Lecture 8 Handout: Geometric quantum theory

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1. Overview

The geometric approach to quantum theory is a natural extension of the Bloch sphere, in which the pure states of a quantum system are expressed as points on a real 2-sphere (Figure 1. But it can be generalised to arbitrary dimensions. And it can go much further than a typical Bloch sphere analysis, to include seemingly all aspects of a quantum system: Schrödinger evolution, uncertainty, and even entanglement have all been naturally expressed using a symplectic form, a metric, and a complex structure on a real manifold, which together form what is known as a Kähler manifold. This approach was first discovered by Kibble [5], developed by local hero Gary Gibbons [4], and rediscovered and further developed by Ashtekar and Schilling [1], forming the basis for Schilling's dissertation at Penn State [8]. The latter two references are an excellent introduction.

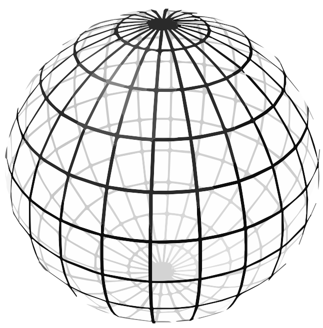

Figure 1. States as points on the Bloch sphere manifold

One advantage of geometric quantum mechanics that it provides controlled context in which to study linear and non-linear extensions of quantum theory [1, 3, 6, 2]. Another is that it allows one to naturally compare quantum structures to those in other geometric theories [9, 7].

2. Geometrising Hilbert space

2.1. **Projective manifolds.** We want to make quantum theory look like symplectic mechanics. To achieve this, we first need a manifold. Happily, quantum theory has an obvious manifold built into it. You may be familiar with this in the Bloch sphere for two-dimensional Hilbert space, where the points on the surface of a sphere are identified with rays on a Hilbert space. This manifold idea can be generalised to arbitrary Hilbert spaces.

Let $\mathcal{H}$ be a separable Hilbert space (i.e. one with a countable basis); a *ray* is an equivalence Ψ of elements of $\mathcal{H}$ such that,

$$\psi, \psi' \in \Psi \text{ iff } \psi = c\psi' \text{ for some } c \in \mathbb{C}.$$

The set of rays of $\mathcal{H}$ has a natural manifold structure, known as the *projective space* $\mathcal{P}_{\mathcal{H}}$, where each point $p \in \mathcal{P}_{\mathcal{H}}$ is a ray in $\mathcal{H}$, $p = \Psi$. To see the manifold structure, we need an atlas of charts. Let $S_\psi^\perp \subseteq \mathcal{H}$ be the subspace of vectors orthogonal to ψ; that is, $\phi \in S_\psi^\perp$ iff $\langle \psi, \phi \rangle = 0$. Now, for each unit vector $|\phi| = 1$, define a coordinate region $U_\phi := \mathcal{H} - S_\Phi^\perp$, together with a coordinate chart $\pi_\phi : U_\Phi \to S_\Phi^\perp$ given by,

$$\pi_\phi(\psi) = \frac{1}{\langle \phi, \psi \rangle}\psi - \phi.$$

This is a surjective map onto $S_\Phi^\perp$, with a rescaling factor that guarantees that vectors on the same ray all get mapped to the same point, i.e., $\pi_\psi(\phi) = \pi_\psi(\phi')$ only if $\Phi = \Phi'$. So, it induces a bijection: it takes each ray that is non-orthogonal to ϕ onto a unique vector in $S_\Phi^\perp$. This induced bijection on the region U_Φ of rays non-orthogonal to ϕ can then be checked[1] to satisfy the manifold axioms with underlying set $\mathcal{P}_{\mathcal{H}}$.

This manifold is complex, since the charts are elements of a vector space over $\mathbb{C}$. However, it can easily be made into a real-manifold. In any basis $\{\varphi_i\}$, we associate each Hilbert space vector $\psi = \sum_k (a_k + ib_k)\varphi_k$ (with $a_k, b_k \in \mathbb{R}$) with a pair of vectors,

$$\psi_1 = \sum_k a_k \varphi_k \qquad\qquad \psi_2 = \sum_k b_k \varphi_k.$$

The result is a real vector space $\mathcal{H}_\mathbb{R}$ of twice the dimension of $\mathcal{H}$, related to the original by $\psi = \psi_1 + i\psi_2$. Using this canonical separation of vectors into pairs $\psi_1 = \sum_k a_k\varphi_k$ and $\psi_2 = \sum_k b_k\varphi_k$, we now define a canonical complex structure $J : \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$, given by the linear extension of the map,

$$J(\psi_1) = \psi_2 \qquad\qquad J(\psi_2) = -\psi_1.$$

This J is a linear operator on $\mathcal{H}_\mathbb{R}$ with the property that $J^2 = -1$. So, although $\mathcal{H}_\mathbb{R}$ is a real-vector space, J gives it some structure usually associated with a complex manifold (hence the name, 'complex structure').

So, in just the way that the rays of $\mathcal{H}$ thus give rise to a complex manifold $\mathcal{P}_{\mathcal{H}}$, the rays of $\mathcal{H}_\mathbb{R}$ give rise to a real manifold $\mathcal{P}$ with a complex structure J. In summary, we have the following:

[1] *Exercise:* Check this, by showing that i) $\bigcup_{\Phi \in \mathcal{P}_{\mathcal{H}}} U_\Phi = \mathcal{P}_{\mathcal{H}}$; ii) π_ϕ is a bijection onto an open subset of $S_\Phi^\perp$; iii) $\pi_\phi(U_\Phi \cap U_{\Phi'})$ is open in $S_\Phi^\perp$; and iv) $\pi_{\phi'} \circ \pi_\phi^{-1}$ is smooth.

Fact. *The rays on a Hilbert space $\mathcal{H}$ form complex manifold $\mathcal{P}_{\mathcal{H}}$ of dimension $n \in [2, \infty]$, which can be viewed as $2n$-dimensional real manifold $\mathcal{P}$ with a complex structure J.*

2.2. **Symplectic form and Riemannian metric.** The Hilbert space inner product is a function from pairs of vectors to the complex numbers, $\langle \dot{,} \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. Just as a complex number can be separate into two real numbers, we can separate $\langle \dot{,} \rangle$ into two real-valued functions g and ω:

$$\langle \psi, \phi \rangle = \tfrac{1}{2} g(\psi, \phi) + \tfrac{i}{2} \omega(\psi, \phi),$$

for all $\psi, \phi \in \mathcal{H}$. The letters for these functions have been chosen for a reason: they have properties that are strikingly similar to a Riemannian metric and a symplectic form.

Begin with g. It is obviously bilinear, and also symmetric, $g(\psi, \phi) = g(\phi, \psi)$. It is also non-degenerate, in that $g(\psi, \phi) = 0$ for all ϕ only if $\psi = \mathbf{0}$. This function is not a tensor, so it is not technically a metric. But, it can be 'lifted' to a $(0, 2)$ tensor on the real manifold $\mathcal{P}$ associated with $\mathcal{H}$, where it turns out to be a full-fledged Riemannian metric. Here we use the fact that $\mathcal{P}$ as we have defined it has an underlying linear structure. The *canonical linear lift* of a linear manifold $\mathcal{P}$ maps each point $p \in \mathcal{P}$ to a vector X_p in the tangent space of every other point p'; this vector a mapping on smooth functions f given by,

$$X_p(f) := \tfrac{d}{ds} f(p' + sp)\big|_{s=0}.$$

This association allows us to lift function g to a real-valued function on pairs of vectors in the tangent space at any point. That is, it is a rank-(0,2) tensor. It inherits the properties of being bilinear, symmetric and non-degenerate, which means that it is a full-fledged Riemannian metric.

A similar exercise can be carried out to lift the function ω to a symplectic form. As a function on $\mathcal{H}$, it is obviously bilinear. It is also skew-symmetric, since,

$$\omega(\psi, \phi) = \mathrm{Im}\left(\langle \psi, \phi \rangle\right) = -\mathrm{Im}\left(\langle \phi, \psi \rangle\right) = -\omega(\phi, \psi).$$

I leave the remaining properties of a symplectic form for you to check.[2]

2.3. **The quantum Kähler manifold.** ur complex structure J lifts to a rank-(1,1) tensor that maps each vector in the tangent space at a point to another vector. Using the definitions of J, g and ω, it is a straightfoward exercise[3] to show that they satisfy the relation,

$$g(X, Y) = \omega(X, J(Y)).$$

[2] *Exercise:* Check that the canonical lift of ω is a symplectic form, in that it is: i) a rank-(0,2) tensor; ii) skew symmetric $\omega(X, Y) = -\omega(Y, X)$; iii) closed, $d\omega = \mathbf{0}$; and iv) non-degenerate, $\omega(X, Y) = 0$ for all Y only if $X = \mathbf{0}$.

[3] *Exercise:* check it for yourself! Hint: use the fact that, viewed as an operator on $\mathcal{H}$, J satisfies $\langle \psi, J\phi \rangle = i\langle \psi, \phi \rangle$.

A manifold $\mathcal{P}$ with a Riemannian metric, symplectic form, and complex structure satisfying this relation is called a *Kähler manifold*. The development above shows that this structure is built naturally into every separable Hilbert space.

3. Observables and Schrödinger Evolution

An *quantum observable* for Kähler quantum mechanics is a function $f : \mathcal{P} \to \mathbb{R}$ of the form,

$$f_A(\psi) = \langle \psi, A\psi \rangle = \tfrac{1}{2} g(\psi, A\psi),$$

where $\psi \in \Phi = p$ is any unit-norm vector associated with the point $p \in \mathcal{P}$, and $A : \mathcal{H} \to \mathcal{H}$ is a densely-defined self-adjoint operator. Not every smooth function on $\mathcal{P}$ can be written in this way; so, the observables of quantum mechanics are in a sense more restrictive than the observables of symplectic mechanics.

To define the dynamical evolution generated by an observable f_H, we now have two choices, which turn out to be equivalent.

In the first place, can define the symplectic evolution associated with the observable f_H. This is given in the usual way by the vector field X_f that satisfies,

$$df = \iota_{X_f} \omega.$$

In the second place, we can define the Schrödinger evolution associated with f_H to be the one that is induced by the unitary operator $U_t = e^{-itH}$ on $\mathcal{H}$. This gives rise to a vector field Y_H on $\mathcal{P}$; to define it at a point $p \in \mathcal{P}$ (containing a unit vector $\psi \in \mathcal{H}$), we take the canonical linear lift (see above) of the function Y_H given by,

$$Y_H(\psi) = -J(H\psi).$$

Remarkably, these two notions of dynamical evolution are the very same: symplectic evolution is just Schrödinger evolution for smooth functions f_H associated with a self-adjoint operator H.

This can be quickly verified: let Y_H be the tangent vector at ψ associated with Schrödinger evolution generated by f_H. And, let Z be any tangent vector at ψ, associated Hilbert space vector $\zeta \in \mathcal{H}$, so at the point ψ, $Z(f_H) = \frac{d}{dt} f_H(\psi + t\zeta)|_{t=0}$. Then,

$$\begin{aligned}
(df_H)(Z) &= Z(f_H) = \tfrac{d}{dt} f_H(\psi + t\zeta) \\
&= \tfrac{d}{dt} \langle \psi + t\zeta, H(\psi + t\zeta) \rangle \big|_{t=0} \\
&= \langle \psi, H\zeta \rangle + \langle \zeta, H\psi \rangle \\
&= g(H\psi, \zeta) \\
&= \omega(-JH\psi, \zeta) \text{ (Kähler property)} \\
&= \omega\left(Y_H(\psi), \zeta\right) \text{ (Schrödinger ev.)} \\
&= \iota_Y \omega(Z).
\end{aligned}$$

This means that the Schrödinger vector field Y_H is the very same one that is given by symplectic evolution generated by f_H.

A further observation: since unitary evolution preserves the inner product, it preserves both the symplectic form *and* the metric: that is, Schrödinger evolution generates a Killing vector field on the Kähler manifold. The reverse is also true: every Killing vector field is generated by a quantum observable of thee form $f_A(\psi) = \langle \psi, A\psi \rangle$ for some self-adjoint operator A on $\mathcal{H}$. A straightforward corollary of this is that $\mathcal{P}$ is geodesically compleete with respect to g_{ab} [1, Theorem II.2]

4. Time observables

4.1. **Review of Pauli's theorem.** We have seen two theorems governing the existence of time observables in symplectic mechanics and quantum theory. The quantum result is:

Proposition 1 (Pauli's theorem)**.** *Let $\mathcal{H}$ be a Hilbert space, let H be a self-adjoint operator with a half-bounded spectrum. For any $\psi \in \mathcal{H}$, let $\psi(t) = e^{-itH}\psi$ for all $t \in \mathbb{R}$. Then there exists no self-adjoint operator T such that, writing $\langle \psi, T\psi \rangle = t_0$, we have for all $t \in \mathbb{R}$ that,*

$$\langle \psi(t), T\psi(t) \rangle = t_0 + t.$$

In symplectic mechanics there is a local existence result for time observables. There is also a 'loophole' allowing global time observables in a situation that is not available in quantum mechanics.

Proposition 2 (Symplectic Paul's theorem)**.** *Let $(\mathcal{P}, \omega)$ be a symplectic manifold, and let $h : \mathcal{P} \to \mathbb{R}$ be a smooth function with a half-bounded range. For any $p \in \mathcal{P}$, let $p(t)$ denote an integral curve of the canonical vector field generated by h with $p_0 = p(0)$. Then there exists no smooth $\tau : \mathcal{P} \to \mathbb{R}$ such that, writing $\tau(p) = t_0$, we have for all $t \in \mathbb{R}$ that,*

$$\tau(p(t)) = t_0 + t,$$

unless the canonical vector field generated by τ is incomplete.

These statements are analogous in structure and so in one sense are easy to compare: symplectic mechanics allows global time observables in a situation that quantum theory does not, namely, when that observable generates an incomplete vector field.

However, it would be nice to make this comparison more rigorous. One way to do this would be to reformulate quantum theory on symplectic manifolds. This is exactly what geometric quantum theory allows us to do.

4.2. **Kähler time observables.** We have seen that, when quantum theory is formulated on a Kähler manifold, the following facts hold:

- symplectic evolution and Schrödinger evolution are one and the same;

- this symplectic evolution is a Killing vector field; and
- the manifold $\mathcal{P}$ is geodesically complete.

Now, it turns out that every Killing field on a geodesically complete Riemannian manifold is a complete vector field; an easy corollary of this fact is the following [7]:

Proposition 3 (Kähler Pauli theorem)**.** *Let $(\mathcal{P}, \omega, g, J)$ be the Kähler manifold associated with a Hilbert space $\mathcal{H}$, and let h be the smooth half-bounded function associated with a self-adjoint operator H. Then there exists no smooth $\tau : \mathcal{P} \to \mathbb{R}$ such that $\{\tau, h\} = 1$. Thus, for any $p \in \mathcal{P}$, let $p(t)$ denote an integral curve of the canonical vector field generated by h with $p_0 = p(0)$. Then there is no τ such that, writing $\tau(p) = t_0$, we have for all $t \in \mathbb{R}$ that,*

$$\tau(p(t)) = t_0 + t.$$

A Kähler manifold is of course just a symplectic manifold, with a little extra structure. That extra structure helps to show exactly how time observables end up being prevented in quantum theory: they can exist in the class of general smooth functions, but not when associated with a self-adjoint operator. This also suggests a way that time observables can be recovered in quantum theory: by relaxing the definition of an observable.

4.3. **Time observables in extensions of quantum theory.** Suppose we relax the requirement that an observable be self-adjoint, by allowing observables to include any smooth function that generates a vector field that covers the phase space. This class of functions has been studied by Ashtekar and Schilling [1, §III.A], who show that they characterize a class of non-linear extensions of quantum theory proposed by [10]. For this reason, they refer to these functions as *Weinberg functions*.

Unlike orthodox quantum observables, the Weinberg functions can be timely. Let us illustrate with a different example due to John D. Norton[4]. Consider the manifold $\mathcal{P} = \mathbb{R}^2$ with a Cartesian coordinate system (q, p) and the standard symplectic form, together with the (half-bounded) Hamiltonian $h(q, p) = e^p$. The integral curves generated by h can be written $(q_t, p_t) = (e^{p_0}t + q_0, p_0)$ for an arbitrary initial point (q_0, p_0). In this system, the smooth function $\tau(q, p) = q/e^p$ is a time observable:

$$\tau(q_t, p_t) = q_t/e^{p_t} = (e^{p_0}t + q_0)/e^{p_0} = q_0/e^{p_0} + t = \tau(q_0, p_0) + t.$$

The Hamiltonian vector field generated by the timely function τ has integral curves given[5] by $q_s = q_0(1 - s/e^{p_0})$ and $p_s = \log(e^{p_0} - s)$. The vector field tangent to these curves is incomplete, because the curve with the initial point $(q_0, p_0) = (0, 0)$ cannot be extended beyond $s = 1$ where $p(s) = \log(1 - s)$ becomes undefined. However, it is smooth and defined on the entire manifold, and therefore counts as a Weinberg function on the definition of Ashtekar and Schilling [1, §III.A].

[4]Private communication.

[5]Check: $dq/ds = \partial\tau/\partial p = -q/e^p$ and $dp/ds = -\partial\tau/\partial q = -1/e^p$. One can easily see by differentiation that these equations are satisfied by $q_s = q_0(1 - s/e^{p_0})$ and $p_s = \log(e^{p_0} - s)$.

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