The Quantization of Linear Dynamical Systems II: Infinite Systems

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This document, and its predecessor, *Quantization of Linear Dynamical Systems I* (mostly about systems with *finitely* many degrees of freedom), expound a rigorous quantization procedure developed by Irving Segal and others in the 1960s. This means we do not here cover algebraic quantum theory; which will include e.g. inequivalent representations, 'getting out of Fock space', Haag's theorem etc. (cf. eg Emch 1972); and which will be used in discussing e.g. the Unruh effect and elements of QFT on curved spacetimes.

The 'bottom-line' for the two documents taken together is that we have a procedure for quantizing (ie. constructing a representation of the Weyl algebra for) any of a special class of classical systems. The simple harmonic oscillator and the free real bosonic field both belong to this class; but of these two, only for the former (the finite system) does this construction pick out a unique representation.

We begin in Section 1 by recalling from Part I:

(i) quantization as the construction of a representation of the *Weyl algebra* associated with some classical system's phase space (endowed with suitable complex structure); and as "unitarizing" a Hamiltonian evolution in a symplectic space so as to give an evolution in a complex Hilbert space; cf. Sections 1-3 of Part I

(ii) the ideas of a *one-particle structure* and of Fock space, i.e. symmetric Fock space built on any one-particle structure without regard to the details of dynamics; cf. Section 4 of Part I;

(iii) the Stone-von Neumann Theorem, which essentially guarantees that the quantization of the paradigm finite classical system, viz. point particles in $\mathbb{R}^n$, is unique (up to unitary equivalence); and its "fermionic cousin" the Jordan-Wigner theorem; cf. Section 6 of Part I.

Then we work up slowly to the free real bosonic Klein-Gordon field. We first look at two ways the premises of the Stone-von Neumann Theorem can fail: viz. with

(a) failure of weak continuity (Section 2);

(b) a classical configuration space other than $\mathbb{R}^n$, e.g. the circle S_1 (Section 3).

Then we look at an infinite spin chain, as an example where the premises of the Jordan-Wigner Theorem fail. This is an instructive system because one can easily show that unitary equivalence (of representations of the CARs) fails (Section 4).

Finally, section 5 focusses exclusively on the free real bosonic field, subject to the Klein-Gordon equation, and various interpretative issues, including particle localization and the interpretation of the local field operators $\Phi(\mathbf{x})$.

Contents

Sections 1 to 4 owe much to Chapters 2 and 3 of Ruetsche (2011). Section 5 is based on Baez *et al* (1992, Section 1) and Halvorson (2001).

1 Canonical quantization of finite systems: recalled

1.1 Quantization as representations of the Weyl algebra

(This summarises Section 1 of Part I.) A familiar way of developing elementary quantum mechanics is to "promote" the classical Poisson bracket relations

$$\{q^i, q^j\} = \{p_i, p_j\} = 0; \qquad \{q^i, p_j\} = \delta^i_j, \tag{1}$$

where $i, j \in \{1, 2, \dots n\}$, to the *Heisenberg relations* (CCRs)

$$[Q^i, Q^j] = [P_i, P_j] = 0; \qquad [Q^i, P_j] = i\delta^i_j \mathbb{1}; \tag{2}$$

(where $\hbar := 1$) and to seek a representation of these quantities as self-adjoint operators on a Hilbert space. However, in hindsight, we know to expect the Q^is and P_js to have unbounded spectra, and therefore to not be fully defined on the space $L^2(\mathbb{R}^n)$ of square-integrable functions. This nuisance can be remedied by instead turning to the *Weyl form* of the CCRs.

Define, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$U(\mathbf{a}) := e^{-i\mathbf{a}.\mathbf{P}}; \qquad V(\mathbf{b}) := e^{-i\mathbf{b}.\mathbf{Q}}; \tag{3}$$

Then, given (2), we have

$$U(\mathbf{a})V(\mathbf{b}) = e^{i\mathbf{a}.\mathbf{b}} V(\mathbf{b})U(\mathbf{a}). \tag{4}$$

Since the Us and Vs are both families of unitaries, their spectra are bounded, and are defined everywhere on $L^2(\mathbb{R}^n)$. We may take (4) as the primitive CCRs; our task is then to find representations of the Us and Vs.

But we are only halfway to our intended framing of the representation problem. Equation (4) can be given a more abstract presentation, which unifies the quantization of particles and bosonic fields. Setting $z := (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$, we define the family of operators

$$W(z) := e^{\frac{1}{2}i\mathbf{a}.\mathbf{b}} U(\mathbf{a}) V(\mathbf{b}). \tag{5}$$

Then the Weyl form of the CCRs (4) are equivalent to the *Weyl algebra*

$$\begin{aligned} W(z_1)W(z_2) &= e^{\frac{1}{2}i\Omega(z_1,z_2)} W(z_1+z_2); \\ W^\dagger(z) &= W(-z); \end{aligned} \tag{6}$$

for all $z, z_1, z_2 \in \mathbb{R}^{2n}$, where Ω is the *symplectic product*:

$$\Omega(z_1, z_2) := \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2, \tag{7}$$

to be explained shortly. Importantly, the Weyl algebra (6), though abstract, may successfully be extended to bosonic fields.

1.2 Symplectic vector spaces and manifolds; linear systems

(This repeats from Part I: Section 3.3 and then part of Section 3.6.) If we are lucky enough for our classical phase space to be vector space (as when $S = \mathbb{R}^{2n}$), then we can make it a *symplectic vector space*, which is a pair (S, Ω), where S is a phase space—also a vector space—and Ω is a symplectic product. The symplectic product $\Omega : S \times S \to \mathbb{R}$ is, by definition, anti-symmetric, linear and non-degenerate (i.e. if $\Omega(z_1, z_2) = 0$ for all z_2, then $z_1 = \mathbf{0}$).

We define the symplectic product on $S = \mathbb{R}^{2n} \ni z_1, z_2$ as in (7). Note that $\Omega(z, \cdot) : S \to \mathbb{R}$ is a real-valued function on S, and so a classical observable. In particular, $\Omega(z, \cdot) = q^i$ iff z has $(n+i)$th component $b_i = 1$ and the rest 0, and $\Omega(z, \cdot) = p_i$ iff z has ith component $a^i = -1$ and the rest 0. In general, $\Omega(z, \cdot)$ is some linear combination of p_is and q^is. In this formulation, the classical Poisson bracket relations (1) may be written

$$\{\Omega(z_1, \cdot), \Omega(z_2, \cdot)\} = -\Omega(z_1, z_2), \tag{8}$$

the corresponding Heisenberg form of the CCRs are

$$[\hat{\Omega}(z_1, \cdot), \hat{\Omega}(z_2, \cdot)] = -i\Omega(z_1, z_2)\mathbb{1}, \tag{9}$$

where (in the sought representation) the map $z \mapsto \hat{\Omega}(z, \cdot)$ takes elements of S to self-adjoint operators, and the Weyl unitaries are defined by

$$W(z) := e^{i\hat{\Omega}(z,\cdot)}. \tag{10}$$

This is Wald's presentation: see Wald (1994, Ch. 2). Later we will use field operators Φ, for which $\Phi(Jz) = \hat{\Omega}(z, \cdot)$, or $\Phi(z) = -\hat{\Omega}(Jz, \cdot) = \hat{\Omega}(\cdot, Jz)$.

Symplectic manifolds, more generally:— In the case where the classical phase space S is not a vector space, we must resort to a longer route. In this case, we seek a group whose action on S is *transitive* and preserves the symplectic form $\omega := \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$. (In the case that S is a vector space, this group is just the (abelian) additive group of translations in S, which is isomorphic to S. That is what allowed us to treat S as a symplectic vector space above.) For illustration, taking the case $S = \mathbb{R}^{2n}$, the group action is a $2n$-parameter family of diffeomorphisms associated with the vector fields (with constant coefficients)

$$X_z = \sum_{i=1}^{n} b_i \frac{\partial}{\partial q^i} - a^i \frac{\partial}{\partial p_i}, \tag{11}$$

for any $z := (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$. We may now act on any two such vector fields with the *symplectic form* ω with which S, being a classical phase space, is equipped. This yields

$$\omega(X_{z_1}, X_{z_2}) = \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2. \tag{12}$$

Our quantization problem then becomes the search for continuous families of unitaries $z \mapsto W(z)$ which respect this symplectic structure, as expressed in the Weyl algebra (6), setting $e^{\frac{1}{2}i\Omega(z_1,z_2)} = e^{\frac{1}{2}i\omega(X_{z_1},X_{z_2})}$. Since the Weyl algebra (6) is unitary up to the phase factor $e^{\frac{1}{2}i\omega(X_{z_1},X_{z_2})}$, it is a *projective unitary representation* of the group of symplectomorphisms on S.

Quadratic Hamiltonians and linear systems— We spell out how a Hamiltonian being quadratic implies that time-evolution preserves linear structure. So let the phase space Γ be a symplectic vector space with global coordinates (q, p). We write ξ^α, with α running from 1 to $2n$.

We now *define* a linear system as one in which the Hamiltonian is a quadratic form $H_{\alpha\beta}$ in the ξs. That is: the energy $= H = (\xi^\alpha)^T[H_{\alpha\beta}\xi^\beta]$. Then taking partial derivatives of the energy H with respect to any ξ^α (holding all other ξ^α constant of course) will give: a linear combination of the various ξ^β, i.e. a linear combination with constant coefficients. Call it $a_\alpha\xi^\alpha$ (with summation convention).Then ∇H is the column of these partial derivatives. Multiplying ∇H by the symplectic matrix keeps it a linear combination. So the Hamiltonian vector field is a linear combination of the various ξ^β with constant coefficients. Call it $b_\alpha\xi^\alpha$ (with summation convention)

So at each point $\xi = (q, p) \in \Gamma$, the infinitesimal flow is: $b_\alpha\xi^\alpha$. Then it is trivial that the time-evolution preserves the linear structure of solutions. For take two points: $\xi_1 = (q_1, p_1)$ and $\xi_2 = (q_2, p_2)$. At the sum-state got by superposing these states, $\xi_{1+2} := (q_1 + q_2, p_1 + p_2)$, the infinitesimal flow is by definition: $b_\alpha\xi^\alpha_{1+2}$. But this is: $b_\alpha(\xi^\alpha_1 + \xi^\alpha_2) = b_\alpha(\xi^\alpha_1) + b_\alpha(\xi^\alpha_2)$.

In short: The sum of two instantaneous states has as its infinitesimal Hamiltonian flow (tangent vector in phase space) the sum of the two states' individual Hamiltonian flows (tangent vectors).

1.3 One-particle structures

(This repeats passages of Section 4 from Part I.) There are two core ideas of the Segal quantization of a linear classical system.

First: there is a map K from the solution space of a classical linear system, i.e. a symplectic vector space, to a Hilbert space. K is required to satisfy conditions that combine the ideas of complex structures and symplectic structures, in such a way that the Hilbert space is determined. In short: we choose a complex structure J that preserves and tames the symplectic form, and thereby complexify the real vector space and define a Hilbert space; (such a complex structure J is not unique). Besides, K is determined as having a unitary dynamics that is the "unitary cousin" of the classical system's dynamics. K, or the Hilbert space to which it leads, is called a *one-particle structure*.

Second: there is the usual Fock space construction, which will be applied to the one-particle structure's Hilbert space (i.e. after the first idea has been implemented). So here, the phrase 'one-particle' signals that the Hilbert space is the first (non-zero, i.e. non-vacuum) summand of the usual Fock space sum of ever larger tensor powers.

In Part I, we saw this illustrated for the harmonic oscillator (in one spatial dimension). Starting with classical harmonic oscillator, the first idea delivers us as the quantum state space— not the familiar quantum harmonic oscillator, with (in one spatial dimension) Hilbert space $L^2(\mathbb{R})$!—but 'merely' the world's simplest complex Hilbert space, viz. $\mathbb{C}$ i.e. the complex plane.

To get the familiar quantum harmonic oscillator, i.e. $L^2(\mathbb{R})$ (equipped with the quantum harmonic oscillator Hamiltonian), we need to take the Fock space built from $\mathbb{C}$. That Fock space will "be" (i.e. be a Hilbert space isomorphic to) $L^2(\mathbb{R})$. So we in effect *factorize* the usual understanding of canonical quantization—viz. (for the 1-dimensional harmonic oscillator) "replace the two-dimensional classical phase space $\mathbb{R}^2 \ni (q,p)$, with $L^2(\mathbb{R})$, i.e. L^2 functions on the configuration space $\mathbb{R}$—into: *first*, build a 1-particle structure; *second*, build the Fock space.

Here is a bit more detail about the first idea. (We postpone review of the second idea until later.)

We begin with the triple, $(\mathcal{S}, \Omega, \Phi_t)$, where $\mathcal{S}$ is a symplectic vector space, the 'phase/solution space' for a Hamiltonian system, Ω is its symplectic product, and Φ_t for the one-parameter group of motions (i.e. symplectomorphisms) along the integral curves of the Hamiltonian vector field X_h. We add a complex structure J that:

1. is a symplectomorphism; i.e. $\Omega(Jz_1, Jz_2) = \Omega(z_1, z_2)$ (it follows that $[J, \Phi_t] = 0$, i.e. J is equivariant under the classical dynamics);
2. "tames" Ω in that $\Omega(z, Jz) > 0$, for all $z \neq 0$.

Given this J, we define a complex inner product on (S, Ω, Φ_t, J):

$$\langle z_1, z_2 \rangle_{\mathcal{S}} := \frac{1}{2}\Omega(z_1, Jz_2) + \frac{1}{2}i\Omega(z_1, z_2), \tag{13}$$

Note: The real part of this definition is using the idea that given a symplectic vector space V, with symplectic product ω, one can define a *complex-linear* but *real-valued* symmetric bilinear form g_J on the complex vector space V_J by: $g_J(u,v) := \omega(u, Jv)$. Then we use the idea that we can define a *sesquilinear*, *complex-valued* function on $V \times V$, i.e. complex inner product, in terms of g_J and ω, by: $\langle u, v \rangle \equiv \langle u, v \rangle_{\omega,J} := g_J(u,v) + i\omega(u,v)$.

1.4 The Stone-von Neumann and Jordan-Wigner uniqueness theorems

(This repeats passages of Section 6 from Part I.)

Theorem 1.1 (Stone-von Neumann Uniqueness Theorem)**.** *Let (S,Ω) be a symplectic vector space, with $S = \mathbb{R}^{2n}$. Every weakly continuous irreducible representation of the Weyl algebra over (S,Ω) is unitarily equivalent to the Schrödinger representation, in which, for all $\psi(\mathbf{x}) \in L^2(\mathbb{R}^n)$,*

$$(W(\mathbf{a},\mathbf{b})\psi)(\mathbf{x}) := e^{-i\mathbf{a}.(\mathbf{x}-\frac{1}{2}\mathbf{b})}\psi(\mathbf{x}-\mathbf{b}). \tag{14}$$

Note as special cases that $(W(\mathbf{a},\mathbf{0})\psi)(\mathbf{x}) \equiv (U(\mathbf{a})\psi)(\mathbf{x}) = \psi(\mathbf{x}-\mathbf{a})$ and $(W(\mathbf{0},\mathbf{b})\psi)(\mathbf{x}) \equiv (V(\mathbf{b})\psi)(\mathbf{x}) = e^{-i\mathbf{b}.\mathbf{x}}\psi(\mathbf{x})$. In fact, the Schrödinger representation is strongly continuous, so by Stone's Theorem there are $2n$ self-adjoint operators, Q^i and P_i, such that $U(\mathbf{a}) = e^{-i\mathbf{a}.\mathbf{P}}$, $V(\mathbf{b}) = e^{-i\mathbf{b}.\mathbf{Q}}$ and for all $\psi(\mathbf{x}) \in L^2(\mathbb{R}^n)$ in suitable domains,

$$(\mathbf{Q}\psi)(\mathbf{x}) = \mathbf{x}\psi(\mathbf{x}); \qquad (\mathbf{P}\psi)(\mathbf{x}) = -i\nabla\psi(\mathbf{x}). \tag{15}$$

For the Jordan-Wigner theorem for the CARs, we consider first a sequence of quantum theories, each corresponding to a chain of spin-$\frac{1}{2}$ systems. The first theory describes a single spin-$\frac{1}{2}$ system, with observables $\{\sigma(x), \sigma(y), \sigma(z)\}$, which satisfy the Pauli relations

$$[\sigma(x), \sigma(y)] = 2i\sigma(z) \quad \text{and cyclic perms;} \qquad \sigma^2 := \sigma(x)^2 + \sigma(y)^2 + \sigma(z)^2 = 3\mathbb{1}. \tag{16}$$

This is equivalent to satisfying the canonical *anti*-commutation relations (CARs; see eg p.60-61 of Ruestsche 2011),

$$d^2 = \left(d^\dagger\right)^2 = 0; \qquad [d, d^\dagger]_+ = 1; \tag{17}$$

where

$$\sigma(x) = d + d^\dagger; \quad \sigma(y) = -i\left(d - d^\dagger\right); \quad \sigma(z) = dd^\dagger - d^\dagger d. \tag{18}$$

We now consider a theory describing a linear chains of n spin-$\frac{1}{2}$ systems, with observables $\{\sigma_k(x), \sigma_k(y), \sigma_k(z) \mid k \in \{1, 2, \ldots n\}\}$, satisfying

$$[\sigma_j(x), \sigma_k(y)] = 2i\delta_{jk}\sigma_k(z) \quad \text{and cyclic perms;} \qquad \sigma_k^2 := \sigma_k(x)^2 + \sigma_k(y)^2 + \sigma_k(z)^2 = 3\mathbb{1}. \tag{19}$$

Of course, our theory falls outside the scope of the Stone-von Neumann theorem, because it is characterized by CARs, rather than CCRs. However, there is an analogous uniqueness theorem:

Theorem 1.2 (Jordan-Wigner Uniqueness Theorem)**.** *For each finite n, every irreducible representation of the CARs (equivalently, the Pauli relations) is unitarily equivalent to the Pauli representation, in which*

$$\begin{aligned} \sigma_k^P(x) &= \underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{k-1}\otimes\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\otimes\underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{n-k}; \\ \sigma_k^P(y) &= \underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{k-1}\otimes\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\otimes\underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{n-k}; \\ \sigma_k^P(z) &= \underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{k-1}\otimes\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\otimes\underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{n-k}. \end{aligned} \tag{20}$$

The Stone-von Neumann theorem fails to apply if either of its antecedent conditions fail; i.e. if either the classical phase space is not $\mathbb{R}^{2n}$, or else the representation of the Weyl algebra is not weakly continuous. Following Ruestche (2011, Ch. 3), it is helpful to break the various possible failures into three cases:

(i) weak continuity fails;

(ii) classical phase space is finite-dimensional, but not $\mathbb{R}^{2n}$;

(iii) classical phase space is infinite-dimensional.

In each of these cases, we have no guarantee that the quantization of our classical system is unique. In fact, for each of these cases we know that the quantization is not unique. We'll investigate case (i) in Section 2, case (ii) in Section 3, and case (iii) in Section 5¿ Before that we will deal Section 4 with infinite spin chains, i.e. the break-down analogous to case (iii) for CARs.

2 Dropping weak continuity—and getting position or momentum eigenstates

There are representations of the CCRs that give up weak continuity (aka: regularity) whose Hilbert space contains exact position eigenstates: but these are not the "improper eigenstates" given by delta-functions (cf. our Hilbert space Review), nor the eigenstates of "rigged Hilbert space". By a parallel construction, one can build a non-regular representation with exact momentum eigenstates. To explain this, we will follow Halvorson, "Complementarity of representations in quantum mechanics", *Studies in History and Philosophy of Modern Physics* 2004.

His paper develops results from the 1970s, e.g. by Beaume et al. A philosophers' review of Halvorson is in Ruetsche 2011, Chapter 3.1.

These constructions are of interest for several reasons:—

(i) They use a non-separable Hilbert space: a kind of quantum state-space relevant to various foundational/interpretative discussions (reviewed by Earman, "Quantum Physics in Non-separable Hilbert spaces", Pittsburgh 2020: Earman's Section 5.2 discusses this case).

(ii) In the representation with exact position eigenstates, there are no momentum eigenstates; indeed, the momentum operator does not exist. And vice versa: the representation with exact momentum eigenstates has no position eigenstates, and no position operator. Besides, these representations are unitarily *inequivalent*. (Non-separable Hilbert spaces and unitarily inequivalent representations will be themes for us below.)

(iii) Building on (ii), Halvorson sees these results as formulating (vindicating!) Bohr's doctrine of complementarity (this theme is also in other contemporary papers of his). We recall from our Hilbert space Review, that in $L^2(\mathbb{R})$, complementarity is usually taken to be formulated by such facts as:

(a) the position-momentum uncertainty relation (i.e. the product of the standard deviations of any function and its Fourier transform is lower bounded); and

(b) the meet of (intersection of the ranges of) any compact-support spectral projector for position with any compact-support spectral projector for momentum is the zero projector (subspace); (cf.: for any function of bounded support, its Fourier transform has unbounded support).

Whether or not Bohr (or we!) really "want" a quantum particle to be able to have a precise/sharp *real-number* position, or momentum—but not both!—in a way that goes beyond (a) and (b) . . . is a matter for discussion! This is taken up by Ruetsche ibid. and e.g.: B Feintzeig et al. Why be regular? Part I, *Studies in History and Philosophy of Modern Physics* 2019; and B Feintzeig and J Weatherall, Why be regular? Part II, *Studies in History and Philosophy of Modern Physics* 2019; and B Feintzeig, The classical limit of a state on the Weyl algebra, *Journal of Mathematical Physics* 2018.

We now construct a representation with position eigenstates, guided by the Schrödinger representation. Unlike the latter, our representation will be carried by the non-separable Hilbert space $l^2(\mathbb{R})$ of all square-summable functions $\psi : \mathbb{R} \to \mathbb{C}$. These are functions ψ supported on a *countable* set of real numbers, $\sigma(\psi) \subset \mathbb{R}$, and that satisfy

$$||\psi||^2 := \sum_{x \in \sigma(\psi)} |\psi(x)|^2 < \infty \tag{21}$$

Here $||\psi||$ is the norm derived from the inner product $\langle \psi, \phi \rangle = \sum_{x \in \sigma(\psi) \cap \sigma(\phi)} \psi^*(x)\phi(x)$. The space $l^2(\mathbb{R})$ is spanned by the continuum-many states (characteristic functions of real numbers)

$$\psi_\lambda(x) = \begin{cases} 1 & \text{if} \quad x = \lambda, \\ 0 & \text{if} \quad x \neq \lambda. \end{cases} \tag{22}$$

The $\{\psi_\lambda : \lambda \in (R)\}$ are an orthonormal basis of $l^2(\mathbb{R})$.

We define the representations of the Weyl unitaries using these basis states, and guided by the Schrödinger representation. We define for each $a, b \in (R)$:

$$(U(a)\psi_\lambda)(x) := \psi_\lambda(x - a) \equiv \psi_{\lambda+a}(x); \qquad (V(b)\psi_\lambda)(x) := e^{-ibx}\psi_\lambda(x) \equiv e^{-ib\lambda}\psi_\lambda(x). \tag{23}$$

Since for each $a, b \in (R)$, $U(a)$ and $V(b)$ map an orthonormal basis to another, they extend to unitaries. One checks that the Weyl relations, eq. 4, hold.

It can now be checked that weak continuity fails for the Us. Recall (from Section 6.1 of Part I) that weak continuity requires that for every vector ψ in the Hilbert space, $\langle \psi, U(a+\varepsilon)\psi\rangle \to \langle \psi, U(a)\psi\rangle$ as $\varepsilon \to 0$. Then we note that

$$\langle \psi_\lambda, U(a)\psi_\lambda\rangle = \begin{cases} 1 & \text{if} \quad a = 0, \\ 0 & \text{if} \quad a \neq 0, \end{cases} \tag{24}$$

and so $U(a)$ is not weakly continuous at $a = 0$. It follows that Stone's Theorem (cf. Section 3.6 of Hilbert space Review) does not apply, and we have no self-adjoint operator, the would-be momentum, to generate spatial translations. More precisely: Stone's theorem does not apply, so that we cannot define the momentum operator in the standard way by taking the derivative $-i(dU(a)/da)|_{a=0}$. (For discussion of a more general conception of "having a momentum operator", cf. e.g. Halvorson 2004, Section 4.)

Note that Stone's theorem is often formulated with an assumption of strong continuity on the 1-parameter group (e.g. De Faria and De Melo, 2010, Appendix A.9, p. 250). But in fact for 1-parameter unitary groups, weak continuity implies strong continuity; (by a simple argument, e.g. Prugovecki 2006, Lemma 6.2, p. 234).

On the other hand, the Vs are weakly continuous. For trivially, on our orthonormal basis, for any $\lambda \in \mathbb{R}$: $\langle \psi_\lambda, V(b)\psi_\lambda\rangle \equiv e^{ib\lambda} \to 1$ as $b \to 0$. So the Vs are weakly continuous, and therefore also strongly continuous; and so by Stone's Theorem, we have a self-adjoint operator Q such that $V(b) = e^{ibQ}$. Its action on our orthonormal basis is as we want:

$$(Q\psi_\lambda)(x) = -i \lim_{b\to 0} b^{-1}(V(b) - I)\psi_\lambda(x) = -i \lim_{b\to 0} b^{-1}(e^{ib\lambda} - I)\psi_\lambda(x) = \lambda\psi_\lambda(x). \tag{25}$$

So much by way of constructing a position representation. Alternatively, we can *mutatis mutandis* build a momentum representation on $l^2(\mathbb{R})$. The situation is then reversed: the Vs fail to be weakly continuous, and so fail to yield a self-adjoint generator, the would-be position operator; while the Us are generated by a momentum operator satisfying the expected eigenvalue equation.

These two representations on $l^2(\mathbb{R})$, the position and momentum representations, are not unitarily equivalent. This can be seen immediately: no unitary A exists such that $AQA^\dagger$, with Q as defined in (25), is the position operator in the momentum representation—no such operator exists!

3 Nontrivial configuration spaces: a particle on the circle

For a particle on the circle, the configuration space is S^1, coordinatized by $\phi \in [0, 2\pi)$ and the phase space is $S = S^1 \times \mathbb{R}$, coordinatized by $(\phi, l) \in [0, 2\pi) \times \mathbb{R}$. This phase space cannot be a symplectic vector space, since S^1 is not a vector space. But it is a symplectic manifold, with symplectic form $\omega = \mathrm{d}l \wedge \mathrm{d}\phi$. Therefore we have to look for the group of symplectomorphisms on S. This is a 2-parameter family, generated by the vector fields

$$X_z = b\frac{\partial}{\partial\phi} - a\frac{\partial}{\partial l}, \tag{26}$$

where $z := (a, b) \in \mathbb{R}^2$. As discussed in Section 3 of Part I, this parameter space can be given the structure of a symplectic manifold by defining

$$\Omega(z_1, z_2) := \omega(X_{z_1}, X_{z_2}) = a_2 b_1 - a_1 b_2. \tag{27}$$

Inspired by the Schrödinger representation on $L^2(\mathbb{R})$, we might want to define the Weyl unitaries on $L^2(S^1) \ni \psi(\phi)$, according to:

$$(V(b)\psi)(\phi) := e^{-ib\phi}\psi(\phi); \qquad (U(a)\psi)(\phi) := \psi(\phi - a). \tag{28}$$

But now we face the problem that ψ is only defined on $[0, 2\pi)$, while a may be any real number. The standard solution (see Morandi 1992, Ch. 3) is to seek representations not in the space of square-integrable functions on S^1, but rather on its *universal covering space*, $\mathbb{R}$, coordinatized by $\tilde{\phi}$. The idea is that a phase θ is picked up for each 2π translation along $\mathbb{R}$; and different choices of θ give unitarily inequivalent representations of the Weyl relations.

In detail:— The group $\pi_1(S^1)$ of homotopy equivalence classes $[\gamma]$ on S^1 ($[\gamma]$ a loop on S^1) acts on the real line in the obvious way. Note that $\pi_1(S^1) \cong \mathbb{Z}$. Namely: if $[k]$ is the class of loops circling S^1 $|k|$ times, clockwise if $k > 0$ and anti-clockwise if $k < 0$ (so $k \in \mathbb{Z}$), then the action is: $[k] \cdot \tilde{\phi} := \tilde{\phi} + 2\pi k$. Given this action, we require the states $\tilde{\psi} \in L^2(\mathbb{R})$ to satisfy the condition

$$\tilde{\psi}([\gamma] \cdot \tilde{\phi}) = a([\gamma])\tilde{\psi}(\tilde{\phi}), \tag{29}$$

where $a : \pi_1(S^1) \to U(1)$ is a 1-dimensional unitary representation of $\pi_1(S^1)$.

Let $[+1]$ be the class of loops circling S^1 once clockwise, and let $a([+1]) =: e^{i\theta}$, where $\theta \in [0, 2\pi)$. Here we see how the choice of the representation a fixes the phase picked up by a single translation by 2π—and thus by any integer number of such translations. That is: this implies that $a([k]) = e^{ik\theta}$, where $k \in \mathbb{Z}$. It then follows, using eq. 28 and 29, that

$$(U(2k\pi)\tilde{\psi})(\tilde{\phi}) = e^{-ik\theta}\tilde{\psi}(\tilde{\phi}) \ . \tag{30}$$

It may be checked that

$$(V(b)\tilde{\psi})(\tilde{\phi}) := e^{-ib\tilde{\phi}}\tilde{\psi}(\tilde{\phi}); \qquad (U_\theta(a)\tilde{\psi})(\tilde{\phi}) = e^{-i\frac{a\theta}{2\pi}}\tilde{\psi}(\tilde{\phi} - a); \tag{31}$$

satisfy the required Weyl relations and condition (30).

The self-adjoint generator of the U_θs is the angular momentum operator

$$L_\theta = -i\frac{\mathrm{d}}{\mathrm{d}\tilde{\phi}} + \frac{\theta}{2\pi}, \tag{32}$$

which, due to (30), has the discrete spectrum $\{k + \frac{\theta}{2\pi} \mid k \in \mathbb{Z}\}$.

Since the spectra of any two $L_{\theta_1}, L_{\theta_2}$, where $\theta_1 \neq \theta_2$, are disjoint, no two representations are unitarily equivalent.

But the value of θ has empirical consequences, as illustrated by the related examples: (i) the Aharonov-Bohm effect; and (ii) anyons. In both of these cases the configuration space's first homotopy group is $\pi_1(\mathcal{Q}) \cong \mathbb{Z}$, like the particle on the circle.

4 Infinite degrees of freedom 1: the infinite spin chain

Recall Section 1.4 above about the Jordan-Wigner theorem, and its specification of the Pauli representation on a *finite* spin: which recalled Section 6.1 of Part I. We now **repeat** more of that Section 6.1.

An alternative to the Pauli representation (though, by the Jordan-Wigner theorem, equivalent to it) is the representation S (for 'switch') that defines the spin matrices according to

$$\sigma_k^S(x) = \sigma_k^P(y); \qquad \sigma_k^S(y) = \sigma_k^P(z); \qquad \sigma_k^S(z) = \sigma_k^P(x); \qquad k = 1, 2, ..., n \tag{33}$$

i.e. the switch representation of $\sigma_k(x)$ in $\mathcal{H}_S$ has the same matrix elements as the Pauli representation of $\sigma_k(x)$ in $\mathcal{H}_P$, etc. Now let $U : \mathbb{C}_P^2 \to \mathbb{C}_S^2$ be the unitary such that $U\sigma_k^P(x)U^\dagger = \sigma_k^S(x)$, etc. Then the unitary $\otimes^n U : \mathcal{H}_P \to \mathcal{H}_S$ establishes the unitary equivalence between the switch and Pauli representations.

This equivalence extends to all operators in $\mathcal{B}(\mathcal{H}_S)$ and $\mathcal{B}(\mathcal{H}_P)$. In particular, let $\{f_i(\{\sigma_k^P(i)\})\}$ be a sequence of linear functions of the $\{\sigma_k^P(i)\}$ which converges in $\mathcal{H}_P$'s weak topology to the operator F_P. Each $f_i(\{\sigma_k^P(i)\}) \in \mathcal{B}(\mathcal{H}_P)$ and $\mathcal{B}(\mathcal{H}_P)$ is closed under weak convergence; so $F_P \in \mathcal{B}(\mathcal{H}_P)$. Similarly, let $\{f_i(\{\sigma_k^S(i)\})\}$ be a sequence of linear functions of the $\{\sigma_k^S(i)\}$, where

$$f_i(\{\sigma_k^S(i)\}) = U f_i(\{\sigma_k^P(i)\}) U^\dagger. \tag{34}$$

Weak convergence is preserved under unitary transformations, so the $\{f_i(\{\sigma_k^S(i)\})\}$ converge in $\mathcal{H}_S$'s weak topology to some operator $F_S \in \mathcal{B}(\mathcal{H}_S)$, and $F_S = U F_P U^\dagger$.

In the Pauli representation $\mathcal{H}_P \cong \mathbb{C}^{2n}$, we may define the *polarization* observable $\hat{\mathbf{m}}^P := (m_x^P, m_y^P, m_z^P)$, where

$$m_x^P := \frac{1}{n} \sum_{k=1}^{n} \sigma_k^P(x), \quad \text{etc.} \tag{35}$$

Clearly, $\hat{\mathbf{m}}^P \in \mathcal{B}(\mathcal{H}_P)$, and the spectrum of $\hat{\mathbf{m}}^P$ is parametrized by points on the unit sphere. From the above considerations, we know that the similarly defined polarization observable $\hat{\mathbf{m}}^S := (m_x^S, m_y^S, m_z^S)$ in the switch representation satisfies

$$\hat{\mathbf{m}}^S = U \hat{\mathbf{m}}^P U^\dagger, \tag{36}$$

and so expectation values in S are identical to corresponding (given U) expectation values in P.

Now consider the theory of the *infinite spin-chain*, in which we have a spin-$\frac{1}{2}$ system for every integer in $\mathbb{Z}$. This theory has observables satisfying the Pauli relations (19). Representations of the Pauli relations in such a theory will be carried by a separable Hilbert space only if we make some hard choices about which of the uncountably many *prima facie* possible states are to be excluded.

(References for what follows include, G. Sewell's books, *Quantum Theory of Collective Phenomena* 1986, and *Quantum Mechanics and its Emergent Macrophysics* 2002; cf. Section 2.3 of each book. The natural proposal to set $\mathcal{H}$ = the infinite tensor product of $\mathbb{C}^2$ leads to a non-separable Hilbert space, since it has $2^{\aleph_0}$ dimensions: cf. Section 2.1 of Earman, "Quantum Physics in Non-separable Hilbert spaces", Pittsburgh archive 2020.)

One way to construct a separable Hilbert space is to pick a single-site state-vector $|\theta, \phi\rangle$ to favour. $|\theta, \phi\rangle$ represents the eigenstate (with eigenvalue 1) for the spin vector's being $\hat{\mathbf{u}}_{(\theta,\phi)}$, which is the unit vector intersecting the unit sphere characterized at latitude $\frac{\pi}{2} - \theta$ and longitude ϕ. Our Hilbert space $\mathcal{H}_{(\theta,\phi)}$ is then constructed as follows. First, it contains the state in which every spin-site has state $|\theta, \phi\rangle$; call this state $\Omega_{(\theta,\phi)}$. Then we generate $\mathcal{H}_{(\theta,\phi)}$ by taking the closed linear span of all states obtained from $\Omega_{(\theta,\phi)}$ by $SU(2)$ rotations on any finite number of the spin sites.

We can do this as follows. First define $\mathcal{H}_{(\theta,\phi)}$ as a fermionic Fock space on $l^2(\mathbb{Z})$:

$$\mathcal{H}_{(\theta,\phi)} := \mathfrak{F}_- \left[l^2(\mathbb{Z}) \right] = \mathbb{C} \oplus l^2(\mathbb{Z}) \oplus \mathcal{A}_2 \left[l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z}) \right] \oplus \dots \tag{37}$$

The subspace $\mathcal{A}_N \left[\otimes^N l^2(\mathbb{Z}) \right]$ corresponds to arbitrary superpositions of states in which exactly N spin sites are in an eigenstate of pointing in the direction $-\hat{\mathbf{u}}_{(\theta,\phi)} \equiv \hat{\mathbf{u}}_{(\pi-\theta,\phi+\pi)}$ and all remaining spin sites are in an eigenstate of pointing in the familiar direction $\hat{\mathbf{u}}_{(\theta,\phi)}$.

We define the "vacuum" state $\Omega_{(\theta,\phi)}$ by

$$\Omega_{(\theta,\phi)} = 1 \oplus \mathbf{0} \oplus \mathbf{0} \oplus \dots \tag{38}$$

We now define fermionic creation and annihilation operators $d_k^\dagger, d_k$ for each spin site $k \in \mathbb{Z}$. $\mathcal{H}_{(\theta,\phi)}$ is the closed linear span of arbitrary combinations of these acting on $\Omega_{(\theta,\phi)}$. First we define the operators ${d_k^{(N)}}^\dagger : \otimes^{N-1} l^2(\mathbb{Z}) \to \otimes^N l^2(\mathbb{Z})$ and $d_k^{(N)} : \otimes^N l^2(\mathbb{Z}) \to \otimes^{N-1} l^2(\mathbb{Z})$ for all $N \in \mathbb{N}$:

$$\begin{array}{rcl} {d_k^{(N)}}^\dagger \left(\psi_1 \otimes \ldots \otimes \psi_{N-1}\right) & := & \chi_k \otimes \psi_1 \otimes \ldots \otimes \psi_{N-1} \\ d_k^{(N)} \left(\psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_N\right) & := & \psi_1(k)\ \psi_2 \otimes \ldots \otimes \psi_N \end{array} \tag{39}$$

where $\chi_k(j) = \delta_{jk}$. Now we may define $d_k^\dagger, d_k : \mathfrak{F}_-\left[l^2(\mathbb{Z})\right] \to \mathfrak{F}_-\left[l^2(\mathbb{Z})\right]$ by

$$\begin{array}{rcccccccc} {d_k}^\dagger & := & {d_k^{(1)}}^\dagger & \oplus & \sqrt{2}\mathcal{A}_2 {d_k^{(2)}}^\dagger & \oplus & \sqrt{3}\mathcal{A}_3 {d_k^{(3)}}^\dagger & \oplus & \ldots \\ d_k & := & 0 & \oplus & d_k^{(1)} & \oplus & \sqrt{2} d_k^{(2)} & \oplus & \sqrt{3} d_k^{(3)} \oplus \ldots \end{array} \tag{40}$$

It may be checked that

$$[d_j, d_k]_+ = [d_j^\dagger, d_k^\dagger]_+ = 0; \qquad [d_j, d_k^\dagger]_+ = \delta_{jk}\ . \tag{41}$$

We may now define

$$\begin{array}{rcl} \sigma_k^{(\theta,\phi)}(x) & := & U_k(\theta,\phi)\left(d_k + d_k^\dagger\right) U_k(\theta,\phi)^\dagger; \\ \sigma_k^{(\theta,\phi)}(y) & := & -iU_k(\theta,\phi)\left(d_k - d_k^\dagger\right) U_k(\theta,\phi)^\dagger; \\ \sigma_k^{(\theta,\phi)}(z) & := & U_k(\theta,\phi)\left(d_k d_k^\dagger - d_k^\dagger d_k\right) U_k(\theta,\phi)^\dagger; \end{array} \tag{42}$$

where

$$U_k(\theta,\phi) := \sin\frac{1}{2}\theta e^{-\frac{1}{2}\phi} d_k + \sin\frac{1}{2}\theta e^{\frac{1}{2}\phi} d_k^\dagger + \cos\frac{1}{2}\theta e^{\frac{1}{2}\phi} d_k d_k^\dagger - \cos\frac{1}{2}\theta e^{-\frac{1}{2}\phi} d_k^\dagger d_k. \tag{43}$$

Intuitively, think of each $U_k(\theta,\phi)$ as rotating eigenstates of spin-direction $\hat{\mathbf{u}}_{(\theta,\phi)}$ to eigenstates of spin-direction $\hat{\mathbf{z}} := \hat{\mathbf{u}}_{(0,0)}$ at spin-site k.

The significant result is now that different choices for (θ,ϕ)—and therefore for $\Omega_{(\theta,\phi)}$—lead to *unitarily inequivalent representations* of the Pauli relations. This can be seen informally by considering that the inner product between any state from $\mathcal{H}_{(\theta,\phi)}$ and any state from $\mathcal{H}_{(\theta',\phi')}$, where $(\theta,\phi) \neq (\theta',\phi')$, involves infinitely many factors of the kind $\langle\theta,\phi|\theta',\phi'\rangle$, each of which is strictly less than one. Therefore, the inner product is zero. This is an instance of representations which are called *disjoint*; we will return to this idea below.

Alternatively, note that, for finite spin-sites, the unitary connecting (the analogues of) $\Omega_{(\theta,\phi)}$ and $\Omega_{(0,0)}$ could be implemented by

$$\prod_{k=1}^{n} U_k(\theta,\phi) = \otimes^N \begin{pmatrix} \cos\frac{1}{2}\theta e^{\frac{1}{2}\phi} & \sin\frac{1}{2}\theta e^{-\frac{1}{2}\phi} \\ \sin\frac{1}{2}\theta e^{\frac{1}{2}\phi} & -\cos\frac{1}{2}\theta e^{-\frac{1}{2}\phi} \end{pmatrix} \tag{44}$$

on $\otimes^N \mathbb{C}^2$. But we cannot make sense of the infinite-site counterpart $\prod_{k=-\infty}^{\infty} U_k(\theta,\phi)$ on a separable Hilbert space.

We can see the unitary equivalence more rigorously by noting that the observables

$$m_{x,n}^{(\theta,\phi)} := \frac{1}{2n+1}\sum_{k=-n}^{n} \sigma_k^{(\theta,\phi)}(x), \quad \text{etc.} \tag{45}$$

defined on $\mathcal{H}_{(\theta,\phi)}$ converge in the weak topology, as $n \to \infty$, to the *global polarization* $\hat{\mathbf{m}}_\infty^{(\theta,\phi)}$, where

$$\langle\Omega_{(\theta,\phi)}, \mathbf{m}_\infty^{(\theta,\phi)}\Omega_{(\theta,\phi)}\rangle = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\hat{\mathbf{u}}_{(\theta,\phi)} = \hat{\mathbf{u}}_{(\theta,\phi)}. \tag{46}$$

Similarly, we can define the global polarization $\hat{\mathbf{m}}_{\infty}^{(\theta',\phi')}$ in $\mathcal{H}_{(\theta',\phi')}$, where

$$\langle \Omega_{(\theta',\phi')}, \mathbf{m}_{\infty}^{(\theta',\phi')} \Omega_{(\theta',\phi')} \rangle = \hat{\mathbf{u}}_{(\theta',\phi')}. \tag{47}$$

But $\hat{\mathbf{u}}_{(\theta,\phi)} \neq \hat{\mathbf{u}}_{(\theta',\phi')}$, so these two representations must be unitarily inequivalent.

Some comments:

(i) We can see unitary inequivalence as arising from *"vacuum" polarization.* I.e., the states on which we build each representation differ "infinitely" from each other, and since any two states in the same representation are accessible by a finite number of transformations, any state in one representation will be inaccessible to any state in the other.

(ii) If $N < \infty$, then all states "fit" into a separable Hilbert space, and there is no superselection. But superselection can be approximated for large N by restricted the algebra of quantities to "local" ones.

(iii) How to choose which representation? Answer: sometimes dynamics, sometimes not. E.g. as we have in effect seen above: the ferromagnetic choice $H = \sum_{k=-\infty}^{\infty} (1 - \sigma_k.\sigma_{k+1})$ does not determine a unique vacuum.

(iv) The idea of "particles" arises as a solution to the problem of defining a Hilbert space of states which is separable, i.e. has a countable basis, for an infinite system (for which we might naturally expect an uncountable number of basis states). That is: here, particles allow us to define finite deviations of the system from a selected "vacuum" state. We say "vacuum" in scare-quotes because (i) we have not invoked a Hamiltonian and (ii) in the spin-chain, the "vacuum" is no more "empty" than any other state. This use for particles also arises in QFT, and is separate from the idea of "particles" associated with finding *normal modes* and their excitations (as discussed in Section 4.1 of Part I).

(v) Unlike in QFT, there is no vacuum entanglement here: i.e. the vacuum state is not entangled between the sites.

(vi) In our *GNS and all that: a rough guide to algebras and states*, we will return to the closing argument above, for unitary inequivalence. We will see it in the context of the facts that (i) the *representations* of a C^*-algebra are given within a Hilbert space, which allows us to define a weak topology; (ii) we can close the set of the C^*-algebra's representatives in this weak topology; and (iii) the new operators so generated (which don't live in the C^*-algebra) have different spectra in different representations; so (iv) they cannot be unitarily equivalent.

(vii) There is also here the general philosophical theme mentioned at the end of Section 2.4.C of Part I: singular limits and emergence.