The Quantization of Linear Dynamical Systems I: Finite Systems

Spoken by JNB, but: taught by Caulton (adam.caulton@balliol.ox.ac.uk), Wikipedia ... ;
Philosophical Aspects of QFT, Lent 2020: version of 10 February 2020

This document, and its succcessor on the Quantization of Linear Dynamical Systems with Infinitely many degrees fo freedom, expound a rigorous quantization procedure developed by Irving Segal and others in the 1960s. This means we postpone to the second half of term, coverage of algebraic quantum theory; which will include e.g. inequivalent representations, 'getting out of Fock space', Haag's theorem etc. (cf. eg Emch 1972). But the present material:

(i) gives a strong grip on the first (forbiddingly concise!) third of Wald 1994, which is the basis for the rest of that book on QFT in curved spacetime and thus e.g. the Unruh effect (an essay!);

(ii) is of intrinsic interest... though please be warned that here you will find: no Lagrangian, no path integrals, no renormalization, no gauge theory, no curved spacetime, no gravitation; indeed, no interactions, and overall, not much physics ... we will focus on the harmonic oscillator (!),the free KG field and spin-chains (and without putting a Hamiltonian on the chain...). Nor will you find much straight-up philosophy ... but perhaps the light here shed on field/wave vs.particle counts as philosophy, since wave vs.particle is, like continuum vs. discrete, a perennial dichotomy of *natural philosophy*...

In this document, we consider only finitely many degrees of freedom, and lead up to the Stone-von Neumann Theorem, which essentially guarantees that the quantization of point particles in $\mathbb{R}^n$ is unique. We begin by introducing the Weyl form of the CCRs; and posing the quest for its representations (Section 1). Then we present the complexification and realification of vector spaces, complex structures etc. (Section 2); and symplectic vector spaces and manifolds (Section 3). Then we present linear systems, both classical and quantum; and thus the harmonic oscillator (Section 4). With all this in hand, we can then see the task of quantization as "unitarizing" a Hamiltonian evolution in a symplectic space so as to give an evolution in a complex Hilbert space. This gives the idea of a *one particle structure*, both in general and for the harmonic oscillator as an example (Section 5). The key to successful quantization, which see at work in the harmonic oscillator example, turns out to be the *two out of three property* of the unitary group: which concerns its relation to certain orthogonal and symplectic groups (Section 6). Then we treat the case of finitely many harmonic oscillators, and so the occupation number representation: which can be described in a "Fock-space way" (Section 7). Finally, we state (i) the Stone-von Neumann Theorem; and (ii) an analogous theorem (the Jordan-Wigner theorem) about the uniqueness of the representation of the CARs (as against CCRs) of a *finite* system, such as a spin chain (Section 8).

Mottoes:

Let us try to introduce a quantum Poisson Bracket which shall be the analogue of the classical one....we are thus led to the following definition for the quantum Poisson Bracket of any two variables u and v: $uv - vu = i\hbar[u, v]$. Dirac (1930/1958, Section 21)

There is thus a complete harmony between the wave and light-quantum descriptions of the interaction. (Dirac, 1927, p. 245).

First quantization is a mystery, but second quantization is a functor. (E. Nelson).

Probably all these connections would have been clarified long ago, if quantum physicists had not been hampered by a prejudice in favor of complex and against real numbers. (Freeman Dyson)

The life of a theoretical physicist consists of solving harmonic oscillator at ever higher levels of abstraction. (S. Coleman)

Contents

1 Canonical quantization introduced

1.1 Commutation relations: from Heisenberg to Weyl

The idea of *canonical quantization* is familiar from elementary quantum mechanics: to "promote" the classical Poisson bracket relations

$$\{q^i, q^j\} = \{p_i, p_j\} = 0; \qquad \{q^i, p_j\} = \delta^i_j, \tag{1}$$

where $i, j \in \{1, 2, \dots n\}$, to the *Heisenberg canonical commutation relations* (CCRs)

$$[Q^i, Q^j] = [P_i, P_j] = 0; \qquad [Q^i, P_j] = i\hbar\delta^i_j \mathbb{1}; \tag{2}$$

(we will usually set $\hbar := 1$). This Poisson bracket-commutator correspondence originated with Dirac (cf. his *Principles of Quantum Mechanics* 1958, Section 21f.) The standard representation of eq. (2) is the familiar Schroedinger representation: namely, for n configurational degrees of freedom, e.g. a spinless particle in Euclidean n-space, or n such particles on a line:

$$Q^i\psi = q_i\psi, \qquad P_j\psi = -\frac{ih}{2\pi}\frac{\partial\psi}{\partial q_j} \qquad \text{for } \psi \in L^2(\mathrm{I\!R}^n, d\mathbf{q}). \tag{3}$$

This prompts four main topics. They are of increasing scope, and we will consider only the first.

(a): To examine canonical quantization as just described for position and momentum in $\mathrm{I\!R}^n$. The big positive result here is the Stone von Neumann theorem, stating (roughly) that for $\mathrm{I\!R}^n$ as the configuration space, the Schroedinger representation of (2) is unique up to unitary equivalence. Cf Section 8. But so as to set the scene for quantum field theory, and more generally so as to get materials useful for contexts other than $\mathrm{I\!R}^n$, we will lead up to this slowly. This will mean expounding some ideas of *Segal quantization*, which is the most straightforward generalization of the above ideas. In short: it replaces $\mathrm{I\!R}^n$ as the classical configuration space, by an arbitrary n-dimensional manifold.

(b): To extend quantization to other quantities, in particular functions (polynomial, or even "arbitrary", functions) of position and momentum.

(c): To consider other methods of quantization.

(d) To pursue the *pure mathematical* interest of quantization. For a glimpse of this, cf. Folland (2008, p. 49; and Vogan 2005, cited there). In short: the interest lies in how it helps one find all the irreducible unitary representations of a connected Lie group G: i.e. in physical language, finding all quantum systems in which G acts irreducibly as a symmetry group. The corresponding classical problem is to find all symplectic manifolds on which G acts transitively as a group of canonical transformations (symplectomorphisms), i.e. all symplectic homogeneous G-spaces. But this classical problem is "under good control". For the orbits of the co-adjoint action of G on $\mathfrak{g}^*$ are symplectic homogeneous G-spaces; and furthermore, all symplectic homogeneous G-spaces can be, more or less, built from orbits of such co-adjoint action. (Here, "more or less" signals issues about central extensions and covering spaces). Thus a "good" quantization procedure for such spaces is likely to be illuminating for the task of finding all the irreducible unitary representations of G.

Of course, we foreswear (d); and for the most part, we foreswear (b) and (c). For an introduction to both, and of course (a), we recommend:

(i): N Landsman, Between Classical and quantum, especially Section 3; in J Butterfield and J Earman eds, *Handbook of Philosophy of Physics* (2006) and: quant-ph:0506082; and for more details:

(ii): S Ali and M Englis, Quantization methods: a guide for physicists and analysts, *Reviews in Mathematical Physics* 2005, math-ph: 0405065.

In particular, as to (b): Ali and Englis Section 1 review the obstructions confronting quantization of (even just a "handful" of polynomial) functions of position and momentum. These obstructions concern ambiguities of operator-ordering. That is: natural general constraints on the quantization map Q ("adding a hat") that sends a classical (real-scalar) quantity $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ to a quantum quantity, i.e. to a self adjoint operator $Q_f : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$, lead to *contradictions*. This topic originates in papers by Groenewold and van Hove. Recent developments include: Gotay et al. Obstructions in quantization theory, *Journal of Nonlinear Science*, volume 6, p. 469-498, 1996; and Gotay, On the Groenewold-van Hove Problem, *Journal of Mathematical Physics* 1999.

As to (c): Ali and Englis review (Section 3f.) geometric quantization, deformation quantization etc. But even their Section 2 gives details of e.g. the inequivalent quantizations involved in the Aharonov Bohm effect.

But the four topics are of course closely related. For example, these obstructions mean that a main motivation to pursue (c)'s other methods of quantization is to extend quantization to as many quantities as possible.

For us, concentrating on (a): the main point about (b), i.e. the obstructions, will be that (cf. Wald 1994, Section 2.2 , pp. 17-18): Segal quantization "works" for:

(i) a classical configuration space that is an arbitrary n-dimensional manifold M (so that classical quantities are real functions of the cotangent bundle T^*M); *provided that*

(ii) we restrict consideration to quantities that are at most linear in the momenta (i.e. the momenta canonically conjugate to arbitrary configurational coordinates q on M).

Here, the word "works" means that the quantization map Q maps Poisson brackets into commutators, divided by $i\hbar$. (In more formal jargon: "Q respects Lie algebra structure"). That is: Q obeys, for classical quantities $f, g : T^*M \rightarrow \mathbb{R}$ that are appropriately restricted by condition (ii) above:

$$[Q(f), Q(g)] = i\hbar\, Q(\{f, g\}) \tag{4}$$

In this sense, Segal quantization is a good framework for the quantization of finite-dimensional systems.

And Segal quantization has other merits. We will also see that for linear classical systems, it "respects" the dynamics. That is: the Segal quantization of the classical Hamiltonian (which is essentially like that of a harmonic oscillator: "$p^2 + q^2$") is the "correct" quantum Hamiltonian. Besides, we will eventually see that it works for (some!) quantum field theories. Specifically, it works for the quantization of the free bose field (e.g. De Faria and De Melo, Section 6.3). Furthermore, it does this in a manner that generalizes readily to constructing quantum field theories on *curved* spacetimes (Wald 1994, p. 31 and Section 3.2).

So much by way of preamble. For our main topic, i.e. (a) above, the first job is to pass from the Heisenberg CCRs to the *Weyl form of the CCRs*. The point here is that since the classical position and momentum quantities, for a phase space $\mathbb{R}^{2n}$, are unbounded, we expect the quantum position and momentum Q^i, P_j to also be unbounded, indeed to have all of $\mathbb{R}$ as their spectra—so that, if they are to be self-adjoint, they cannot be defined on all of $L^2(\mathbb{R}^n)$.

Indeed, setting aside the physical desideratum that the spectra should be unbounded: there is a simple theorem that if two *bounded* self-adjoint operators Q, P have a commutator that is proportional to the identity, they must *commute*. That is: If $[Q, P] = \alpha I$ for some $\alpha \in \mathbb{C}$, then $\alpha = 0$. (De Faria and De Melo, Lemma 2.11; Jauch 1968, p. 205, Problem 4).

In short: we face issues of domains. We remedy this by formulating to the *Weyl form* of the CCRs. These govern unitary exponentiations of linear combinations of the position, and similarly, of the momentum operators.

Thus we define, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$U(\mathbf{a}) := e^{-i\mathbf{a}.\mathbf{P}/\hbar} \quad ; \qquad V(\mathbf{b}) := e^{-i\mathbf{b}.\mathbf{Q}/\hbar}; \tag{5}$$

Since the Us and Vs are both families of unitaries, their spectra are bounded, and are defined everywhere on $L^2(\mathbb{R}^n)$. In the Schroedinger representation, we have

$$(U(\mathbf{a})\psi)(\mathbf{x}) = \psi(\mathbf{x} - \mathbf{a}) \;\; ; \;\; (V(\mathbf{b})\psi)(\mathbf{x}) = e^{-i\mathbf{b}.\mathbf{x}/\hbar}\psi(\mathbf{x}) \tag{6}$$

so that U represents translations in space, and V represents translations in momentum-space.

We have, of course, commutation for each of position and momentum, alone:

$$U(\mathbf{a})U(\mathbf{b}) = U(\mathbf{b})U(\mathbf{a}) = U(\mathbf{a} + \mathbf{b}) \; ; \; V(\mathbf{a})V(\mathbf{b}) = V(\mathbf{b})V(\mathbf{a}) = V(\mathbf{a} + \mathbf{b}) \tag{7}$$

To deduce the commutation relations of U and V operators, we need the *Campbell-Baker-Hausdorff formula* for products of exponentials of non-commuting operators. Given a self-adjoint operator A, we say that a vector $\psi \in \mathcal{H}$ is *analytic* if for all n, $A^n(\psi)$ is defined, and so is $e^A\psi$. Then the version of the Campbell-Baker-Hausdorff formula which is appropriate here (De Faria and De Melo, Lemma 2.12) says that if:

(i) A, B and $A + B$ have a common dense domain D of analytic vectors, and

(ii) [A,B] commutes with A and with B:

then in D:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \equiv e^{A+B} e^{\frac{1}{2}[A,B]} \tag{8}$$

To apply (8) to (5), we set $A := -i\mathbf{a}.\mathbf{P}/\hbar$ and $B := -i\mathbf{b}.\mathbf{Q}/\hbar$, to deduce that

$$U(\mathbf{a})V(\mathbf{b}) = \exp(\frac{1}{2}i(\mathbf{a} \cdot \mathbf{b})/\hbar). \exp(-i(\mathbf{a}.\mathbf{P}/\hbar + \mathbf{b}.\mathbf{Q}/\hbar)) \; ; \tag{9}$$

and *mutatis mutandis*, we set $A := -i\mathbf{b}.\mathbf{Q}/\hbar$ and $B := -i\mathbf{a}.\mathbf{P}/\hbar$, to deduce that

$$V(\mathbf{b})U(\mathbf{a}) = \exp(-\frac{1}{2}i(\mathbf{a} \cdot \mathbf{b})/\hbar). \exp(-i(\mathbf{a}.\mathbf{P}/\hbar + \mathbf{b}.\mathbf{Q}/\hbar)). \tag{10}$$

Combining these immediately gives the *Weyl commutation relations*:[1]

$$U(\mathbf{a})V(\mathbf{b}) = e^{i\mathbf{a}.\mathbf{b}/\hbar} V(\mathbf{b})U(\mathbf{a}). \tag{11}$$

[1]Beware: (i) many authors 'flip' the notation of U and V, so that V represents translations in space; and (ii) some authors (even rigorous ones e.g. Prugovecki 1981, Chapter IV, Sections 6.2, 6.4!) also put the $\hbar$ in the numerator of the exponent, so that the exponent is in dire danger of having dimension action-squared! Besides, (iii): various texts also get the sign of the exponent in (11) wrong. (See later for discussion of different choices of sign in the two definitions of (5).) I am following S. Summers (2001: in *John von Neumann and the Foundations of quantum mechanics*, ed. M. Redei and M. Stoeltzner). Summers puts the $\hbar$ in the denominator of the exponent, is perfectionist about signs; and his use of U for translation in space, is like Weyl himself (1932, Chapter IV, Section 14, building on Chapter II, Section 11): this last text being no doubt correct, but—with all due respect!—incomprehensible.

1.2 The Weyl algebra

So from now on, we take as our CCRs, not the Heisenberg form (2), but (11) together with the trivial commutations of Us and Vs alone i.e. (7).

We have so far built the Us and Vs concretely from given $\mathbf{Q}, \mathbf{P}$. But in the usual tradition of physics, we can:

(i) consider an abstract algebra of Us and Vs subject to the relations (11) and (7); any such algebra is called *the Weyl algebra*; and then

(ii) try to classify the representations of this algebra, especially the unitary representations on some Hilbert space.

As already announced at the start of Section 1.1, the main result about (ii), for finite-dimensional systems, will be the Stone-von Neumann uniqueness theorem. But as that discussion also suggested: the Weyl algebra, and Segal quantization, will also be centre-stage for quantizing fileds (including on curved spacetime) and for the pure mathematical topic (d) of Section 1.1.

Now, we first make two comments about this endeavour (in order of increasing importance for us); and then develop a more abstract formulation of the Weyl relations, which will be central in all that follows.

(1): *The relation between the Heisenberg and Weyl forms*:– The Weyl form of the CCRs implies the Heisenberg form, and so a representation of the Weyl form is also a representation of the Heisenberg form. But uniqueness (up to unitary equivalence) of a representation of the Weyl form does not imply uniqueness of the implied representation of the Heisenberg form. The reason lies in the simple theorem above, that two *bounded* self-adjoint operators Q, P cannot obey the Heisenberg form. In fact, the Heisenberg form does not imply the Weyl form, even if Q and P are essentially self-adjoint on their respective domains; though conditions can be added that make the implication go through (e.g. Dixmier's condition (1958: in French!), discussed by Jauch (1968, p. 204-205)).

(2): *Allowing for projective unitary representations*:— Of course, the quantum state is *non-redundantly* represented by a *ray* rather than a unit vector. This motivates considering *projective* representations of groups, rather than "true" representations. Such representations allow a phase to occur in equations stating the group composition law for the representing operators. Indeed, we see this even for elementary abelian groups, like the phase-space translation groups we are concerned with: cf. the phase in (11), and in (54) below.

Equation (11) can be given a more abstract formulation, which both:

(i) brings out the role being played by the symplectic structure in the underlying framework of Hamiltonian mechanics, and

(ii) underpins how Segal quantization succeeds in quantizing linear classical systems, both finite-dimensional and infinite-dimensional.

Setting $z := (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$, we define the family of operators

$$W(z) := e^{\frac{1}{2}i\mathbf{a}.\mathbf{b}}U(\mathbf{a})V(\mathbf{b}). \tag{12}$$

Then the Weyl form of the CCRs, i.e. (11) and and (7), are equivalent to the following, which is thus also called the *Weyl algebra*: for all $z, z_1, z_2 \in \mathbb{R}^{2n}$,

$$\begin{aligned} W(z_1)W(z_2) &= e^{\frac{1}{2}i\Omega(z_1,z_2)}W(z_1+z_2); \\ W^\dagger(z) &= W(-z); \end{aligned} \tag{13}$$

where Ω is the *symplectic product*:

$$\Omega(z_1, z_2) := \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2, \tag{14}$$

The symplectic meaning of Ω will be explained in Section 3. But as a preliminary to that, we spell out some elementary ideas and results about complexification and complex structures: which are often treated very concisely if at all, (e.g. Wald 1994, p. 190).

2 Complexification, complex structures—and all that

There is a circle of ideas which can be traversed starting from almost any point... We begin with complexification, then describe complex structures, then complex conjugation of spaces, and then the compatibility of a complex structure with a bilinear form, such as an inner product or symplectic form.

2.1 Complexification

The *complexification* $V^{\mathbb{C}}$ of a real vector space V is defined as the tensor product of V with the complex numbers $\mathbb{C}$

$$V^{\mathbb{C}} := V \otimes \mathbb{C} \ . \tag{15}$$

Here we think of $\mathbb{C}$ as a copy of $\mathbb{R}^2$, with a basis $\{(1,0),(0,i)\}$. So far, this is just a real vector space. Every vector in $V^{\mathbb{C}}$ can be written uniquely as

$$v = v_1 \otimes 1 + v_2 \otimes i \tag{16}$$

and the (real) dimension of $V^{\mathbb{C}}$ is twice the dimension of V. But we make it into a complex vector space, by defining complex scalar multiplication by

$$\alpha(v \otimes \beta) = v \otimes (\alpha\beta) \ \text{ for all } v \in V \text{ and } \alpha, \beta \in \mathbb{C} \ ; \tag{17}$$

where we also of course require scalar multiplication to distribute over addition, i.e. we 'extend by linearity':

$$\alpha(v \otimes \beta + u \otimes \gamma) := \alpha(v \otimes \beta) + \alpha(u \otimes \gamma) \equiv v \otimes (\alpha\beta) + u \otimes (\alpha\gamma) \ . \tag{18}$$

Since every vector in $V^{\mathbb{C}}$ can be written uniquely as $v = v_1 \otimes 1 + v_2 \otimes i$, it is usual to drop the tensor product symbol and just write

$$v = v_1 + iv_2. \tag{19}$$

One then checks that the definition eq. 15 implies that the complex scalar multiplication defined by eq. 17, can be written in the usual-looking form. Namely: for a complex number $\alpha = a + ib$ with $a, b \in \mathbb{R}$

$$(a + ib)(v_1 + iv_2) = (av_1 - bv_2) + i(bv_1 + av_2). \tag{20}$$

So we regard $V^{\mathbb{C}}$ as the direct sum of two copies of V, equipped with a complex scalar multiplication defined by eq. 20 .

There is a natural embedding of V in to $V^{\mathbb{C}}$ given by

$$v \mapsto v \otimes 1 \ . \tag{21}$$

V may thus be regarded as a *real* subspace of $V^{\mathbb{C}}$. If V has a basis $\{e_i\}$ over $\mathbb{R}$ then a correponding basis for $V^{\mathbb{C}}$ is given by $\{e_i \otimes 1\}$ over $\mathbb{C}$. The *complex* dimension of $V^{\mathbb{C}}$ is therefore equal to the *real* dimension of V:

$$\dim_{\mathbb{C}} V^{\mathbb{C}} = \dim_{\mathbb{R}} V. \tag{22}$$

Alternatively: We can *define* the complexification of V as the direct sum

$$V^{\mathbb{C}} := V \oplus V \tag{23}$$

equipped with a *complex structure* (cf. below for details) given by the operator $J : V^{\mathbb{C}} \to V^{\mathbb{C}}$, where J is defined by

$$J(v, w) := (-w, v) \ . \tag{24}$$

Here J encodes multiplication by i in the sense that setting $a = 0, b = 1$ in eq. 20 yields

$$i(v_1 + iv_2) = -v_2 + iv_1 = -v_2 \otimes 1 + v_1 \otimes i \tag{25}$$

where the last expression on the right is in the notation of eq. 16.

Let $\dim_{\mathbb{R}} V = n$. Then in matrix form, J is given by a $2n \times 2n$ matrix J by

$$J := \begin{pmatrix} \mathbf{0} & -\mathbf{1_V} \\ \mathbf{1_V} & \mathbf{0} \end{pmatrix} \ . \tag{26}$$

where $-\mathbf{1_V}$ is the identity map on V. Thus $V^{\mathbb{C}}$ can be written as $V \oplus JV$ or as $V \oplus iV$, so as (i) to avoid the tensor product notation, and (ii) to signal the fact that the direct sum in eq. 23 is endowed with J. J swaps the summands in the sense that $J(v, 0) = (0, v)$.

Examples: (i) the complexification of $\mathbb{R}^n$ is $\mathbb{C}^n$; (ii) if V is the $m \times n$ matrices with real entries, then $V^{\mathbb{C}}$ is the $m \times n$ matrices with complex entries.

Again we have (cf. eq. 22): the *complex* dimension of $V^{\mathbb{C}}$ is equal to the *real* dimension of V, which is half the *real* dimension of $V \oplus V$:

$$\dim_{\mathbb{C}} V^{\mathbb{C}} = \dim_{\mathbb{R}} V = \frac{1}{2} \dim_{\mathbb{R}} (V \oplus V) \ . \tag{27}$$

2.2 Complex structures

2.2.A Basics:— A *complex structure* on a real vector space V is an automorphism J of V that squares to minus the identity map, - I. That is: $J^2 = -1 \equiv$ - I. Such a structure on V allows one to define multiplication by complex scalars in a canonical fashion so as to regard V as a complex vector space. Namely:

$$(x + iy)v := xv + yJ(v) \ \text{ for all } v \in V \text{ and } x, y \in \mathbb{R} \ ; \tag{28}$$

which (check!) makes V into a complex vector space, denoted V_J.

If V is any real vector space, there is a canonical complex structure J on the direct sum $V \oplus V$: namely, the complex structure on the complexification $V^{\mathbb{C}}$ of V, i.e. on the tensor product $V \otimes \mathbb{C}$, written as $V \oplus JV$ or as $V \oplus iV$. That is, J is given by $J(v, w) := (-w, v)$, i.e. by eq. 24, ; and the matrix form of J is as in eq. 26. In this notation for complexification—i.e. the notation, $V \oplus JV$ or $V \oplus iV$—we can write: $V \oplus JV = (V \oplus V)_J$ or similarly $V \oplus iV = (V \oplus V)_J$.

One can go in the other direction. Any complex vector space W is also a real vector space, with the same vector addition and real scalar multiplication. On this underlying real vector space, one defines a complex structure J by $J(w) := iw$ for all $w \in W$; where the right-hand-side is given us by W being a complex vector space. With this complex structure defined, we of course get back the original complex vector space W.

In fact, if V_J has complex dimension n, then V must have real dimension $2n$. That is, a finite-dimensional real space V admits a complex structure only if it is even-dimensional. If $\{v_1, ..., v_m\}$ is a basis of the complex vector space V_J, then $\{v_1, J(v_1)..., v_m, J(v_m)\}$ is a basis of the underlying real vector space V.

Every even-dimensional real vector space V admits a complex structure. Indeed, many. For any basis $\{e_1, e_2, \ldots, e_{2n}\}$ of V can be divided in to n pairs, say $\{e_1, e_2\}, \ldots, \{e_{2n-1}, e_{2n}\}$, and then one can define J as the 'swap with a minus' on each such pair, i.e. $J(e_1) := e_2, J(e_2) := -e_1, \ldots, J(e_{2n-1}) := e_{2n}, J(e_{2n}) := -e_{2n-1}$, and then one extends by linearity to all of V. So $J^2 = -1$.

Suppose that we are given a real linear transformation $A : V \to V$ on a real vector space V, and that V admits a complex structure J. Then A defines a complex linear transformation of the complex space V_J if and only if A commutes with J, i.e. if and only if $AJ = JA$: (trivial check, cf. eq. 28).

Likewise, a real subspace U of V is a complex subspace of V_J (i.e. is closed under complex-linear combinations) if and only if J preserves U, i.e. if and only if $J(U) < U$; (trivial check).

2.2.B: Basic example:— Obviously, the main example of a complex structure is the structure on $\mathbb{R}^{2n}$ coming from the complex structure on $\mathbb{C}^n$. That is, the complex n-dimensional space $\mathbb{C}^n$ is also a real $2n$-dimensional space. Here, one uses the same vector addition and real scalar multiplication: while multiplication by the complex number i is not only a *complex* linear transform of the space, thought of as a complex vector space, but also a *real* linear transform of the space, thought of as a real vector space. This is just because scalar multiplication by i:

(a) commutes with scalar multiplication by real numbers, i.e. $i(\lambda v) = (i\lambda)v = (\lambda i)v = \lambda(iv)$, and

(b) distributes across vector addition.

As a complex $n \times n$ matrix, this complex structure is simply the diagonal matrix with i on the diagonal. The corresponding real $2n \times 2n$ matrix is denoted J. What this matrix J looks like will depend on how we order the basis: cf. eq. 30 and 31 in (1) and (2) below.

Again, there is the general equation that counts dimensions, with $V^{\mathbb{C}} = (V \oplus V)_J$ (cf. eq. 27):

$$\frac{1}{2}\dim_{\mathbb{R}}(V \oplus V)_J = \dim_{\mathbb{C}}(V \oplus V)_J = \dim_{\mathbb{R}} V = \frac{1}{2}\dim_{\mathbb{R}}(V \oplus V) \ . \tag{29}$$

And in this example, with $V = \mathbb{R}^n$: these numbers are all n.

2.2.C: The "look" of J:— Suppose given a complex vector space, of complex dimension n, and a basis $\{e_1, e_2, \ldots, e_n\}$. This set, together with these vectors multiplied by i, namely $\{ie_1, ie_2, \ldots, ie_n\}$, form a basis for the underlying real vector space. (Cf. 2.2.A, paragraph 4, above.) There are two natural ways to order this basis.

(1): If one orders the basis as $\{e_1, ie_1, e_2, ie_2, \ldots, e_n, ie_n\}$, then the matrix for J takes the following block-diagonal form, where the blocks are the 2×2 matrix $J_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. That is: J is (with subscript $2n$ added, so as to indicate dimension):

$$J_{2n} := \begin{pmatrix} J_2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & J_2 & \dots & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & \mathbf{0} & \dots & J_2 \end{pmatrix} . \qquad (30)$$

(2): If one orders the basis as $\{e_1, e_2, \dots, e_n, ie_1, , ie_2, \dots, ie_n\}$, then the matrix for J is block-antidiagonal:

$$J_{2n} := \begin{pmatrix} \mathbf{0} & -\mathbf{1_n} \\ \mathbf{1_n} & \mathbf{0} \end{pmatrix} : \qquad (31)$$

This is more natural when one thinks of the real space as a direct sum of real spaces, as in the second, alternative, approach to complexification at the end of Section 2.1. Thus eq. 31 is the same as eq. 26.

2.2.D: Compatibility with bilinear forms:— Later we will be much concerned with vector spaces that have: either an inner product (like a Hilbert space) or a symplectic product (as in Hamiltonian mechanics; cf. Section 3). So we here consider, in general, the "meshing" of a complex structure with bilinear forms.

If B is a bilinear form on a real vector space V, i.e. $B : V \times V \to \mathbb{R}$, then we say that J *preserves* B if for all $u, v \in V$

$$B(Ju, Jv) = B(u, v) . \qquad (32)$$

Recall that since J is an automorphism with $J^2 = -1$, we have $J^{-1} = -J$. This implies that eq. 32 is equivalent to J being skew-adjoint with respect to B. That is:

$$B(Ju, v) = -B(u, Jv) . \qquad (33)$$

Examples of bilinear forms are inner products and symplectic products. If g is an inner product on V then J preserves g if and only if J is an orthogonal transformation. Likewise, J preserves a non-degenerate, skew-symmetric form ω, i.e. a symplectic product, if and only if J is a symplectic transformation, i.e. $\omega(Ju, Jv) = \omega(u, v)$. If ω and J obey, for all non-zero $u \in V$, $\omega(u, Ju) > 0$, we say that J *tames* ω.

A symplectic form ω on a real vector space V, together with a complex structure J that preserves ω, define: a symmetric bilinear form g_J on V_J. Namely, by:

$$g_J(u, v) := \omega(u, Jv) \qquad (34)$$

g_J is symmetric because J being skew-adjoint with respect to ω, i.e. eq. 33, implies that the rhs of eq. 34, i.e. $\omega(u, Jv) = -\omega(Ju, v) \equiv \omega(v, Ju) =: g_J(v, u)$. One similarly checks trivially that: (i) J preserves g_J; (ii) if J *tames* ω, then g_J is positive-definite, i.e. an inner product.

2.3 Complex conjugation of spaces

2.3.A: Basics:— The *complex conjugate of complex vector space* W is the complex vector space $\overline{W}$ that has the same elements and additive group structure as $\overline{W}$, but whose scalar multiplication involves conjugation. That is: we define the scalar multiplication $*$ in $\overline{W}$ in terms of the scalar multiplication $\cdot$ in W by:

$$\alpha * w := \overline{\alpha} \cdot w \ , \quad \text{for all } \alpha \in \mathbb{C}, w \in \overline{W} \qquad (35)$$

Various properties and results ensue!

(1) $\overline{\overline{W}} = W$.

(2) W and $\overline{W}$ have the same complex dimension. Note that the identity map $id : W \to \overline{W}$ is an antilinear map, since

$$id(\alpha \cdot w) = \alpha \cdot w \equiv \overline{\alpha} * w = \overline{\alpha} * id(w) \tag{36}$$

and id maps any basis of W into a basis of $\overline{W}$. So id is an *anti-isomorphism* from W to $\overline{W}$. It is a "canonical" one in the sense that its definition needs no choice of basis. That is: it is defined in terms of the underlying identity of vectors.

But of course, there are countless anti-isomorphisms defined in terms of such bases (just like there are countless isomorphisms!). For given any two bases, $\{e_i\}$ and $\{f_i\}$, of W and $\overline{W}$ respectively, the map $\Theta : e_i \to f_i$ can be extended by *antilinearity* to be an antilinear map, an *anti-isomorphism*, from W to $\overline{W}$.

(3) If W and U are complex vector spaces, an antilinear map $f : W \to U$ can be regarded as an ordinary linear map $f : \overline{W} \to U$, since:

$$f(\alpha * w) = f(\overline{\alpha} \cdot w) = \overline{\overline{\alpha}} \cdot f(w) = \alpha \cdot f(w) \; ; \tag{37}$$

where in the last two expressions, $\overline{\overline{\alpha}} \cdot f(w)$ and $\alpha \cdot f(w)$, the $\cdot$ is of course scalar multiplication in the codomain space U.

Conversely, any linear map g defined on $\overline{W}$, $g : \overline{W} \to U$, gives rise to an antilinear map from W to U, which again we write with a g. That is, we write: $g : W \to U$. For if we write the scalar multiplication in W as $\cdot$ (as before) and the scalar multiplication in U as $\cdot\cdot$, then the map $g : W \to U$ obeys:

$$g(\alpha \cdot w) \equiv g(\overline{\alpha} * w) = \overline{\alpha} \cdot\cdot g(w) \; , \tag{38}$$

since $g : \overline{W} \to U$ is linear. So the defined map $g : W \to U$ is antilinear.

(4) A linear map between complex vector spaces, $f : W \to U$, gives rise to a corresponding *also!* linear map $\overline{f} : \overline{W} \to \overline{U}$ which has the same action as f. For $\overline{f}$ preserves scalar multiplication, since

$$\overline{f}(\alpha * w) := f(\overline{\alpha} \cdot w) = \overline{\alpha} \cdot f(w) = \alpha * \overline{f}(w) \; . \tag{39}$$

If W, U are finite-dimensional, and the matrix of f with respect to bases $\{e_i\}$ of W and $\{g_j\}$ of U is (c_{ij}), i.e. $f(e_i) = c_{ij} g_j$, then the matrix of the linear map $\overline{f} : \overline{W} \to \overline{U}$ with respect to the *same* (as regards the underlying identity of vectors!) bases, i.e. $\{e_i\}$ of $\overline{W}$ and $\{g_j\}$ of $\overline{U}$, is the matrix whose entries are the complex conjugates of the c_{ij}. For in U, $c_{ij} g_j$ is short for $c_{ij} \cdot g_j$. But $c_{ij} \cdot g_j = \overline{c_{ij}} * g_j$. In short: to get the matrix of $\overline{f}$ from the matrix of f, we take complex conjugates of entries—but we do not transpose!

(5) *The complex conjugate of a Hilbert space.* That a Hilbert space $\mathcal{H}$ has extra structure additional to being a vector space, viz. the inner product, implies that there *is* a canonical aka natural, i..e basis-independent, isomorphism between $\mathcal{H}$ and $\overline{\mathcal{H}}$.

Indeed, recall *Riesz' theorem*: for a separable Hilbert space $\mathcal{H}$, every continuous linear functional $F : \mathcal{H} \to \mathbb{C}$ is given by taking the inner product with a unique vector $\psi_F \in \mathcal{H}$. That is:

$F(\cdot) = (\psi_F, \cdot)$. Since this inner product is *sesquilinear*, i.e. $(\alpha\psi, \beta\phi) = \overline{\alpha}\beta(\psi, \phi)$, there is natural *antilinear* bijection between continuous linear functionals and vectors in $\mathcal{H}$: $F \mapsto \psi_F$. This is antilinear because $(\alpha F) \mapsto \psi_{(\alpha F)} \equiv \overline{\alpha}.\psi_F$. (Here, the . is good old scalar multiplication in $\mathcal{H}$!).

So there is natural *linear* bijection—i.e. an isomorphism!—between continuous linear functionals and vectors in the complex conjugate Hilbert space $\overline{\mathcal{H}}$. That is the dual space of linear functionals, $\mathcal{H}^*$ can be identified with $\overline{\mathcal{H}}$. It then follows that if we identify $\mathcal{H}^{**}$ with $\mathcal{H}$, there is natural isomorphism between $\mathcal{H}^{**} \equiv \mathcal{H}$ and $(\overline{\mathcal{H}})^*$.

Exercise! : Is there a natural isomorphism between $(\overline{\mathcal{H}})^*$ and $\overline{\mathcal{H}^*}$?

3 Symplectic structure

We first recall elements of the symplectic structure underlying Hamiltonian mechanics (Section 3.1). This will show us how to write the classical Poisson brackets in terms of the symplectic product (Section 3.2). Thus we will return to the ideas of the Weyl algebra (cf. Section 1.2), in the form using operators W—which combine the translations in position and in momentum that were given separately by the operators U and V. Then we generalize to symplectic manifolds (Section 3.3).

3.1 Symplectic vector spaces

We will rewrite the classical Poisson brackets, eq. 1, but repeated here:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \; ; \; \{q_i, p_j\} = \delta_{ij} \tag{40}$$

in terms of a symplectic product on a vector space.

We begin with *Hamilton's equations*

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad ; \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \; . \tag{41}$$

Defining

$$\xi^\alpha = q^\alpha, \;\; \alpha = 1, ..., n \quad ; \quad \xi^\alpha = p_{\alpha - n}, \;\; \alpha = n+1, ..., 2n \tag{42}$$

Hamilton's equations become

$$\dot{\xi}^\alpha = \frac{\partial H}{\partial \xi^{\alpha+n}}, \;\; \alpha = 1, ..., n \quad ; \quad \dot{\xi}^\alpha = -\frac{\partial H}{\partial \xi^{\alpha-n}}, \;\; \alpha = n+1, ..., 2n \; . \tag{43}$$

Writing $\mathbf{1}$ and $\mathbf{0}$ for the $n \times n$ identity and zero matrices respectively, we define the $2n \times 2n$ *symplectic matrix* ω by

$$\omega := \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \; . \tag{44}$$

ω is antisymmetric, and has the properties, writing $\tilde{}$ for the transpose of a matrix, that

$$\tilde{\omega} = -\omega = \omega^{-1} \quad \text{so that} \quad \omega^2 = -\mathbf{1} \quad ; \quad \text{also} \quad \det \omega = 1. \tag{45}$$

Using ω, Hamilton's equations eq. 43 get the more symmetric form, in matrix notation

$$\dot{\xi} = \omega \frac{\partial H}{\partial \xi} \; . \tag{46}$$

In terms of components, writing $\omega^{\alpha\beta}$ for the matrix elements of ω, and $\partial_\alpha := \partial\ /\partial\xi^\alpha$, eq. 43 become

$$\dot{\xi}^\alpha = \omega^{\alpha\beta}\partial_\beta H. \tag{47}$$

Eq. 46 and 47 show how ω forms, from the naive gradient (column vector) ∇H of H on the phase space Γ of qs and ps, the vector field on Γ that gives the system's evolution: the *Hamiltonian vector field*, often written X_H. At a point $z = (q, p) \in \Gamma$, eq. 46 can be written

$$X_H(z) = \omega\nabla H(z). \tag{48}$$

Interpretation in terms of areas: Let us begin with the simplest possible case: $\mathbb{R}^2 \ni (q, p)$, representing the phase space of a particle constrained to one spatial dimension. Here, the 2×2 matrix

$$\omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{49}$$

defines the antisymmetric bilinear form on $\mathbb{R}^2$:

$$A : ((q^1, p_1), (q^2, p_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto q^1p_2 - q^2p_1 \in \mathbb{R} \tag{50}$$

since

$$q^1p_2 - q^2p_1 = \begin{pmatrix} q^1 & p_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q^2 \\ p_2 \end{pmatrix} = \det\begin{pmatrix} q^1 & q^2 \\ p_1 & p_2 \end{pmatrix} \ . \tag{51}$$

It is easy to prove that $A((q^1, p_1), (q^2, p_2)) \equiv q^1p_2 - q^2p_1$ is the signed area of the parallelogram spanned by $(q^1, p_1), (q^2, p_2)$, where the sign is positive (negative) if the shortest rotation from (q^1, p_1) to (q^2, p_2) is anti-clockwise (clockwise).

Similarly in $\mathbb{R}^{2n}$: the matrix ω of eq. 44 defines an antisymmetric bilinear form on $\mathbb{R}^{2n}$ whose value on a pair $(q, p) \equiv (q^1, ...q^n; p_1, ..., p_n), (q', p') \equiv (q'^1, ...q'^n; p'_1, ..., p'_n)$ is the sum of the signed areas of the n parallelograms formed by the projections of the vectors $(q, p), (q', p')$ onto the n pairs of coordinate planes labelled $1, ..., n$. That is to say, the value is:

$$\Sigma_{i=1}^n\, q^ip'_i - q'^ip_i \ . \tag{52}$$

3.2 Returning to the Weyl algebra

If we are lucky enough for our classical phase space to be vector space (as when $S = \mathbb{R}^{2n}$), then we can make it a *symplectic vector space*, which is a pair (S, Ω), where S is a phase space—also a vector space—and Ω is a symplectic product. The symplectic product $\Omega : S \times S \to \mathbb{R}$ is, by definition, anti-symmetric, linear and non-degenerate (i.e. if $\Omega(z_1, z_2) = 0$ for all z_2, then $z_1 = \mathbf{0}$).

We define the symplectic product Ω on $S = \mathbb{R}^{2n} \ni z_1, z_2$ as in (14): which we repeat here:

$$\Omega(z_1, z_2) := \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2, \tag{53}$$

Then as we saw in Section 1.2, the Weyl form of the CCRs, i.e. (11) and (7), are equivalent to the following: for all $z, z_1, z_2 \in \mathbb{R}^{2n}$,

$$\begin{aligned} W(z_1)W(z_2) &= e^{\frac{1}{2}i\Omega(z_1,z_2)}W(z_1 + z_2); \\ W^\dagger(z) &= W(-z) \ . \end{aligned} \tag{54}$$

Note that $\Omega(z,\cdot) : S \to \mathbb{R}$ is a real-valued function on S, and so a classical observable. In particular, $\Omega(z,\cdot) = q^i$ iff z has $(n+i)$th component $b_i = 1$ and the rest 0, and $\Omega(z,\cdot) = p_i$ iff z has ith component $a^i = -1$ and the rest 0. In general, $\Omega(z,\cdot)$ is some linear combination of p_is and q^is.

In this formulation, the classical Poisson bracket relations (1: repeated as 40) may be written

$$\{\Omega(z_1,\cdot), \Omega(z_2,\cdot)\} = -\Omega(z_1, z_2) \ . \tag{55}$$

So the corresponding Heisenberg form of the CCRs are

$$[\hat{\Omega}(z_1,\cdot), \hat{\Omega}(z_2,\cdot)] = -i\Omega(z_1, z_2)\mathbb{1} \ . \tag{56}$$

Thus we seek a representation in which the map $z \mapsto \hat{\Omega}(z,\cdot)$ takes elements of S to self-adjoint operators, and in which the Weyl unitaries defined by

$$W(z) := e^{i\hat{\Omega}(z,\cdot)}. \tag{57}$$

obey the Weyl algebra, eq. 54.

This is Wald's presentation: see Wald (1994, Ch. 2). Later we will use field operators Φ, for which $\Phi(Jz) = \hat{\Omega}(z,\cdot)$, or $\Phi(z) = -\hat{\Omega}(Jz,\cdot) = \hat{\Omega}(\cdot, Jz)$.

3.3 Symplectic manifolds, more generally

In the case where the classical phase space S is not a vector space, we must resort to a longer route. In this case, we seek a group whose action on S is *transitive* and preserves the symplectic form $\omega := \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$. (In the case that S is a vector space, this group is just the (abelian) additive group of translations in S, which is isomorphic to S. That is what allowed us to treat S as a symplectic vector space above.) For illustration, taking the case $S = \mathbb{R}^{2n}$, the group action is a $2n$-parameter family of diffeomorphisms associated with the vector fields (with constant coefficients)

$$X_z = \sum_{i=1}^{n} b_i \frac{\partial}{\partial q^i} - a^i \frac{\partial}{\partial p_i}, \tag{58}$$

for any $z := (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$. We may now act on any two such vector fields with the *symplectic form* ω with which S, being a classical phase space, is equipped. This yields

$$\omega(X_{z_1}, X_{z_2}) = \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2. \tag{59}$$

Our quantization problem then becomes the search for continuous families of unitaries $z \mapsto W(z)$ which respect this symplectic structure, as expressed in the Weyl algebra (54), setting $e^{\frac{1}{2}i\Omega(z_1,z_2)} = e^{\frac{1}{2}i\omega(X_{z_1},X_{z_2})}$. Since the Weyl algebra (54) is unitary up to the phase factor $e^{\frac{1}{2}i\omega(X_{z_1},X_{z_2})}$, it is a *projective unitary representation* of the group of symplectomorphisms on S.