the eigenspace for a.
Then $E_x I_a = 0$ for $x < a$ and $E_x I_a = I_a$ for $x \ge a$
And for $\varepsilon > 0$, any ψ $\quad E_a\psi - E_{a-\varepsilon}\psi \to I_a\psi$ as $\varepsilon \to 0$

Define spectrum $:= \{x \in \mathbb{R} \mid E_x \text{ increases}\}$ i.e. x not in (a, b) on which E_x const. — Jordan p7

point spectrum $:= \{x \in \mathbb{R} \mid E_x \text{ jumps}\}$

cont's spectrum $:= \{x \in \mathbb{R} \mid E_x \text{ increases continuously}\}$

Thm 14.5 A self-adjoint oper is bdd iff its spectrum is bdd

A s-adjt oper is positive if $(\phi, A\phi) \geq 0$ ~~att~~ for all ϕ.

Thm 14.6 A self-adjt oper is positive iff its spectrum is non-negative.

3.15 Functions of an Operator, Stone's Theorem

Given A self-adjoint $A = \int x \, dE_x$. Let $f: \mathbb{R} \to \mathbb{C}$. We define $f(A)$ by:

$$(\varphi, f(A)\psi) = \int_{-\infty}^{\infty} f(x) \, d(\varphi, E_x \psi)$$

(for functions this is Riemann integral)

Clearly a) for $f(x) = x$, we have $f(A) = A$

b) for $f(x) = 1$, we have $f(A) = \mathbb{1}$, since $\int_{-\infty}^{\infty} d(\phi, E_x \psi) = (\varphi, \psi)$

c) $(f+g)(A) = f(A) + g(A)$; $(cf)(A) = c\, f(A)$

d) Also for products: (fg) is defined by $(fg)(x) := f(x)g(x)$

Then $(fg)(A) = f(A)g(A)$

e) So polynomials: $f(x) = c_0 + c_1 x + c_2 x^2 \ldots + c_n x^n$

gives $f(A) = c_0 \mathbb{1} + c_1 A + c_2 A^2 + \ldots + c_n A^n$

f) $[f(A)]^\dagger = (f^*)A$. If $f: \mathbb{R} \to \mathbb{R}$, then $f(A)$ is self-adjt

If $f^* f = 1$, then $[f(A)]^\dagger f(A) = \mathbb{1} = f(A)[f(A)]^\dagger$

i.e. $f(A)$ is unitary.

g) $f(A)$ is positive if $f(x) \geq 0$ on spectrum of A

h) $f(A)$ is bdd if $|f(x)|$ is bdd over spectrum of A.

Given $H = \int_{-\infty}^{\infty} x \, dE_x$, define for all t: $(\phi, U_t \psi) := \int e^{itx} d(\phi, E_x \psi)$

U_t is an operator, viz $U_t = e^{itH}$, which is unitary since $(e^{itx})^* e^{itx} = 1$.

Evidently $U_0 = \mathbb{1}$ and since $e^{itx} e^{it'x} = e^{i(t+t')x}$, we have

$U_t U_{t'} = U_{t+t'}$. The converse is

Thm 15.1 Stone's theorem

$\forall t \in \mathbb{R}$, let U_t be unitary and suppose (i) $(\phi, U_t\psi)$ is continuous function of t, $\forall\phi$ and $\forall\psi$. (ii) $U_0 = 1$, $U_t U_{t'} = U_{t+t'}$

(a unitary representation of $(\mathbb{R}, +)$).

Then ① there is a unique self-adjoint operator H such that $U_t = \exp(itH)$ for all t.

② the domain of H is $\left\{\psi \,\middle|\, \frac{1}{(it)}(U_t - 1)\psi \text{ converges as } t \to 0\right\}$ and the limit vector is $H\psi$.

③ If a bdd oper commutes with all U_t, it commutes w H.

Using ②, we say:-

If $U_t\psi \in \mathrm{dom}(H)$, then $\frac{1}{i\Delta t}(U_{\Delta t} - 1)U_t\psi \to H U_t\psi$ as $\Delta t \to 0$

That is: $\frac{1}{i\Delta t}(U_{t+\Delta t} - U_t)\psi \to H U_t \psi$

So we write $-i\frac{d}{dt}(U_t\psi) = H U_t \psi$ "Schrödinger equation"