Is a particle an irreducible representation of the Poincaré group?

Adam Caulton

Balliol College, University of Oxford
Faculty of Philosophy, University of Oxford

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Outline

'Ever since the fundamental paper of Wigner on the irreducible representations of the Poincare group, it has been a (perhaps implicit) definition in physics that an elementary particle "is" an irreducible representation of the group, G, of "symmetries of nature".'
— Ne'eman and Sternberg (1991, p. 327).

Ne'eman & Sternberg have in mind that

$$G = ISO(3,1)^{\uparrow} \times SU(3) \times SU(2) \times U(1) \ .$$

But I will ignore all internal d.o.f.s (apart from spin).

"is"?

- *happens* to be represented by?
- *happens* to be represented by in certain regimes?
- is *characterised* by its being represented by?

Wigner's identification: a clarification

A particle's space of possible states is represented by a Hilbert space which is the carrier space for an irreducible, projective, unitary representation of the Poincaré group, and its possible properties are represented by the algebra of self-adjoint linear operators, defined on that Hilbert space, which generate this representation.

A particle's space of possible states is represented by a phase space which is the carrier space for an irreducible, symplectic group action, up to neutral elements, of the Poincaré group, and its possible properties are represented by the algebra of well-behaved real-valued functions, defined on that phase space, which generate this representation.

The general idea: a reconstruction

- A physical system (such as a particle) is completely characterised by its possible states, and by its associated quantities (essentially, the properties it may or may not possess).
- In quantum mechanics (and in Hamiltonian classical mechanics), a state space and algebra of quantities can be determined by considering representations (or realisations) of some group, typically a Lie group.
- The deep reason that this is possible is that quantities live a double life in these frameworks: as physical determinables (essentially, properties) and as *generators* of transformations. (The same deep fact lies behind Noether's theorem.)
- The needs of each framework place constraints on acceptable representations/realisations. In quantum mechanics, we seek projective unitary representations. (In Hamiltonian classical mechanics, we seek symplectic realisations up to neutral elements.)

So: why the Poincaré group?

The Poincaré algebra

	H	P_j	J_j	K_j
H	0	0	0	P_j
P_i	0	0	$\epsilon_{ijk} P_k$	$\delta_{ij} H$
J_i	0	$\epsilon_{ijk} P_k$	$\epsilon_{ijk} J_k$	$\epsilon_{ijk} K_k$
K_i	$-P_i$	$-\delta_{ij} H$	$\epsilon_{ijk} K_k$	$-\epsilon_{ijk} J_k$

These generate the proper orthochronous Poincaré group $ISO(3,1)^{\uparrow}$, with Casimir invariants $P_\mu P^\mu = m^2 \in \mathbb{R}$ and $W_\mu W^\mu = -m^2 s(s+1)\hbar^2$ (if $m^2 > 0$), where $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$.

We may also wish to include the discrete symmetries C, P, T, yielding the "full" Poincaré group $IO(3,1)$.

The Bargmann (extended Galilei) algebra

	M	H	P_j	J_j	G_j
M	0	0	0	0	0
H	0	0	0	0	P_j
P_i	0	0	0	$\epsilon_{ijk} P_k$	$\delta_{ij} M$
J_i	0	0	$\epsilon_{ijk} P_k$	$\epsilon_{ijk} J_k$	$\epsilon_{ijk} G_k$
G_i	0	$-P_i$	$-\delta_{ij} M$	$\epsilon_{ijk} G_k$	0

These generate (a central extension of) the Galilei group $Gal(3)$, with Casimir invariants $M, H - \frac{1}{2M}\mathbf{P}^2$ and $\mathbf{W}^2 = M^2 s(s+1)\hbar^2$ (if $M > 0$), where $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$.

Outline

'The wave functions form a description of the physical state, not an invariant however, since the same state will be described in different coordinate systems by different wave functions. In order to put this into evidence, we shall affix an index to our wavefunctions, denoting the Lorentz frame of reference for which the wave function is given. ***Thus φ_l and $\varphi_{l'}$ represent the same state****, but they are different functions. The first is the wave function of the state in the coördinate system l, the second in the coördinate system l'.*

. . .

'Because of the invariance of the transition probability we have

$$|(\varphi_l, \psi_l)|^2 = |(\varphi_{l'}, \psi_{l'})|^2 \tag{1}$$

. . .

'We see thus that there corresponds to every invariant quantum mechanical system of equations such a representation of the inhomogeneous Lorentz group [i.e., the Poincaré group].'
—Wigner (1939, 150-1)

'The principle of relativistic invariance, or of any invariance, then makes three postulates. These have been stated, with admirable clarity, a short time ago by R. Haag in an article which I have not yet seen in print. They are:

(a) ***It should be possible to translate a complete description of a physical system from one coordinate system into every equivalent coordinate system.***
(b) *That the translation of a dynamically possible description be again dynamically possible. Expressed in a somewhat more simple language: a succession of events which appears possible to one observer should appear possible also to any other observer.*
(c) *That the criteria for the dynamical possibility of complete descriptions be identical for equivalent observers.'*

—Wigner (1958, p. 519)

Outline

Wigner's identification: problems

Problems with Wigner's identification and its motivation:

- *Symmetry confusion.*

  If the Poincaré algebra is taken to generate transformations which *preserve the physical state*, then we are bound to interpret the *entire* irrep as representing a single physical state. (See also McCabe 2004.)

  Yet the opposite is the case: distinct rays represent distinct physical states, and the irrep *exhausts* the possibilities for the particle.
- *Over-restrictive constraint on "equivalent" observers.*

  Relatedly, why the focus on only *inertial* observers? Why not *arbitrary* observers?
- *Over-restrictive constraint on particles.*

  A system's algebra obeys the Poincaré algebra only if the corresponding system is free. What about interacting particles?

The first two problems may be remedied; the third problem is fatal.

What kind of symmetry?

- *At one end:* representational redundancies.
- *At the other end:* the most general transformations definable on the system.

Compare:
Irreps of the symmetric group S_N (only interesting for paraparticles).

- Whole irrep identified with a *single physical state*.
- Symmetries preserve *all* physical information.
- Casimir invariants as humdrum physical properties (not just the "intrinsic" or "essential" ones).

Irreps of the Heisenberg group (generated by **Q** and **P**):

- Single irrep identified with the *entire state space*.
- Symmetries preserve the *least* amount of physical information.
- Casimir invariants as state-independent, or "intrinsic", or "essential" (Castellani 1998) properties.

What justifies seeking projective unitary representations?

- Wigner's justification hangs on transition amplitudes being Poincaré-invariant, because rival Lorentz frames should agree on the physical information, encapsulated in transition probabilities. This loses its motivation if the Poincaré transformations are interpreted as genuine physical changes.
- An alternative justification might be afforded by the demands that: (i) the transformations be linear; (ii) they preserve the length of all vector states; and (iii) each transformation has an inverse. (But this wasn't Wigner's justification.)

$$
\begin{aligned}
\|U\psi\|^2 &= \|\psi\|^2, \quad \text{for all } \psi \\
U^\dagger U &= \mathbb{1} \\
U^\dagger U U^{-1} &= U^{-1} \\
U^\dagger &= U^{-1}
\end{aligned}
$$

Why the spacetime symmetry group? Pros

An irrep of the spacetime symmetry group—properly interpreted—has the right sort of flavour to be connected to *particulate* behaviour.

- Particles (plus instantaneous velocity) can be shunted around spacetime and boosted, hence the spacetime transformations.
- Particles are structureless, so can *only* be shunted around spacetime and boosted, hence irreducibility.

These point to irreps of the Poincaré group as providing an exhaustive catalogue of the different ways a particle can be. I take it that this is the received understanding—but it is very different from Wigner's conception.

Why the spacetime symmetry group? Cons

But the use of the spacetime symmetry group prompts two key worries:

- (*Dynamical innocence.*) We want to find the broadest space of possibilities for a particle. This demands "dynamical innocence". But how could space*time* transformations $(H, \mathbf{K})$ be dynamically innocent?
- (*Geometrical innocence.*) Relatedly, why should a particle be defined in terms of any particular background spacetime structure?

  This should be particularly unsatisfactory for Brownites about spacetime structure: if spacetime structure is a *codification* of the behaviour of systems—including particles—then it seems to have things backwards to constrain particles *ab initio* to obey that structure.

'I suggest that principles linking dynamical structure and spacetime structure [such as Earman's SP1 and SP2] need not rest on dubious metaphysical principles, but are, rather, analytically true.'—Myrvold (2017)

Wigner as a "principle theorist" about geometry?

'The point of view which I shall adopt is that the problems of physics are still rather far from their solution and that the role of symmetry and invariance is that of a guide in the development of the proper physical concepts, rather than something that one simply reads off from the ready equations. To illustrate this, I shall not say that physical equations are invariant under rotations in space because the Hamiltonian contains only distances and the absolute values of the momenta, Rather, my point of view would be to search for those Hamiltonians which give a physical theory that is invariant under spatial rotations and other relativistic transformations.'—Wigner (1956, p. 518).

Why the spacetime symmetry group? Cons (II)

Irreps of the group of symmetries of spacetime are constrained to be free.

- *Galilei spacetime:*
  $H - \frac{1}{2M}\mathbf{P}^2$ is a Casimir invariant (as is M); so $H = \frac{1}{2M}\mathbf{P}^2 + \textit{const.}$ on any irrep. (Bargmann 1954, Levy-Leblond 1967.)
- *Minkowski spacetime:*
  $P_\mu P^\mu \equiv H^2 - \mathbf{P}^2$ is a Casimir invariant; so $H^2 = \mathbf{P}^2 + \textit{const.}$ on any irrep.

- Is there a characterisation of particles which does not constrain them to be free?
- Is there a characterisation which embraces both Galilei and Minkowski spacetime (and more besides)?

Alternative spacetimes

- **Aristotelian spacetime.**
  Symmetries: t-translations, rotations.
  Generators: energy, ang. momentum.
  Casimirs: energy, ang. mom. magnitude.
  (What happened to momentum, position?)
- **Galilei spacetime.**
  Symmetries: t-translations, rotations, s-translations, Galilei boosts.
  Generators: energy, ang. momentum, momentum, Galilei "kicks".
  Casimirs: mass, "internal energy", spin.
  (Position is a derived quantity, using the "kicks" and mass)
- **Leibnizian spacetime.**
  Symmetries: t-translations, arbitrary (time-dependent) displacements.
  Generators: energy, ang. momentum, ang. momentum "kicks", . . . , momentum, Galilei "kicks", acceleration "kicks", . . .
  Casimirs: ??
  (An embarrassment of riches!)
- Consider also a generic GR spacetime, with no spacetime symmetries. . .

A lingering question

It may remain perfectly correct to say of any *free* particle that it is (represented by) an irreducible representation of the Galilei/Poincaré group. (We shall see later that this is true.)

E.g. it is true of the 1-particle sector of the Hilbert space describing any RQFT, or any asymptotic particle states.

If the Poincaré group (or any particular spacetime symmetry group) is the wrong one to characterise particles, why does it work in the case that the particles are free?

Outline

Field and particle pictures

Field picture		**Particle picture**
field and momentum configurations $(f,g) \in \Gamma$	$\xrightarrow{K}$	*1-particle wavefunctions* $\psi \in \mathcal{H}_J$
field operators $\Phi(f,g) = \hat{\pi}(f) - \hat{\phi}(g)$		*creation/annihilation operators* $a_J(\psi), a_J^\dagger(\psi)$
$[\hat{\phi}(g), \hat{\pi}(f)] = i\hbar(f,g)$ $[\hat{\phi}(g), \hat{\phi}(g')] = 0$ $[\hat{\pi}(f), \hat{\pi}(f')] = 0$		$[a_J(\phi), a_J^\dagger(\psi)] = \langle\phi,\psi\rangle$ $[a_J(\phi), a_J(\psi)] = 0$ $[a_J^\dagger(\phi), a_J^\dagger(\psi)] = 0$

$$
\begin{aligned}
a_J(K(f,g)) &= \frac{1}{2\hbar}\left[i\Phi(f,g) - \Phi(J(f,g))\right] \\
\Phi(f,g) &= -i\hbar\left[a_J(K(f,g)) - a_J^\dagger(K(f,g))\right] \\
KJ(f,g) &= iK(f,g)
\end{aligned}
$$

Finding the 1-particle structure

- Start with the classical phase space (a symplectic vector space) (Γ, Ω);
- Impose relativistic equations of motion (i.e. specify a Hamiltonian H);
- Define a complex structure $J : \Gamma \to \Gamma$—equivalently, define a real metric $\mu : \Gamma \times \Gamma \to \mathbb{R}$—that *unitarizes* the classical dynamics (equivalent because $\mu(\mathfrak{z}_1, \mathfrak{z}_2) = \frac{1}{2\hbar}\Omega(\mathfrak{z}_1, J\mathfrak{z}_2)$);
- Define an embedding $K : \Gamma \to \mathcal{H}$ into some Hilbert space
- Define a inner product using Ω, J and μ: $\langle K\mathfrak{z}_1, K\mathfrak{z}_2 \rangle = \mu(\mathfrak{z}_1, \mathfrak{z}_2) + \frac{i}{2\hbar}\Omega(\mathfrak{z}_1, \mathfrak{z}_2)$. Then $\overline{K[\Gamma]} = \mathcal{H}$.
- For any classical quantity Q, its quantum counterpart $\hat{Q}$ is such that $\langle K\mathfrak{z}, \hat{Q}K\mathfrak{z} \rangle = Q(\mathfrak{z})$;
- The generators of the irrep of the PG on $\mathcal{H}$ are the quantum counterparts of certain Noether charges on (Γ, Ω), given H.

Finding the 1-particle structure: an example

- Let $\Gamma = C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \ni (f, g) \equiv \mathfrak{z}$
- Let $H(f,g) = \int \mathrm{d}^3\mathbf{x}\ \frac{1}{2}g^2 + \frac{1}{2}f\left(-\nabla^2 + m^2\right) f \equiv \int \mathrm{d}^3\mathbf{x}\ \frac{1}{2}g^2 + \frac{1}{2}f\omega^2 f$, where $\omega := \sqrt{-\nabla^2 + m^2}$ (an *anti-local* operator)
- Define $J(f,g) := (-\omega^{-1}g, \omega f)$
- Define $K(f,g) := \frac{1}{\sqrt{2\hbar}}\left(\omega^{\frac{1}{2}} f + i\omega^{-\frac{1}{2}} g\right) \equiv \psi(\mathbf{x})$
- Define $\langle \psi_1, \psi_2 \rangle = \int \mathrm{d}^3\mathbf{x}\ \psi_1^*(\mathbf{x})\psi_2(\mathbf{x})$
- The classical Hamiltonian flow in Γ is mirrored by unitary evolution in $\mathcal{H}$.
- For any Noether charge of the classical theory, we can identify its quantum counterpart, e.g. we are led to $\mathbf{P} = -i\hbar\nabla$.

Outline

The received wisdom on interacting QFTs

- Insofar as legitimate particle talk is tied to a Fock representation, there are no particles whenever there are interactions (which is always). (Haag 1955; Greenberg & Licht 1963; Earman & Fraser 2006; Fraser 2008.)
- Particle talk is vindicated as approximate and emergent by Haag-Ruelle scattering theory (e.g. Wallace 2001) or LSZ theory (Bain 2000). The relevant particles are free, and so transform as irreps under the PG.

A tempting line

Wigner's identification seems to work well enough, provided that the particles in question are free. The particle picture in QFT is tied to Fock representations, which are appropriate only when the field is free $\Leftrightarrow$ *the particles are free. In the case of interacting QFT, the particle picture breaks down, exactly when Wigner's identification fails to apply. So Wigner's identification gets it right in both the free and the interacting cases.*

The truth about the field and particle pictures

- There are many (unitarily inequivalent) irreps of the Φ. Among them: the various Fock representations. But there are many more irreps besides.
- There is an obvious sense in which $a_J, a_J^\dagger$ are tied to a particular J, and so a particular 1-particle structure, and so a particular Fock rep. This is not true of Φ; we should think of Φ as only partially represented in each irrep.
- Nevertheless, the identities linking Φ with any $a_J, a_J^\dagger$ hold in *all* irreps. The CCRs expressed in terms of Φ are equivalent to the CCRs expressed in terms of $a_J, a_J^\dagger$, and they hold in all irreps of Φ.
- The $a_J, a_J^\dagger$ CCRs are sufficient to yield $N_J(\Pi)a_J^\dagger(\phi) = \left(a_J^\dagger(\phi) + 1\right) N_J(\Pi)$, where $\phi \in \text{ran}(\Pi)$, and $\text{sp}(N_J(\Pi)) = \mathbb{N}$, where $\dim(\Pi) < \infty$, in *every* irrep.
- Insofar as particulate structure is encapsulated in the $a_J, a_J^\dagger$ CCRs, *all* irreps of Φ have particulate structure (but not nec. Fock structure).
- (Compare with the good old 1D harmonic oscillator.)

"Thin" particles

- So far, this is an *extremely thin* notion of particle. It allows us to talk about particles associated with J in *any* irrep of the Φ, *for any J you like.*
- (I do not wish to question the supremacy of the field picture. Particles are emergent phenomena.)
- Sanity can be restored by placing further constraints on admissible particle representations.
- *Proposal:* J must be such that the resulting particles survive over a reasonable time-scale, during the evolution of the quantum field.
- 'Reasonable' is left deliberately vague.
- Particles are emergent from the field when/if the dynamics of the field (and our specification of reasonable lifetimes) make them a salient structures. (Dennettian/Wallacean wisdom.)
- (Note that we still need the thin notion of particle in order to provide the options among which the field dynamics, and our interests, choose.)

In defence of interacting particles

The regime of legitimate particle talk encompasses—but surpasses—just free particles. Consider:

- bound states;
- particles in weak fields (e.g. the electron in the relativistic Hydrogen atom);
- resonances.

If we want particle-talk to be legitimate in these cases, then we have to abandon Wigner's identification.

Outline

What is the alternative?

For inspiration I turn to:

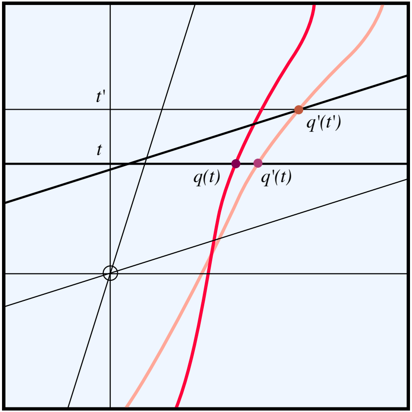
- classical Hamiltonian mechanics;
- non-relativistic quantum mechanics.

In both cases one has an algebra of quantities that:

- are generated by dynamically & geometrically innocent primitives: namely, $\mathbf{Q}$ and $\mathbf{P}$;
- are adequate to capture (e.g.) generators of the Poincaré group (as we'll see).

Wouldn't it be nice to find $\mathbf{Q}, \mathbf{P}$ for relativistic particles too?

LTs in classical Hamiltonian theory



(Due to Currie, 1963.)

To first order:

$$\begin{aligned} q'(t) &\approx q'(t') - (t' - t)\dot{q}'(t') \\ &\approx \gamma(v)(q(t) + vt) \\ &\qquad - [(\gamma(v) - 1)t + vq(t)]\,\dot{q}'(t') \\ &\approx q(t) + v\,[t - q(t)\dot{q}(t)] \end{aligned}$$

$$\lim_{v \to 0} \frac{\mathrm{d}q'}{\mathrm{d}v} = t - q\dot{q}$$

Also:

$$\lim_{v \to 0} \frac{\mathrm{d}p'}{\mathrm{d}v} = H - q\dot{p}$$

LTs in classical Hamiltonian theory, contd.

So let K be the Hamiltonian-like generator of this transformation:

$$\lim_{v \to 0} \frac{\mathrm{d}q'}{\mathrm{d}v} = t - q\dot{q} = t - q\frac{\partial H}{\partial p} = \frac{\partial K}{\partial p} = \{q, K\}$$

Similarly,

$$\lim_{v \to 0} \frac{\mathrm{d}p'}{\mathrm{d}v} = H - q\dot{p} = H + q\frac{\partial H}{\partial q} = -\frac{\partial K}{\partial q} = \{p, K\}$$

These equations for K are integrable:

$$K = tp - qH$$

- In 3 spatial dimensions: $\mathbf{K} = t\mathbf{p} - \mathbf{q}H$. In QM, $\mathbf{K} = t\mathbf{P} - \mathbf{Q} \circ H$.
- (This expression is "dynamically innocent" insofar as H is treated as a super-variable.)

The Foldy form

Plan of action: look at irreps of the PG, postulate sensible re-expressions of the generators in terms of alternative quantities which are plausibly dynamically & geometrically innocent.

Hope is provided by Foldy (1956), who showed that the quantities

$$\begin{aligned} H &= \Lambda\sqrt{\mathbf{P}^2 + m^2}\ ; \\ \mathbf{J} &= \mathbf{Q} \times \mathbf{P} + \mathbf{S}\ ; \\ \mathbf{K} &= t\mathbf{P} - \mathbf{Q} \circ H + \frac{\Lambda \mathbf{S} \times \mathbf{P}}{H + m}\ ; \end{aligned}$$

on $\mathcal{L}_2(\mathbb{R}^3) \otimes \mathbb{C}^{2(2s+1)}$ with $\mathbf{Q}, \mathbf{P}$ the usual operators on $\mathcal{L}_2(\mathbb{R}^3)$, S_i familiar spin operators on $\mathbb{C}^{2s+1}$, and $\Lambda := H(H^2)^{-\frac{1}{2}}$, constitute an (m^2, s) irrep of the *full* Poincaré group $IO(3,1)$.

(Worries about negative energy can be remedied by opting for the suitable complex structure $i \mapsto -i$ on negative-frequency states; see Baez *et al* 1992.)

Outline

Spin-orbit decompositions: definitions

Let a *spin-orbit decomposition* of the Poincaré generators of any (m^2, s) irrep be self-adjoint operators $\mathbf{Q}, \mathbf{P}, \mathbf{S}$ such that:

- $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L} = \mathbf{Q} \times \mathbf{P}$;
- $\mathbf{K} = \mathbf{M} + \mathbf{N}$, where $\mathbf{M} = t\mathbf{P} - \mathbf{Q} \circ H$ and $\mathbf{N}$ is a function of $\mathbf{S}$ and $\mathbf{P}$ alone;
- $[\mathbf{Q}, \Lambda] = \mathbf{0}$, where $\Lambda := H(H^2)^{-\frac{1}{2}}$.

Let a spin-orbit decomposition be *canonical* iff $\mathbf{Q}$ and $\mathbf{P}$ obey the usual Heisenberg algebra, $\mathbf{S}$ obeys the spin algebra, and $[Q_i, S_j] = [P_i, S_j] = 0$.

Let a spin-orbit decomposition be *proper* iff $H, \mathbf{P}, \mathbf{L}, \mathbf{M}$ generate a (perhaps reducible) $(m^2, 0)$ rep of the Poincaré group.

Spin-orbit decompositions: results

- Any spin-orbit decomposition is canonical iff it is proper.
- Any (m^2, s) irrep has a unique canonical spin-orbit decomposition. (Essentially proved by Jordan 1980.) (Ditto for proper spin-orbit decomposition.)
- In this decomposition, $\mathbf{Q}$ is the Newton-Wigner (1949) position operator and H and $\mathbf{N}$ both take Foldy form.
- $\mathbf{Q}$ transforms covariantly under $\mathbf{M}$ (but typically not $\mathbf{K}$).
- With $\mathbf{J}, \mathbf{K}$ defined as before, $\mathbf{Q}, \mathbf{P}, \mathbf{S}$ provides an (m^2, s) irrep iff $H^2 = \mathbf{P}^2 + m^2$. But of course $\mathbf{Q}, \mathbf{P}$ and $\mathbf{S}$ are defined independently of H.
- The Pauli-Lubanski axial vector W^μ is definable in terms of $\mathbf{S}, H, \mathbf{P}$ alone (so it is independent of $\mathbf{Q}$), and is interpretable as a four-vector lying in the simultaneity plane of the particle's rest frame.

Poincaré meets Heisenberg

	H	P_j	J_j	K_j
H	0	$-\dot{P}_j$	$-\dot{J}_j$	$\dot{Q}_j \circ H - t\dot{P}_j$
P_i	$\dot{P}_i$	0	$i\hbar\epsilon_{ijk}P_k$	$\delta_{ij}H + \dot{P}_i \circ Q_j$
J_i	$\dot{J}_i$	$\epsilon_{ijk}P_k$	$\epsilon_{ijk}J_k$	$\epsilon_{ijk}K_k - \dot{J}_i \circ Q_j$
K_i	$t\dot{P}_i - \dot{Q}_i \circ H$	$-\delta_{ij}H - Q_i \circ \dot{P}_j$	$\epsilon_{ijk}K_k - Q_i \circ \dot{J}_j$	$t(Q_i \circ \dot{P}_j - Q_j \circ \dot{P}_i)$ $-(Q_i \circ \dot{Q}_j - Q_j \circ \dot{Q}_i) \circ H$

These commutation relations realise the Poincaré algebra iff:

$$\dot{\mathbf{P}} = -\frac{i}{\hbar}[\mathbf{P}, H] = 0 \quad \text{and} \quad \dot{\mathbf{Q}} = -\frac{i}{\hbar}[\mathbf{Q}, H] = \mathbf{P}H^{-1}$$

which determines that

$$H^2 = \mathbf{P}^2 + const.$$

(Aside: Particle localisation—a fly in the ointment?)

- Insofar as legitimate particle talk is tied to some relativistically acceptable scheme for exact localisation, there are no particles. In particular, the Newton-Wigner scheme is not relativistically acceptable. (Hegerfeldt 1974, 1994; Malament 1996; Halvorson & Clifton 2002.)
- If particles are taken as emergent field phenomena, then there is a relativistically acceptable scheme for *approximate* localisation (Wallace 2001.)
  - Approximate, because of a mismatch between localisation in the 1-particle structure and localisation in terms of EVs of field quantities, originating with the complex structure:

$$\langle 1_\psi|\hat{\phi}(f)^2|1_\psi\rangle - \langle 0|\hat{\phi}(f)^2|0\rangle = \hbar|\langle f, \omega^{-\frac{1}{2}}\psi\rangle|^2 \ .$$

  - This mismatch engenders a peaceful coexistence between Hegerfeldt spreading in the 1PS and microcausality. (And provides an answer to Halvorson & Clifton 2002 on behalf of Fleming & Butterfield 1999.)

Upshots

- A dynamically innocent characterisation of particles is available, along the same group-representational lines as Wigner's identification. The characterisation offered by the Heisenberg group $(\mathbf{Q}, \mathbf{P})$ and the spin algebra $(\mathbf{S})$.
- Mass as dynamical in origin, not intrinsic or essential. Spin remains intrinsic.
- One can incorporate both Poincaré and Galilei transformations, since the generators of both may be defined in terms of $\mathbf{Q}, \mathbf{P}$ and $\mathbf{S}$:

$$\mathbf{G} = t\mathbf{P} - M\mathbf{Q} \; ; \qquad H_{Gal} = \frac{1}{2M}\mathbf{P}^2 + const.$$

These, with $\mathbf{P}$ and $\mathbf{J}$ as usual, obey the Galilei algebra.

Further upshots

- The characterisation of particles in terms of the Heisenberg group can be extended even to discrete space or spacetime: in this case there is no Heisenberg group, but there is a (discrete) Weyl algebra. (See Santhanam & Tekumalla 1976.) *This is very handy for fully regularised QFTs.*
- Assuming Hamiltonian determinism, we can see why the Poincaré group or Galilei group get it approximately right: these groups are characterised by changes in position and momentum, which specify the initial value data sufficient for determining a unique trajectory.

Outline

Other subtleties

What happens when the mass goes to zero?

- Investigated by Lomont & Moses (1962), Moses (1968) and Jordan (1982).
- Take some (m^2, s) irrep of $ISO(3,1)^\uparrow$, and express $\mathbf{Q}, \mathbf{P}, \mathbf{S}$ in a representation that diagonalises helicity.
- Send $m^2 \to 0$. $H, \mathbf{P}, \mathbf{J}$ and $\mathbf{K}$ all diagonalise. If $s \neq 0$, the representation becomes a *reducible* $(0, s)$ rep of $ISO(3,1)^\uparrow$.
- $\mathbf{Q}$ and $\mathbf{S}$ fail to diagonalise.
- If $s = \frac{1}{2}$, the rep remains an *ir*rep of $IO(3,1)^\uparrow$.
- Otherwise, $\mathbf{Q}$ and $\mathbf{S}$ fail to be defined.

Summary

So, is a particle an irreducible representation of the Poincaré group?

- Sometimes. Specifically: when the particle is free.
- But even then, it is wrong to think of the Poincaré group (or any spacetime symmetry group) as *characterising* any particle.

What is a particle then?

- If $s = 0$ or $\frac{1}{2}$, or $m^2 > 0$ an irreducible representation of the familiar Heisenberg group/Weyl algebra associated with $\Gamma = \mathbb{R}^6$.
- Otherwise, we may need another characterisation.

Thank you!