WEEK 8 HANDOUT: SYMMETRY AND UNITARY INEQUIVALENCE

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Background Reading: Baker and Halvorson (2013),

https://arxiv.org/abs/1103.3227

1. Introduction

Spontaneous symmetry breaking occurs when a ground state is not preserved by some symmetry of the system. For example, the so-called Mexican Hat Potential is a toy model with multiple ground states are related by a rotation (Figure 1). Or, in the Higgs mechanism, degenerate ground states in the electroweak interaction are related by an $SU(2) \times U(1)$ symmetry.

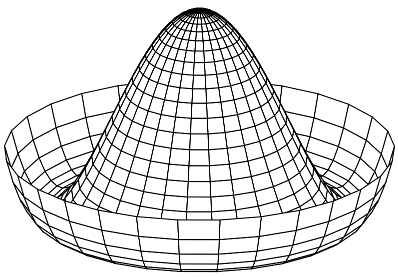

Figure 1. Degenerate ground states in a Mexican hat potential.

A precise description of spontaneous symmetry breaking requires unitarily inequivalent Hilbert space representations, because the vacuum state in a given irreducible representation is generally unique. But that seems to give rise to an apparent paradox: by Wigner's theorem, every symmetry can be implemented by a unitary operator. Shouldn't that all representations related by a symmetry are unitarily equivalent — and hence that spontaneous symmetry breaking is impossible?

Baker and Halvorson propose to resolve this paradox by just thinking carefully about the definitions involved. Let's review them now.

Date: March 2020.

2. Unitary equivalence

When do two mathematical descriptions represent the same physical situation? It's not always easy to answer this. For example, arch-relationist Leibniz argued that two descriptions related by a spatial flip are physically equivalent; arch-substantivalists Newton and Clarke disagreed. Or: Einstein initially failed to understand that the apparent $r = R_S$ singularity in Schwarzschild coordinates (t, r, θ, ϕ) can be removed by a change of coordinates, while still representing the same physical spacetime.

Suppose we describe the kinematics of a quantum system with a preferred vacuum state using the triple $(\mathcal{H}, \mathcal{A}, \Omega)$, where $\mathcal{H}$ is a Hilbert space (like Fock space), $\mathcal{A}$ is an algebra of bounded operators on $\mathcal{H}$, and $\Omega \in \mathcal{H}$ is a preferred vector on $\mathcal{H}$, which we interpret as representing the vacuum. Usually Ω is assumed to be such that the set $\{A\Omega \mid A \in \mathcal{A}\}$ is *dense* in $\mathcal{H}$, to capture the fact that every state is accessible through some operation on the vacuum.

When are two such quantum systems equivalent? The standard answer is: when they are related by a unitary[1] intertwiner or *unitarily equivalent.*

Definition 1 (Unitarily equivalence)**.** Two quantum systems $(\mathcal{H}, \mathcal{A})$ and $(\mathcal{H}', \mathcal{A}')$ are *unitarily equivalent* iff there exists a bijection $\alpha : \mathcal{A} \to \mathcal{A}'$ and a unitary or antiunitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that $\alpha(A) = UAU^*$ for all $A \in \mathcal{A}$.

Why is this the right standard of physical equivalence in quantum theory? It can be motivated in different ways, depending on how we construe the predictions of quantum theory. Here are two (for more see Aniello; 2018). First, we can use the notion of a *transition probability* $|\langle \phi, \psi \rangle|^2$. If we view the self-adjoint operators as observables, then the spectral theorem assigns each observable a set of basis vectors, each of which can be interpreted as the outcome of an experiment. The transition probability then gives the probability of that a prepared state ψ will be measured in the state ϕ associated with a basis vector for an observable. Thinking of the

[1]Recall that if $A : \mathcal{H} \to \mathcal{H}'$ is a linear operator, then for all $\psi' \in \mathcal{H}$ there exists a $\phi \in \mathcal{H}$ such that $\langle \psi', A\chi \rangle_{\mathcal{H}'} = \langle \phi, \chi \rangle_{\mathcal{H}}$. This ϕ is denoted $\phi = A^*\psi'$, which defines an operator $A^* : \mathcal{H}' \to \mathcal{H}$ called the *adjoint.* A *unitary operator* is a linear operator U such that $U^*U = I_{\mathcal{H}}$ and $UU^* = I_{\mathcal{H}'}$.

predictions of quantum theory as transition probabilities of this kind, we can formulate the following property:

Definition 2 (equal transition probabilities)**.** Two quantum systems $(\mathcal{H}, \mathcal{A})$ and $(\mathcal{H}', \mathcal{A}')$ have *equal transition probabilities* iff there is a bijection $\alpha : P_1(\mathcal{H}) \to P_1(\mathcal{H}')$ between their one-dimensional projections that preserves transition probabilities, in that if $\psi \in \mathcal{H}$ denotes a unit vector contained in a projection $E_\psi \in P_1(\mathcal{H})$, and $\alpha(\psi) \in \mathcal{H}'$ denotes a unit vector contained in $\alpha(E_\psi) \in P_1(\mathcal{H}')$, then,

$$|\langle \alpha(\psi), \alpha(\phi) \rangle|^2 = |\langle \psi, \phi \rangle|^2 \text{ for all } E_\psi, E_\phi \in P_1(\mathcal{H}).$$

A version of Wigner's theorem says that this property holds if and only if unitary equivalence does (Aniello; 2018):

Proposition 1. $(\mathcal{H}, \mathcal{A})$ *and* $(\mathcal{H}', \mathcal{A}')$ *are unitarily equivalent if and only if they have equal transition probabilities.*

Proof. Assuming unitary equivalence, define $\alpha(\psi) = U\psi$ to immediately find that $|\langle \alpha(\psi), \alpha(\phi) \rangle|^2 = |\langle U\psi, U\phi \rangle|^2 = |\langle \psi, \phi \rangle|^2$. Conversely, assume equal transition probabilities. Then by Wigner's theorem there exists a unitary or antiunitary operator U such that $\alpha(\psi) = U$. Associating each ψ with a one-dimensional projection E in $\mathcal{A}$, we find that the map $E \mapsto UEU^*$ gives rise to bijection from the projections in $\mathcal{A}$ to those in $\mathcal{A}'$, which extends uniquely to a bijection $A \mapsto UAU^*$ from $\mathcal{A}$ to $\mathcal{A}'$. □

Another way to look at the predictions of quantum theory is in terms of the structure of the density matrices. Recall that a density matrix ρ is associated with a *mixed state* if and only if $\rho = \epsilon\rho_1 + (1 - \epsilon)\rho_2$ for some $\epsilon \in (0, 1)$ and density matrices ρ_1, ρ_2; otherwise it is a *pure state*. A mapping that preserves the density matrix structure is called a *density matrix automorphism* or a *Kadison automorphism*. This is another general sense in which we might say that two quantum systems are equivalent: they have the same density matrices. We formulate this as follows:

Definition 3 (equivalent density matrices)**.** Two quantum systems $(\mathcal{H}, \mathcal{A})$ and $(\mathcal{H}', \mathcal{A}')$ have *equivalent density matrices* iff there is a bijection Φ from the density matrices of one to the density matrices of the other that preserves convex structure, $\Phi(\epsilon\rho_1 + (1-\epsilon)\rho_2) = \epsilon\Phi(\rho_1) + (1-\epsilon)\Phi(\rho_2)$ for all ρ_1, ρ_2 and for all $\epsilon \in [0,1]$.

We then have the following result due to Kadison (1965).

Proposition 2. *Two quantum systems $(\mathcal{H}, \mathcal{A})$ and $(\mathcal{H}', \mathcal{A}')$ have equivalent density matrices iff they are unitarily equivalent.*

An even more general way to look at a quantum system is as an abstract unital C^* algebra $\mathfrak{A}$, together with its set of 'states', where a *state* $\omega : \mathcal{A} \to \mathbb{C}$ is any linear function that is positive ($\rho(A^*A) \geq 0$ for all $A \in \mathfrak{A}$) and satisfies $\omega(I) = 1$. The self-adjoint elements of $\mathfrak{A}$ are again interpreted as observables, while the states $\omega \in S_{\mathcal{A}}$ have the general properties associated with the expectation value of an observable in ordinary quantum mechanics; thus, we interpret refer to $\omega(A)$ as the 'expectation value' of A in the state ω. Quantum systems in this sense make the same probabilistic predictions if their respective states make the same assignments to operators in $\mathfrak{A}$:

Definition 4 (equal expectation values)**.** Two C^* algebras $\mathfrak{A}$ and $\mathfrak{A}'$ have *equal expectation values* iff there is a bijection between their states $\alpha : \omega \mapsto \omega'$ and between their elements $\beta : A \mapsto A'$ that preserves expectation values, $\omega'(A') = \omega(A)$ for all $A \in \mathfrak{A}$ and all states ω on $\mathfrak{A}$.

Roberts and Roepstorff (1969) proved that this property holds iff the two algebras are related by a C^*-algebra automorphism:

Proposition 3. *Two C^* algebras $\mathfrak{A}$ and $\mathfrak{A}'$ have equal expectation values iff they are related by a $*$-automorphism.*

If $(\mathcal{H}, \mathcal{A})$ is a Hilbert space representation of $\mathfrak{A}$ defined by $\pi : \mathfrak{A} \to \mathcal{A} \subseteq B(\mathcal{H})$, then a $*$-automorphism $\alpha : \mathfrak{A} \to \mathfrak{A}$ is *unitarily implementable* iff there exists a Hilbert space $\mathcal{H}'$ and a unitary U such that $A \mapsto \pi(\alpha(A))$ defines a representation of $\mathfrak{A}$, which is unitarily equivalent to the first: $\pi(\alpha(A)) = U\pi(A)U^*$. Note that not every $*$-automorphism is unitarily implementable. This is our first insight about spontaneous symmetry breaking: a 'symmetry' in the sense of a $*$-automorphism may not be unitarily implementable.

Nevertheless, the upshot of all these results is strong reason to associate physically equivalent systems with unitary equivalence — except in the most abstract case of preserving expectation values, in which case all we get is a $*$-automorphism. As Earman (2003) points out, spontaneous symmetry breaking typically deals with symmetries in this sense, and which are not unitarily implementable. However, this automorphism on states will typically still satisfy the requirements of Wigner's theorem expressed above.

3. Baker and Halvorson's resolution

An apparent paradox threatens: spontaneous symmetry breaking is associated with two quantum systems $(\mathcal{H}, \mathcal{A}, \Omega)$ and $(\mathcal{H}', \mathcal{A}', \Omega)$ that are related by a symmetry, but still manage to be unitarily inequivalent, in order for there to be some sense in which two systems have an inequivalent vacuum. Baker and Halvorson respond by pointing out that an automorphism $\alpha : \mathfrak{A} \to \mathfrak{A}$ on a C^*-algebra gives rise to a sister-automorphism $\alpha' : \omega \to \omega'$ that acts on states, defined by, $\alpha'(A) := \alpha(A)$. They then propose the following solution to the paradox:

> *It is possible for an automorphism $\alpha : \mathfrak{A} \to \mathfrak{A}$ to be be unitarily implemented as it acts on states, but not as it acts on operators.*

They propose that the former is the result of Wigner's theorem, while the latter occurs in the context of spontaneous symmetry breaking. So, given two quantum systems $(\mathcal{H}, \pi(\mathfrak{A}), \Omega)$ and $(\mathcal{H}', \pi'(\mathfrak{A}), \Omega')$ that are representations of the same C^*-algebra $\mathfrak{A}$, Baker and Halvorson point out that, for each symmetry α (a $*$-automorphism) of $\mathfrak{A}$, there is a unitary operator U such that α' acting on states is implemented by a unitary operator $W : \mathcal{H} \to \mathcal{H}$. This W may still have the effect of mapping the two representations to each other, in that for any fixed $\psi \in \mathcal{H}$, the following two sets of vectors in $\mathcal{H}$ are the same:

$$\{W^*\pi'(A)W\psi \mid A \in \mathfrak{A}\} = \{\pi(A)\psi \mid A \in \mathfrak{A}\}, \tag{1}$$

and W is unitary. However, W still may not relate the operators in the two algebras element-by-element, in that the following statement of unitary equivalence *need not be true* for all $\psi \in \mathcal{H}$:

$$W^*\pi(A)W\psi = \pi(A)\psi \text{ for all } A \in \mathfrak{A}. \tag{2}$$

To illustrate, Baker and Halvorson then exhibit the following corollary to Wigner's theorem.

Theorem 1 (Baker and Halvorson)**.** *Let* $(\mathcal{H}, \pi, \Omega)$ *be the GNS representation for* $(\mathfrak{A}, \omega)$*, and let* $(\mathcal{H}', \pi, \Omega')$ *be a GNS representation for* $\omega \circ \alpha^{-1}$*. Then there is a unique unitary operator* $W = W_{\pi,\pi'} : \mathcal{H} \to \mathcal{H}'$ *such that* $W\Omega = \Omega'$ *and* $W\pi(\alpha^{-1}(A)) = \pi'(A)W$ *for all* $A \in \mathfrak{A}$*.*

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