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Why is $\mathcal{CPT}$ Fundamental?

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Lüders and Pauli proved the $\mathcal{CPT}$ theorem based on Lagrangian quantum field theory almost half a century ago. Jost gave a more general proof based on "axiomatic" field theory nearly as long ago. The axiomatic point of view has two advantages over the Lagrangian one. First, the axiomatic point of view makes clear why $\mathcal{CPT}$ is fundamental—because it is intimately related to Lorentz invariance. Secondly, the axiomatic proof gives a simple way to calculate the $\mathcal{CPT}$ transform of any relativistic field without calculating $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ separately and then multiplying them. The purpose of this pedagogical paper is to "deaxiomatize" the $\mathcal{CPT}$ theorem by explaining it in a few simple steps. We use theorems of distribution theory and of several complex variables without proof to make the exposition elementary.

KEY WORDS: CPT theorem; general quantum field theory; Lorentz covariance.

1. INTRODUCTION

The notion of $\mathcal{CPT}$ symmetry, where $\mathcal{C}$ is charge conjugation, $\mathcal{P}$ is parity (space inversion) and $\mathcal{T}$ is time reversal in the sense of Wigner,[2] as a symmetry that holds for any relativistic quantum field theory evolved from the observation of Lüders[(1)] that charge conjugation symmetry and space–time inversion symmetry both impose the same constraints on the form of the interaction Hamiltonian so that $\mathcal{CPT}$ symmetry has a more fundamental basis than either $\mathcal{C}$, $\mathcal{P}$ or $\mathcal{T}$. Pauli[(2)] gave a clear formulation of $\mathcal{CPT}$ symmetry in the context of conditions on the interaction Hamiltonian or Lagrangian. Pauli's formulation is the form of the $\mathcal{CPT}$ symmetry that is usually discussed, the "Lagrangian $\mathcal{CPT}$ theorem." Jost[(3)] gave a general

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[2] $\mathcal{C}$ and $\mathcal{P}$ are unitary; $\mathcal{T}$ and $\mathcal{CPT}$ are antiunitary.

proof of $\mathcal{CPT}$ symmetry based on the fact that space–time inversion is connected to the identity in the complex Lorentz group. By contrast this inversion is not connected to the identity in the real Lorentz group. Jost's analysis is usually called the "axiomatic $\mathcal{CPT}$ theorem." Schwinger[(4)] discussed the $\mathcal{CPT}$ and spin-statistics theorems from the point of view of his differential action principle.

Jost's proof has been labeled as belonging to axiomatic field theory as though that made his proof both incomprehensible and of no practical value. The field theory aspects of Jost's proof require only elementary information about quantum field theory. It suffices to know that c-number fields are promoted to operators on a Hilbert space, that charge conjugation is essentially implemented by hermitian conjugation, and that the Wightman functions,[(5)] defined below in terms of vacuum matrix elements of the field operators, completely determine the theory. This last fact is the "Wightman reconstruction theorem."[(5)] The ideas of Jost's proof are easy to understand if one is willing to accept theorems about distributions and analytic functions of several complex variables without proof. To make notation simple in this pedagogical paper I have exorcised all the test functions that usually appear in discussions of singular functions (distributions). To make clear my lack of rigor I have used the word "analytic" rather than the word "holomorphic" in connection with the functions of several complex variables that appear.

Jost's proof has the practical value that it gives a very simple and general result for the $\mathcal{CPT}$ transformation acting on any relativistic quantum field. Jost discussed the $\mathcal{CPT}$ theorem in three publications, his original paper in *Helvetica Physica Acta*,[(3)] his contribution (in German) to the Pauli memorial volume[(6)] and his book[(7)] on quantum field theory. Jost's theorem also is discussed in the books by Streater and Wightman,[(8)] Bogoliubov et al.,[(9)] and Haag.[(10)]

The standard textbooks of quantum field theory all get to the $\mathcal{CPT}$ theorem by calculating each transformation and then calculating their product. This is not incorrect (except for the technical fact that each of $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ can have an arbitrary phase since they are not connected to the identity while $\mathcal{CPT}$, which is connected to the identity, cannot have an arbitrary phase). However, calculating $\mathcal{CPT}$ by multiplying each of the three discrete symmetries is a very complicated way to calculate $\mathcal{CPT}$. More important, calculating $\mathcal{CPT}$ in that way obscures why $\mathcal{CPT}$ is fundamental but none of the individual symmetries is.

The purpose of this expository note is to explain why $\mathcal{CPT}$ is fundamental and to calculate it for a general relativistic quantum field without worrying about the mathematical issues connected with functions of several complex variables and their relation to tempered distributions whose support in momentum space lies in or on a cone. $\mathcal{CPT}$ is fundamental because it is closely related to Lorentz covariance. We will pay attention to how far we can get with Lorentz covariance alone and where we must use an additional property of the theory. The reader will also see that the calculation of $\mathcal{CPT}$ using general arguments is greatly simpler than the pedestrian calculation of $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ separately and then multiplying them. To make this note self-contained we will explain ideas connected with group theory and field theory that many readers will already understand. Those readers are encouraged to skip the introductory explanations and go directly to the $\mathcal{CPT}$ theorem itself in Sec. 6.

We give a brief summary of the Lagrangian $\mathcal{CPT}$ theorem in Sec. 2, discuss the representations of the real and complex Lorentz groups in Sec. 3, describe the vacuum matrix elements of fields and their relation to analytic functions in Sec. 4, discuss the enlargement of the domain of analyticity of the Wightman functions in Sec. 5, derive the general formula for the $\mathcal{CPT}$ transformation in Sec. 6, discuss $\mathcal{CPT}$ for the S-matrix in Sec. 7 and give a summary in Sec. 8. Appendices give an alternative description of the irreducible representations of the Lorentz group in Appendix A, the detailed form of the transformation law of the fields in Appendix B, the qualitative difference between domains of analyticity for functions of several complex variables and for a single complex variable in Appendix C, and an example to show that Lorentz covariance alone does not suffice for $\mathcal{CPT}$ in Appendix D.

2. THE LAGRANGIAN $\mathcal{CPT}$ THEOREM

The theorem states that $\mathcal{CPT}$ is a symmetry of any local relativistic quantum field theory. This implies that if there is a state $|\Psi\rangle$ then the $\mathcal{CPT}$-conjugate state $|\Psi\rangle_{\text{conjugate}}$ is also present. Some states may be their own $\mathcal{CPT}$-conjugates. In particular every particle has an antiparticle; some particles may be their own antiparticles. The Lagrangian and Hamiltonian densities must be $\mathcal{CPT}$ invariant in the sense that $\mathcal{CPT}\ \mathcal{L}(x)\ (\mathcal{CPT})^\dagger = \mathcal{L}(-x)$ and $\mathcal{CPT}\ \mathcal{H}(x)\ (\mathcal{CPT})^\dagger = \mathcal{H}(-x)$ so that the action is invariant.

The Lagrangian $\mathcal{CPT}$ theorem assumes that the theory is local in the sense that the fields in the Lagrangian occur at the same space–time point with only finite order derivatives. The proof uses the transformations of each field and current, more generally each tensor, based on the individual $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations in each case, to show that Lorentz covariance requires that the Lagrangian and Hamiltonian densities transform as given above. For example, a scalar field $\phi(x)$ transforms as $\mathcal{CPT}\ \phi(x)\ (\mathcal{CPT})^\dagger = \phi(-x)^\dagger$. The transformation of the Lagrangian density given above is a special case of this transformation for a scalar field where the Hermiticity of the Lagrangian required for unitarity makes $\mathcal{L}(-x)^\dagger = \mathcal{L}(-x)$. Details of the Lagrangian theorem are given in standard texts on quantum field theory, for example Refs. 11–13.

3. REPRESENTATION OF THE REAL AND COMPLEX LORENTZ GROUPS

Since the heart of the argument is the fact that the connected component of the complex Lorentz group, $L(C)$, which is the proper complex Lorentz group, $L_+(C)$, contains space–time inversion, we will discuss the Lorentz group first.[7] The (real) Lorentz group can be taken as the group, $SO(1,3)$, of real 4×4 matrices Λ that preserve the metric g that we take to have the form $g = \mathrm{diag}(1, -1, -1, -1)$,

$$\Lambda^T g \Lambda = g. \tag{1}$$

You can check that this condition is equivalent to saying that a Lorentz transformation $x' = \Lambda x$ preserves the scalar product $x{\cdot}x = (x^0)^2 - \sum_1^3 (x^i)^2$, i.e., $\Lambda x{\cdot}\Lambda x = x{\cdot}x$. (To avoid confusion with the component x^2 of a vector, I use $x \cdot x$ for the square of the vector x.) By taking the determinant of Eq. (1) we see that $\det\Lambda = \pm 1$. By looking at the 00 element of Eq. (1) we find $(\Lambda^0{}_0)^2 - \sum_1^3 (\Lambda^0{}_i)^2 = 1$, so either $\Lambda^0{}_0 \geqslant 1$ or $\Lambda^0{}_0 \leqslant -1$. Thus the Lorentz group falls into four disconnected components, $L_+^\uparrow$, $L_+^\downarrow$, $L_-^\uparrow$, and $L_-^\downarrow$ according to the sign of the determinant of Λ and the sign of $\Lambda^0{}_0$. Only the first of these is a group since only $L_+^\uparrow$ contains the identity. We use $x \in V_+$ if x is in the open forward light cone, $x \cdot x > 0, x^0 > 0$; $x \in V_-$ if x is in the open backward light cone, $x \cdot x > 0, x^0 < 0$; $x \in N_+$ if x is on the forward light cone (i.e., is a positive energy null vector), $x \cdot x = 0, x^0 > 0$; $x \in N_-$ if x is on the backward light cone (i.e., is a negative energy null vector), $x \cdot x = 0, x^0 < 0$; $x \sim 0$ or $x \in S$ if x is spacelike, $x \cdot x < 0$; and $x = 0$ or $x \in \mathcal{O}$ if x is the origin. Minkowski space is the union of the disjoint sets V_+, V_-, N_+, N_-, S, and $\mathcal{O}$. The closed forward light cone $\bar{V}_+$ is the union of V_+, N_+, and $\mathcal{O}$; the analogous statement holds for the closed backward light cone $\bar{V}_-$ with $+$ replaced by $-$.

We also have to consider the complex Lorentz group, the group of complex 4×4 matrices that obey Eq. (1). For the complex Lorentz group the sign of the determinant still cannot be changed continuously, but the matrix -1 is now connected to the identity, so there are only two disconnected components. The easiest way to find the continuous family of complex Lorentz transformations that connect the matrices 1 and -1 is by considering the covering groups of the real and complex Lorentz groups, to which we now turn.

We are familiar with the fact that a spin-1/2 state transforms under a rotation by an angle θ with a phase $\theta/2$ rather than the phase θ of a scalar state. Thus a rotation by 2π changes the phase of a spin-1/2 state even though such a rotation should be equivalent to the identity. Therefore for a spin 1/2 state the identity element in the rotation group can be represented by either of the 2×2 matrices 1 or -1. Thus a spin-1/2 state does not transform as a true representation of the rotation group, but rather as a representation up to a factor. The idea of a covering group is to find a larger group whose representations are true representations without additional phases. For the rotation group the covering group is $SU(2)$, the group of 2×2 unitary complex matrices with determinant 1. For the connected component of the Lorentz group the covering group is $\bar{L}_{+}^{\uparrow} \equiv SL(2, C)$, the group of 2×2 complex matrices of determinant 1.

We introduce the two fundamental representations of $SL(2, C)$ as

$$u'_{\alpha} = A_{\alpha\beta} u_{\beta} \tag{2}$$

and

$$\dot{v}'_{\dot{\alpha}} = A^{*}_{\dot{\alpha}\dot{\beta}} \dot{v}_{\dot{\beta}}, \tag{3}$$

where $A \in SL(2, C)$ and $*$ stands for complex conjugate. Van der Waerden introduced the spinors with undotted and dotted indices.[14] We can define a scalar product for these representations using the 2×2 antisymmetric Levi–Civita symbol $\epsilon_{\alpha\beta}$ for the undotted spinors and $\epsilon_{\dot{\alpha}\dot{\beta}}$ for the dotted spinors. We choose $\epsilon_{12} = 1$, $\epsilon_{\dot{1}\dot{2}} = 1$. Any finite-dimensional irreducible representation of $SL(2, C)$ has the form of a spinor with k undotted and l dotted indices, each transforming as given above. Because the only constraint on the $SL(2, C)$ matrices is that the determinant must be one, the only way we can reduce these representations is by contracting with the ϵ's just described, so the irreducible representations of $SL(2, C)$ are spinors with k symmetrized undotted and l symmetrized dotted indices. Each index corresponds to spin-1/2 so these spinors have spin $k/2$ and $l/2$ under the $SU(2) \otimes SU(2)$ formed by taking the groups whose generators are $J \pm iK$, where J are the rotation generators of the real Lorentz group and K are the generators of pure Lorentz transformations (boosts). (See Appendix A for this description.) We focus on irreducible representations because any linear representation can be constructed as a sum of irreducible representations. (The superposition principle of quantum theory requires that representations be linear.)

We take the Pauli matrices to have one undotted and one dotted index, $(\sigma_\mu)_\alpha^{\dot\beta}$, where σ_0 is the unit 2×2 matrix and σ_i are the usual Pauli matrices. Then we can uniquely associate a 2×2 hermitian matrix X with a real vector x^μ by $(X)_\alpha^{\dot\beta} = x^\mu (\sigma_\mu)_\alpha^{\dot\beta}$. To invert this, trace with the σ matrices. The reader should check that $\det X = x \cdot x$. Recalling that matrices in $SL(2, C)$ have determinant 1, we see that $X' = AXA^\dagger$, where $\dagger$ stands for hermitian conjugation, is again Hermitian and is a Lorentz transformation on x. The matrices A and $-A$ stand for the same Lorentz transformation; thus the group $SL(2, C)$ covers the connected component of the Lorentz group twice.

To cover the complex Lorentz group we allow two independent $SL(2, C)$ matrices to enter so that $X' = AXB^T$. This X' is no longer Hermitian, but it still has the same Minkowski metric length, so the covering group of the complex Lorentz group, $L_+(C)$, is $SL(2, C) \otimes SL(2, C)$. Since now we have two independent matrices A and B at our disposal, we can achieve $x \to -x$ either by choosing $A = 1,\ B = -1$ or $A = -1,\ B = 1$. We can go continuously from the identity $A = 1,\ B = 1$ to $A = 1,\ B = -1$ in the first case by choosing $A = 1,\ B(\phi) = \mathrm{diag}(\exp i\phi/2, \exp -i\phi/2)$. We can find the 4×4 complex Lorentz transformations $\Lambda(\phi)$ from the definition of X'. The result, which is a continuous family of complex Lorentz transformations going from the identity to space–time inversion, is

$$\begin{pmatrix} x'^0 \\ x'^3 \\ x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\phi}{2} & i\ \sin\frac{\phi}{2} & 0 & 0 \\ i\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ 0 & 0 & \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \begin{pmatrix} x^0 \\ x^3 \\ x^1 \\ x^2 \end{pmatrix}. \tag{4}$$

Are there other ways to achieve space–time inversion? In $SL(2, C) \otimes SL(2, C)$ we need $AXB^T = -X$, or $AX = -XB^{T-1}$. Thus we need this relation where X is replaced by each of the Pauli matrices σ_μ. For $\sigma_0 = 1$ we need $A = -B^{T-1}$. Then we need $AX = XA$ where for X we can choose any of the space σ's. This requires $A = \omega 1$ and for $A \in SL(2, C)$ we need $\omega^2 = 1$ or $\omega = \pm 1$. Thus the *only* possibilities to invert x^μ are the ones given above. Now we have the group theory we need to discuss the $\mathcal{CPT}$ theorem.

4. VACUUM MATRIX ELEMENTS OF PRODUCTS OF FIELDS DEFINE ANALYTIC FUNCTIONS

Next we have to discuss vacuum matrix elements of products of fields, often called Wightman functions or distributions. Let $\phi^{(k,l)}(x)$ be a field with k undotted and l dotted indices, each set symmetrized, that transforms as the irreducible representation of $SL(2, C)$ described above. We will use the active point of view in which a Poincaré transformation (a, A) acts as[3]

$$U(a, A)\phi^{(k,l)}(x)U(a, A)^{\dagger} = S^{(k,l)}(\Lambda)^{(-1)}\phi^{(k,l)}(\Lambda x + a). \tag{5}$$

The only case for which we need the detailed form of $S^{(k,l)}(A, B)$ is when $\Lambda \in L_{+}(C)$ produces space–time inversion and for that case $S^{(k,l)}(A, B)$ is just a multiple of the identity. Thus the detailed form of $S^{(k,l)}$ is not necessary here. For this reason we have suppressed the indices belonging to the matrices $S^{(k,l)}$ as well as the indices belonging to the field $\phi^{(k,l)}(x)$.[4] We assume the vacuum $|0\rangle$ is invariant under Poincaré transformations,

$$U(a, A)|0\rangle = |0\rangle. \tag{6}$$

We were tempted first to use scalar fields in discussing Jost's proof in order to avoid cumbersome notation and then to give the argument again for the general case. Instead, in order to make clear how simple Jost's argument is, we decided to streamline the notation and give the general case directly. (For some properties such as the support in momentum space which does not depend on the spin we will use the scalar case to illustrate the issue.) Let the single index (p) (for "pair" of indices) stand for (k, l). We will use (p) and (k, l) interchangeably to label fields and other objects. Then the general field becomes $\phi^{(p)}(x)$, the matrices are $S^{(p)}(A)$, and the transformation law, again suppressing indices, is

$$U(a, A)\phi^{(p)}(x)U^{\dagger}(a, A) = S^{(p)}(A)^{-1}\phi^{(p)}(\Lambda x + a). \tag{7}$$

Next we write the vacuum matrix element of an arbitrary product of fields and use this transformation law to find

[3] We are using the covering group of the Poincaré group, so in (a, Λ) we replaced $\Lambda \in L_{+}^{\uparrow}$ by $A \in SL(2, C)$. In the argument of the fields on the right-hand side on the next equation we replaced Λ by $\Lambda(A)$ where $\Lambda(A) \in L_{+}^{\uparrow}$ is the homomorphic image of $A \in SL(2, C)$. Where we use the covering group of the complex Lorentz group we should replace $\Lambda \in L_{+}(C)$ by $\Lambda(A, B)$ where $\Lambda(A, B)$ is the homomorphic image of $(A, B) \in SL(2, C) \otimes SL(2, C)$. To simplify notation we will write Λ instead of $\Lambda(A)$ or $\Lambda(A, B)$ in both cases.

[4] See Appendix B for the transformations with all indices exhibited.

$$
\begin{aligned}
&\mathcal{W}^{(n,p_1p_2\ldots p_n)}(x_1,x_2,\ldots,x_n)\\
&\quad\equiv \langle 0|\phi^{(p_1)}(x_1)\phi^{(p_2)}(x_2)\ldots\phi^{(p_n)}(x_n)|0\rangle\\
&\quad= (|0\rangle, \phi^{(p_1)}(x_1)\phi^{(p_2)}(x_2)\ldots\phi^{(p_n)}(x_n)|0\rangle)\\
&\quad= (U(a,A)|0\rangle, U(a,A)\phi^{(p_1)}(x_1)\phi^{(p_2)}(x_2)\ldots\phi^{(p_n)}(x_n)|0\rangle)\\
&\quad= \left(|0\rangle, \left[\prod_1^n S^{(p_i)}(A)^{-1}\right]\phi^{(p_1)}(\Lambda x_1+a)\phi^{(p_2)}(\Lambda x_2+a)\right.\\
&\qquad\left.\times\cdots\phi^{(p_n)}(\Lambda x_n+a)|0\rangle\right)\\
&\quad= \left[\prod_1^n S^{(p_i)}(A)^{-1}\right]\langle 0|\phi^{(p_1)}(\Lambda x_1+a)\phi^{(p_2)}(\Lambda x_2+a)\cdots\phi^{(p_n)}(\Lambda x_n+a)|0\rangle\\
&\quad= \left[\prod_1^n S^{(p_i)}(A)^{-1}\right]\mathcal{W}^{(n,p_1p_2\ldots p_n)}(\Lambda x_1+a,\Lambda x_2+a,\ldots,\Lambda x_n+a). \qquad (8)
\end{aligned}
$$

This matrix element obeys translation invariance, i.e., it remains the same when all n vectors x_j are translated by the same vector a. Therefore this matrix element depends on only $n-1$ differences of the space–time coordinates. We define the Wightman function[(5)] which is a generalized function or distribution defined on the difference vectors,

$$
W^{(n;p_1p_2\ldots p_n)}(x_1-x_2,x_2-x_3,\ldots,x_{n-1}-x_n)\equiv\langle 0|\phi^{(p_1)}(x_1)\phi^{(p_2)}(x_2)\times\ldots\phi^{(p_n)}(x_n)|0\rangle. \qquad (9)
$$

Since we will have to deal with three kinds of difference vectors, we will use different letters to distinguish them: ξ for real vectors, ρ for those special real vectors (called "Jost points" defined below) in the domain of analyticity $\mathcal{T}'_{n-1}$, and ζ for complex vectors. To streamline the notation we compress the indices $(p_1p_2\cdots p_n)$ on $W^{(n)}$ to a single index $(\wp)$. We define $\xi_j = x_j - x_{j+1}$. Then invariance under the connected component of the Lorentz group (the proper orthochronous component, $L_+^\uparrow$) gives

$$
W^{(n;\wp)}(\Lambda\xi_1,\Lambda\xi_2,\ldots,\Lambda\xi_{n-1}) = \left[\prod_1^n S^{(p_i)}(A)\right]W^{(n;\wp)}(\xi_1,\xi_2,\ldots,\xi_{n-1}). \qquad (10)
$$

We use $\mathcal{W}^{(n)}$, space–time vectors x_i, and Fourier transform conjugate vectors k_i, which are energy–momentum vectors, but in general are not energy–momenta of a physical state, for the distributions that have n vector arguments. We use $W^{(n)}$, space–time difference vectors ξ_i, and Fourier transform conjugate vectors q_i, which are physical energy–momentum vectors, for the distributions that have $n-1$ vector arguments.

The requirement that physical states have positive energy, except the vacuum which has zero energy, implies that the momenta q_i in the Fourier transform of the $W^{(n;\wp)}$'s lie in the closed forward light cone $\bar{V}_+$. To see this, since the support in momentum space depends only on the translation subgroup of the Poincaré group, we drop all indices and consider the case of a scalar field. We use $\phi(x) = \exp(iP\cdot x)\phi(0)\exp(-iP\cdot x)$ and the derivative of this relation, $[P^\mu, \phi(x)] = -i\partial^\mu\phi(x)$, together with the Fourier transform, $\phi(x) = \int d^4k\ \exp(ik\cdot x)\tilde{\phi}(k)$, to find

$$P^\mu\tilde{\phi}(k_{j+1})\ldots\tilde{\phi}(k_n)|0\rangle = -\left(\sum_{j+1}^{n} k_i\right)\tilde{\phi}\left(k_{j+1}\right)\ldots\tilde{\phi}(k_n)|0\rangle. \tag{11}$$

We introduce the Fourier transform of $\mathcal{W}^{(n)}$,

$$\begin{aligned}\mathcal{W}^{(n)}(x_1, x_2, \ldots, x_n) &= \int d^4k_1\ldots d^4k_n\\ &\times\exp\left(-i\sum_1^n k_i\cdot x_i\right)\tilde{\mathcal{W}}^{(n)}(k_1,\ldots,k_n)\end{aligned} \tag{12}$$

and the Fourier transform of $W^{(n)}$,

$$\begin{aligned}W^{(n)}(x_1-x_2,\ldots,x_{n-1}-x_n) &= \int d^4q_1\ldots d^4q_{n-1}\\ &\times\exp\left(-i\sum_1^{n-1} q_i\cdot\left(x_i - x_{i+1}\right)\right)\tilde{W}^{(n)}(q_1,\ldots,q_{n-1}).\end{aligned} \tag{13}$$

This last Fourier transform can also be written

$$\begin{aligned}&W^{(n)}(x_1-x_2,\ldots,x_{n-1}-x_n) = \int d^4q_1\ldots d^4q_{n-1}\\ &\times\exp\left(-iq_1\cdot x_1 - i\sum_2^{n-1}\left(q_i - q_{i-1}\right)\cdot x_i + iq_{n-1}\cdot x_n\right)\tilde{W}^{(n)}(q_1,\ldots,q_{n-1}).\end{aligned} \tag{14}$$

Comparison of the inverse Fourier transforms of Eqs. (12), (13) and (14) shows that

$$\tilde{\mathcal{W}}^{(n)}(k_1,k_2,\ldots,k_n) = \tilde{W}^{(n)}\left(k_1, k_1+k_2,\ldots,\sum_1^{n-1}k_i\right)\delta\left(\sum_1^n k_i\right) \tag{15}$$

and that $k_1 = q_1$, $k_2 = q_2 - q_1$, ..., $k_{n-1} = q_{n-1} - q_{n-2}$, $k_n = -q_{n-1}$, and $q_1 = k_1$, $q_2 = k_1 + k_2$, ..., $q_{n-1} = \sum_1^{n-1} k_i$, $\sum_1^n k_i = 0$, and in particular $q_j = -(\sum_{j+1}^n k_i)$. This last sum is the momentum of a physical state (see Eq. (11)) which completes the argument that the q_i must lie in or on the forward light cone.

Now use the intuitive criterion that a Fourier transform that is a distribution becomes an analytic function when the external variable is made complex in such a way as to provide a damping factor so that the Fourier transform becomes a Laplace transform. Thus we must examine when the factor $\exp(-iq_j \cdot \xi_j)$ becomes decreasing for $\xi_j \to \zeta_j = \xi_j + i\eta_j$ with ξ_j and η_j real vectors. What counts is the absolute value of the factor which is $\exp(q_j \cdot \eta_j)$. This becomes decreasing if η_j is in the backward light cone since the physical momentum q_j is in or on the forward light cone. Thus the Wightman function,

$$W^{(n)}(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}), \tag{16}$$

is an analytic function of $4(n-1)$ complex variables (in four-dimensional space–time) when $\mathrm{Im}\zeta_j \in V_-$. It is also single-valued.

5. ENLARGEMENT OF THE DOMAIN OF ANALYTICITY

As it stands, $W^{(n)}$ is analytic only when $\mathrm{Im}\zeta_i \neq 0$; i.e., its domain of analyticity has no real points. Call this domain, which has the form of a tube with $\mathrm{Re}\zeta_i$ arbitrary and $\mathrm{Im}\zeta_i \in V_-$, the tube $\mathcal{T}_{n-1}$. Now we restore the labels of the fields and use a profound result due to Bargmann, Hall and Wightman[15] which applies to this case. If

$$W^{(n;\wp)}(\Lambda\zeta_1, \Lambda\zeta_2, \ldots, \Lambda\zeta_{n-1}) = \left[\prod_1^n S^{(p_i)}(A)\right] W^{(n;\wp)}(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}) \tag{17}$$

for the covering group, $SL(2, C)$, of real $\Lambda \in L_+^\uparrow$ then $W^{(n;\wp)}(\zeta_1, \zeta_2, \ldots, \zeta_{n-1})$ has a (unique) single-valued analytic continuation to the domain $\mathcal{T}'_{n-1}$, that we call the extended tube,[5] that is the union of all $\Lambda\mathcal{T}_{n-1}$ where now we have complex matrices $\Lambda \in L_+(C)$. This enlargement of the domain of analyticity leads to two crucial results. First, in contrast to $\mathcal{T}_{n-1}$, our new, larger domain of analyticity, $\mathcal{T}'_{n-1}$, contains real points of analyticity ρ_j that we will discuss below. Secondly, since $\mathcal{T}'_{n-1}$ is invariant under complex

[5] Although technically it is not a tube in the variables $(\zeta_1, \ldots, \zeta_{n-1})$.

Lorentz transformations in $L_+(C)$, one of which is space–time inversion, we have the relation

$$W^{(n;\wp)}(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}) = (-1)^L W^{(n;\wp)}(-\zeta_1, -\zeta_2, \ldots, -\zeta_{n-1}), \;\; L = \sum l_i \tag{18}$$

in $\mathcal{T}'_{n-1}$. To see where the factor $(-1)^L$ comes from we repeat that for $\Lambda \in L_+^\uparrow$, Λ depends on the matrices A and A^* in $SL(2, C)$; however, as mentioned in a footnote above, now that we have the extension to $L_+(C)$, Λ depends on two independent matrices A and B in $SL(2, C)$ and we can transform continuously from the identity to space–time inversion. The $S^{(k,l)}$ matrices for the representation of space–time inversion are just powers of (-1); thus if we choose $A = 1$, $B = -1$ then $S^{(k,l)}(1, -1) = (-1)^l \mathbf{1}$.[6] Note that the representation of the complex Lorentz group, including space–time inversion, that we are discussing here is unitary, not antiunitary.

Jost gave a precise characterization of the real points, ρ_j, in $\mathcal{T}'_{n-1}$: $\sum_1^{n-1} \lambda_i \rho_i \sim 0$, for all real $\lambda_i \geqslant 0$ such that $\sum_1^{n-1} \lambda_i > 0$. This requires that each $\rho_i \sim 0$.

Jost's result is particularly simple for the $W^{(2)}$ function for a single scalar field in which there is one complex difference vector ζ. Since $W^{(2)}(\Lambda\zeta) = W^{(2)}(\zeta)$ we can find the extended tube $\mathcal{T}'_1$ by finding the values of ζ^2 that can be obtained from $\Lambda\zeta$ with $\zeta = \xi + i\eta$, $\eta \in V_-$. Then $\zeta^2 = \xi^2 - \eta^2 + 2i\xi \cdot \eta$. The real points are those for which $\xi \cdot \eta = 0$, with $\eta \in V_-$. These points are the space–like points $\xi \sim 0$, in agreement with Jost's general result.

6. THE GENERAL FORMULA FOR $\mathcal{CPT}$.

When we write the relation between Wightman functions at Jost points that comes from space–time inversion, Eq. (18), in terms of vacuum matrix elements we find

$$\begin{aligned}&\langle 0|\phi^{(k_1,l_1)}(x_1)\phi^{(k_2,l_2)}(x_2)\ldots\phi^{(k_n,l_n)}(x_n)|0\rangle \\ &\quad = (-1)^L\langle 0|\phi^{(k_1,l_1)}(-x_1)\phi^{(k_2,l_2)}(-x_2)\ldots\phi^{(k_n,l_n)}(-x_n)|0\rangle.\end{aligned} \tag{19}$$

We should not expect that this relation which comes from a unitary representation of space–time inversion will lead to a useful symmetry of the

[6] Here $\mathbf{1}$ is the direct product of a $(2k+1)\times(2k+1)$ unit matrix and a $(2l+1)\times(2l+1)$ unit matrix and we find the result just stated for L.

theory, because if the spectrum has the momenta q_i in or on the forward light cone so that the time dependence of the Wightman function on the left-hand side for the difference coordinate goes as $\exp(-iq_i^0(x_i^0-x_{i+1}^0))$ as we expect for positive energy then the corresponding time dependence on the right-hand side will go as $\exp(iq_i^0(x_i^0-x_{i+1}^0))$ which occurs for negative energy. This problem will arise for any transformation that includes $t \to -t$. Wigner resolved this problem by realizing that transformations that involve time reversal must be represented by antiunitary operators rather than by unitary operators.(16) In our context this situation is reflected by the fact that although each side of this last equation can be analytically continued; the left-hand side to complex ζ_i with $\eta = \mathrm{Im}\zeta \in V_-$ and the right-hand side to complex $-\zeta_i$ with $-\eta = -\mathrm{Im}\zeta \in V_+$, we cannot take the limit as $\eta_i \to 0$ to get a relation between vacuum matrix elements of products of the fields, because, as just noted, if ζ_i has its imaginary part in the backward cone, then $-\zeta_i$ has its imaginary part in the forward cone and then the analytic continuations of the functions on the two sides of Eq. (18) are not valid in the same domain. On the other hand, if we consider the vacuum matrix element with the fields in completely reversed order,

$$\langle 0|\phi^{(p_n)}(-x_n)\ldots\phi^{(p_2)}(-x_2)\phi^{(p_1)}(-x_1)|0\rangle, \tag{20}$$

which corresponds to

$$W^{(n;i\wp)}(\xi_{n-1},\ldots,\xi_2,\xi_1) \tag{21}$$

in terms of difference variables, where $i\wp$ stands for $p_n \ldots p_2 p_1$, both functions will have the same domain, $\mathcal{T}'_{n-1}$, of analyticity. This is precisely where we have to assume something beyond Lorentz covariance.[7] To reverse the order of all the fields we assume that at a Jost point the two vacuum matrix elements are related by a sign $(-1)^I$ where I is the number of transpositions of Fermi fields necessary to invert the order of the fields. This is implied by the spin-locality theorem[8] but is weaker than that theorem since we need this relation only at Jost points (or even only in a neighborhood of a Jost point) in each matrix element, rather than as an operator relation. With F Fermi fields $I = (F-1)+(F-2)+\cdots+1 = F(F-1)/2$. Since the number of Fermi fields in a non-vanishing vacuum matrix element must be even, $F-1$

[7] If we choose Lorentz covariance of time-ordered products as our condition of Lorentz covariance of the theory instead of Lorentz covariance of Wightman products, then this additional assumption is unnecessary.(17)

[8] This condition is usually called the spin-statistics theorem. We have argued that in the present context it should be called the spin-locality theorem.(18)

must be odd; thus the phase that enters is $(-1)^{F(F-1)/2} = ((-1)^{(F-1)})^{F/2} = (-1)^{F/2} = i^F$. The condition on matrix elements that

$$\begin{aligned}&\langle 0|\phi^{(k_1,l_1)}(x_1)\phi^{(k_2,l_2)}(x_2)\ldots\phi^{(k_n,l_n)}(x_n)|0\rangle\\&\quad= i^F\langle 0|\phi^{(k_1,l_1)}(x_n)\ldots\phi^{(k_2,l_2)}(x_2)\phi^{(k_n,l_n)}(x_1)|0\rangle\end{aligned} \tag{22}$$

at Jost points is called "weak local commutativity." Clearly local commutativity (sometimes called microcausality) implies weak local commutativity. Combining space–time inversion and weak local commutativity and collecting phases, we have the following relation between vacuum matrix elements valid in an open neighborhood of Jost points,

$$\begin{aligned}&\langle 0|\phi^{(k_1,l_1)}(x_1)\phi^{(k_2,l_2)}(x_2)\ldots\phi^{(k_n,l_n)}(x_n)|0\rangle\\&\quad= i^F(-1)^L\langle 0|\phi^{(k_n,l_n)}(-x_n)\ldots\phi^{(k_2,l_2)}(-x_2)\phi^{(k_1,l_1)}(-x_1)|0\rangle.\end{aligned} \tag{23}$$

Because an open neighborhood of Jost points is in the domain of analyticity $\mathcal{T}'_{n-1}$ this last relation holds for the analytic functions,

$$W^{(n;\wp)}(\zeta_1,\zeta_2,\ldots,\zeta_{n-1}) = i^F(-1)^L W^{(n;i\wp)}(\zeta_{n-1},\ldots,\zeta_2,\zeta_1). \tag{24}$$

Now we can take $\mathrm{Im}\zeta_i \to 0, \mathrm{Im}\zeta \in V_-$, on both sides and get an equality between distributions for all ξ_i,

$$W^{(n;\wp)}(\xi_1,\xi_2,\ldots,\xi_{n-1}) = i^F(-1)^L W^{(n;i\wp)}(\xi_{n-1},\ldots,\xi_2,\xi_1). \tag{25}$$

Translated back into vacuum matrix elements this says

$$\begin{aligned}&\langle 0|\phi^{(p_1)}(x_1)\phi^{(p_2)}(x_2)\ldots\phi^{(p_n)}(x_n)|0\rangle\\&\quad= i^F(-1)^L\langle 0|\phi^{(p_n)}(-x_n)\ldots\phi^{(p_2)}(-x_2)\phi^{(p_1)}(-x_1)|0\rangle.\end{aligned} \tag{26}$$

Replacing (p_j) by (k_j, l_j) we have

$$\begin{aligned}&\langle 0|\phi^{(k_1,l_1)}(x_1)\phi^{(k_2,l_2)}(x_2)\ldots\phi^{(k_n,l_n)}(x_n)|0\rangle\\&\quad= i^F(-1)^L\langle 0|\phi^{(k_n,l_n)}(-x_n)\ldots\phi^{(k_2,l_2)}(-x_2)\phi^{(k_1,l_1)}(-x_1)|0\rangle.\end{aligned} \tag{27}$$

We can restore the original order of the fields on the right-hand side by using the hermiticity of the scalar product, $(\Psi, \Xi) = (\Xi, \Psi)^*$. The appearance of complex conjugation is fine, since we know that $\mathcal{CPT}$ is antiunitary. We could not use a unitary operator to represent space–time inversion because that would relate positive energy states to negative energy states. To correct this problem we had to use an antiunitary operator to represent space–time inversion and this antiunitary operator produced the adjoint of the fields, i.e., charge conjugation and we ended up with $\mathcal{CPT}$. We find

$$\langle 0|\phi^{(k_1,l_1)}(x_1)\phi^{(k_2,l_2)}(x_2)\ldots\phi^{(k_n,l_n)}(x_n)|0\rangle = i^F(-1)^L\langle 0|\phi^{(k_1,l_1)\dagger}(-x_1)\phi^{(k_2,l_2)\dagger}(-x_2)\ldots\phi^{(k_n,l_n)\dagger}(-x_n)|0\rangle^*, \quad (28)$$

where $F=\sum_1^n f_i$ and f is zero for a Bose field (with $k+l$ even) and one for a Fermi field (with $k+l$ odd). Now we can read off what $\mathcal{CPT}$, which for brevity we call Θ, must be,

$$\Theta\phi^{(k,l)}(x)\Theta^\dagger=(-1)^l i^f\phi^{(k,l)\dagger}(-x). \quad (29)$$

Because a non-vanishing vacuum matrix element must have an even number of Fermi fields, we could have chosen $(-i)^f$ instead of i^f in Eq. (29). This ambiguity is equivalent to the ambiguity in choosing $A=1, B=-1$, rather than $A=-1, B=1$ when we inverted the space–time coordinates, since for a Fermi field $k+l=1, \text{mod } 2$. When we embed Eq. (29) in an arbitrary vacuum matrix element and use the invariance of the vacuum, $\Theta|0\rangle=|0\rangle$, we find precisely Eq. (28)! When we run this sequence of relations the other way, we conclude that weak local commutativity in the neighborhood of a Jost point is necessary and sufficient for $\mathcal{CPT}$.

Note that $\mathcal{CPT}$ takes each irreducible of $L_+^\uparrow$ to a phase times its adjoint, for example, the part $\phi^{(1,0)}$ of the Dirac spin-1/2 field is mapped to $i\phi^{(1,0)\dagger}$ and the part $\phi^{(0,1)}$ is mapped to $-i\phi^{(0,1)\dagger}$. Both the vector and axial vector fields have the form $\phi^{(1,1)}$ so these fields are indistinguishable under Θ and both get the phase (-1). The analogous statements hold for the scalar and pseudoscalar fields, $\phi^{(0,0)}$, which both get phase 1. The antisymmetric rank two tensor field $T^{\mu\nu}$, $\phi^{(2,0)}+\phi^{(0,2)}$, and the traceless symmetric tensor of rank two, $\phi^{(2,2)}$, also both get phase 1.

The $\mathcal{CPT}$ operator Θ interchanges undotted and dotted indices, so that the Θ transform of a field $\phi^{(k,l)}(x)$ transforms as a field $\psi^{(l,k)}(x)$. Under Θ, particles and antiparticles are interchanged (some particles may be identical to their antiparticles). Energies and momenta stay the same; spin components and helicities are reversed.

When we act twice by Θ we use

$$\Theta\phi^{(k,l)\dagger}(x)\Theta^\dagger=(-1)^l(-i)^f\phi^{(k,l)}(-x). \quad (30)$$

and find

$$\begin{aligned}\Theta^2\phi^{(k,l)}(x)\Theta^{\dagger 2} &= \Theta(-1)^l i^f \phi^{(k,l)\dagger}(-x)\Theta^\dagger \\ &= (-1)^l(-i)^f \Theta\phi^{(k,l)\dagger}(-x)\Theta^\dagger \\ &= (-1)^f \phi^{(k,l)}(x). \end{aligned} \tag{31}$$

so Θ^2 commutes with Bose fields and anticommutes with Fermi fields. The reader can check that the phase of Θ^2 cannot be changed by changing a phase in the definition of Θ. This is true for all antiunitary operators.

7. $\mathcal{CPT}$ FOR THE S-MATRIX

Because Θ reverses time, in and out states are interchanged. Taking the antiunitarity of Θ into account, the S-matrix obeys

$$S_{\alpha,\beta} \equiv_{\text{out}} \langle\alpha|\beta\rangle_{\text{in}} =_{\text{out}} \langle\hat{\beta}|\hat{\alpha}\rangle_{\text{in}} = S_{\hat{\beta},\hat{\alpha}}, \tag{32}$$

where $|\hat{\alpha}\rangle$ has particles and antiparticles exchanged, spin components and helicities reversed, and energies and momenta the same as in $|\alpha\rangle$.

In terms of the S-operator this is

$$\Theta S \Theta^\dagger = S^{-1}, \quad \text{or} \quad \Theta S = S^{-1}\Theta. \tag{33}$$

The results of the axiomatic and Lagrangian $\mathcal{CPT}$ theorems are the same. Masses of particles and antiparticles must be equal. Total lifetimes and widths of particles and antiparticles must be equal. Energies and three-momenta of particles are preserved under $\mathcal{CPT}$; spins and helicities are reversed. Reactions proceed in the reverse direction under $\mathcal{CPT}$.

8. SUMMARY

Jost's general proof of the $\mathcal{CPT}$ theorem leads directly to a definition of the $\mathcal{CPT}$ transformation applied to fields belonging to an arbitrary irreducible representation of $SL(2, C)$, the covering group of the real proper orthochronous Lorentz group, $L_+^\uparrow$. In the real Lorentz group space–time inversion is not connected to the identity; however in the complex Lorentz group $x \to -x$ is connected to the identity. Using the Wightman analytic functions that are analytic continuations of vacuum matrix elements of products of the fields, the Bargmann–Hall–Wightman theorem allows analytic continuation of the Wightman functions from (the covering group) of

$L_+^\uparrow$ to (the covering group of) the complex Lorentz group $L_+(C)$ which now allows space–time inversion. In the larger domain of analyticity given by the Bargmann–Hall–Wightman theorem there are real points of analyticity, the Jost points; however we cannot take limits to get a general relation between the matrix elements at the original points and at the space–time inverted points. However, if we invert the order of the fields, we can get the general relation. This is the only step where we must assume something beyond Lorentz covariance: we must assume weak local commutativity at Jost points to allow the reordering. We can restore the original order of the fields by using the hermiticity of the scalar product, which is not an additional assumption. This step complex conjugates the matrix elements which means, as we expect, that $\mathcal{CPT}$ is antiunitary rather than unitary. Now we are able to read off the general, simple result Eq. (29) for $\mathcal{CPT}$ on each irreducible of (the covering group of) the Lorentz group, $L_+^\uparrow$.

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APPENDIX

A. Alternative Description of the Irreducible Representations of the Lorentz Group

An alternative way to describe the irreducible representations of the Lorentz group is to combine the rotation generators, J_i, with the boost (pure Lorentz transformation) generators, K_i, into a pair of commuting $SU(2)$ generators A_i and B_i. The irreducible fields are $\psi^{(A,B)}$ where A and B are the spins associated with each of the $SU(2)$ algebras.[19] The relation between these two descriptions is $\phi^{(2A,2B)} = \psi^{(A,B)}$ The reader can check that the adjoint of $\phi^{(k,l)}$ transforms as a field $\chi^{(l,k)}$; that is the dotted and undotted indices get interchanged. For this reason we don't have to talk about adjoints of the fields; they are taken care of if we allow an arbitrary irreducible—they don't introduce anything new.

B. Detailed Form of the Transformation Law for the Fields

$$U(a, A)\phi^{(k,l)}_{\alpha_1\ldots\alpha_k\dot{\beta}_1\ldots\dot{\beta}_l}(x)U^{\dagger}(a, A)$$
$$= A^{-1}_{\alpha_1\alpha'_1}\ldots A^{-1}_{\alpha_k\alpha'_k}A^{-1*}_{\dot{\beta}_1\dot{\beta}'_1}\ldots A^{-1*}_{\dot{\beta}_l\dot{\beta}'_l}\phi^{(k,l)}_{\alpha'_1\ldots\alpha'_k\dot{\beta}'_1\ldots\dot{\beta}'_l}(\Lambda(A)x + a), \quad \text{(B.1)}$$

for $\Lambda(A) \in L^{\uparrow}_{+}$, and

$$U(a, A, B)\phi^{(k,l)}_{\alpha_1\ldots\alpha_k\dot{\beta}_1\ldots\dot{\beta}_l}(x)U^{\dagger}(a, A, B)$$
$$= A^{-1}_{\alpha_1\alpha'_1}\ldots A^{-1}_{\alpha_k\alpha'_k}B^{T}_{\dot{\beta}_1\dot{\beta}'_1}\ldots B^{T}_{\dot{\beta}_l\dot{\beta}'_l}\phi^{(k,l)}_{\alpha'_1\ldots\alpha'_k\dot{\beta}'_1\ldots\dot{\beta}'_l}(\Lambda(A, B)x + a), \quad \text{(B.2)}$$

for $\Lambda(A, B) \in L_{+}(C)$, where ϕ is symmetric in the α's and in the $\dot{\beta}$'s separately for both cases.

C. Qualitative Difference of Domains of Analyticity for Functions of Several Complex Variables Compared to those of a Single Complex Variable

Readers can ignore this appendix which is not necessary for our discussion of the $\mathcal{CPT}$ theorem. Functions of several complex variables differ qualitatively from functions of a single complex variable in their possible domains of analyticity. For a single complex variable, for every domain in the complex plane bounded by a smooth curve there is an analytic function that cannot be continued outside this domain. Thus any such region is the domain of analyticity for some function. For several complex variables this is not true. Domains of analyticity must be "holomorphically convex." Intuitively such domains must not have "dimples" that can be removed by analytically continuing *any* analytic function across the dimple using a Cauchy contour that surrounds the dimple. The additional dimensions available for several complex variables is what allows this analytic continuation. This possibility of analytic continuation comes into play when the commutativity or anticommutativity of fields at space-like separation is imposed on the Wightman functions. In the case of interest for the $\mathcal{CPT}$ theorem, which involves reversing all the fields in the vacuum matrix element, the new and old extended tubes agree, so further analytic continuation is not possible.

D. Lorentz Covariance Alone does not Suffice

Lorentz covariance alone is not sufficient for $\mathcal{CPT}$. A single example suffices to show this. A free or generalized free field can be Lorentz covariant but not obey $\mathcal{CPT}$ invariance if the particle and antiparticle masses are different.[20] What fails in that case is that weak local commutativity (see Eq. (22)) does not hold at Jost points. This possibility is associated with the purely time-like support in momentum space of free or generalized free fields; for time-like momenta positive and negative energies can be separated in a covariant way. By contrast positive and negative energies can be transformed into each other for space-like momenta. Note that although the fields in these examples transform covariantly their time-ordered products are not covariant. Thus if we require that time-ordered products be covariant as part of Lorentz covariance of a theory then, as shown in Ref. 20, free fields that violate $\mathcal{CPT}$ are not covariant. See Ref. 21 for a detailed analysis of hybrid Dirac fields ("homeotic" fields[22]) which can be covariant only when they are non-interacting but even in the free case have time-ordered products that are not covariant.

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