EC402 (20/21) - MT WK 7 - ASYMPTOTICS TRILOGY: PART 2 "PAIN IN THE ASYMPTOTICS"

RAGVIR'S SPEAKING NOTES (UNOFFICIAL CONTENT)

(I) "Raguir, I don't like you anymore because...

(i) your "gentle intro" was not gentle; but ok, I learned about "convergence in p"; (consistency)

(ii) I still don't understand "convergence in d"; (asymp. Normality)

(iii) you will probably ambush me with more slide packs in future. (asymp. Efficiency)

...and your jokes are really not funny!"

NOTE: AS USUAL, THE HERO OF OUR TRILOGY WILL CONTINUE TO BE $\bar{X}_N$.

(II) RECAP

We learned something really neat in "gentle intro":

Consider $x_i \overset{\text{IID}}{\sim} \mathcal{N}(\mu, \sigma^2)$ for $i = 1, \ldots, N$

Assume $0 < \sigma^2 < \infty$. Object of inference: μ, for $|\mu| < \infty$

IMPORTANT: I made an assumption of Normality last time but I did not actually use it anywhere in the proof of Chebyshev's WLLN.

So, what was it exactly that we learned?

(Answer below...)

Regardless of the distribution of X_i,

"$\bar{X}_N \xrightarrow{P} \mu$ as $N \to \infty$"; OR

"$\operatorname*{plim}_{N\to\infty} \bar{X}_N = \mu$"; OR

"For any $\varepsilon > 0$, $\lim_{N\to\infty} P\left(|\bar{X}_N - \mu| > \varepsilon\right) = 0$"; OR

"$\bar{X}_N$ is a consistent estimator for μ".

This is an incredible result for (point) estimation; but it does not really help us with inference, does it?

(see what I mean below...)

One may think: well, we know from Ragvir's "walks/quacks" tricks (PS2, Q4) that if $X_i \overset{\text{IID}}{\sim} N(\mu, \sigma^2)$, then $\bar{X}_N \sim N(\mu, \sigma^2/N)$.

i.e. $M_{\bar{X}_N}(t) = E\left[e^{t\bar{X}_N}\right]$

$$= E\left[\exp\left(\frac{t}{N}\sum_{i=1}^{N} X_i\right)\right]$$

$$\overset{\text{Indep.}}{=} E\left[\exp\left(\frac{t}{N}X_1\right)\right] \cdot \ldots \cdot E\left[\exp\left(\frac{t}{N}X_N\right)\right]$$

$$\overset{\text{Ident. dist.}}{=} \left(E\left[\exp\left(\frac{t}{N}X_1\right)\right]\right)^N$$

$$= \left(\exp\left((t/N)\mu + \frac{1}{2}(t/N)^2\sigma^2\right)\right)^N$$

$$= \exp\left(t\mu + \frac{1}{2}t^2(\sigma^2/N)\right).$$

i.e. $\bar{X}_N \sim N(\mu, \sigma^2/N)$.

So indeed $\frac{\bar{x}_N - \mu}{\sigma/\sqrt{N}} \sim N(0,1)$ is a fantastic pivotal

quantity (provided we know σ^2); and we can do all the interval estimation and/or hypothesis testing for μ that our heart desires. But life is not so simple...

WE ALMOST NEVER CAN ASSUME THAT X_i IS NORMAL. So, does that mean all inference is lost?

Answer: No, but we empirical economists need theoretical statisticians to bail us out. Let's see how...

(III) So what can we assume about X_i?

- Say $X_i \overset{IID}{\sim} ?(\mu, \sigma^2)$ for $i = 1, \ldots, N$; $\sigma^2 > 0$.
- Further, say $M_{X_i}(t)$ exists for $|t| < h$ for some $h > 0$ (this gives us finite μ and σ^2).

$$\boxed{\begin{array}{l}\textbf{PAUSE} \\ \text{- Just try and appreciate the above for a second.} \\ \text{- Starting from virtually no assumptions (other than IID and finite mean and variance, which can be weakened)} \\ \text{we will end up with Asymptotic Normality of } \ldots \end{array}}$$

...well, let's see.

(IV) SIMPLE ALGEBRA

STEP 1

let $y_i := (x_i - \mu)/\sigma$ for $i = 1, \dots, N$.

Then, $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} y_i = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (x_i - \mu)/\sigma$

$$= \frac{1}{\sqrt{N}} \left(\sum_{i=1}^{N} x_i - N\mu \right) / \sigma$$

$$= \frac{1}{\sqrt{N}} \left(N \bar{x}_N - N\mu \right) / \sigma$$

$$= \sqrt{N} \left(\bar{x}_N - \mu \right) / \sigma$$

(V) USUAL "WALKS|QUACKS" TRICKS (FROM PS2, Q4)

STEP 2

$$M_{\sqrt{N}(\bar{X}_N-\mu)/\sigma}(t) = M_{\frac{1}{\sqrt{N}}\sum_{i=1}^{N} Y_i}(t)$$

$$= E\left[\exp\left(\left(\frac{t}{\sqrt{N}}\right)\sum_{i=1}^{N} Y_i\right)\right]$$

$$\overset{IID}{=} \left(E\left[\exp\left(\left(\frac{t}{\sqrt{N}}\right)Y_1\right)\right]\right)^N$$

$$= \left(M_{Y_1}(t/\sqrt{N})\right)^N.$$

(VI) STEP 3 & 4 ARE HARD.
BUT IF IT'S TOO TOUGH, PLEASE
DON'T QUIT JUST YET...

IN THAT CASE... FOCUS ONLY
ON THE HIGHLIGHTED PART
UNDER STEP 4 FOR KEY INTUITION.

Step 3

- Say $\tilde{t} := t/\sqrt{N}$
- let's do a Taylor series expansion around $\tilde{t}=0$.

$$e^{\tilde{t}y_1}\Big|_{\tilde{t}=0} = e^{\tilde{t}y_1}\Big|_{\tilde{t}=0}\frac{(\tilde{t}-0)^0}{0!} + y_1 e^{\tilde{t}y_1}\Big|_{\tilde{t}=0}\frac{(\tilde{t}-0)^1}{1!} + y_1^2 e^{\tilde{t}y_1}\Big|_{\tilde{t}=0}\frac{(\tilde{t}-0)^2}{2!} + \dots$$

$$= 1 + y_1\frac{\tilde{t}}{1!} + y_1^2\frac{\tilde{t}^2}{2!} + \dots$$

$$\therefore E\left[e^{(t/\sqrt{N})y_1}\right] = 1 + E(y_1)(t/\sqrt{N}) + E(y_1^2)\frac{(t/\sqrt{N})^2}{2!} + \dots$$

since $E(y_1)=0$
$Var(y_1)=1$

$$= 1 + 0 + \frac{(t/\sqrt{N})^2}{2!} + E(y_1^3)\frac{(t/\sqrt{N})^3}{3!} + \dots$$

$$= 1 + 0 + \left[\left(\frac{t}{\sqrt{N}}\right)^2\frac{1}{2}\right] + \frac{t}{\sqrt{N}}\left[\left(\frac{t}{\sqrt{N}}\right)^2\frac{E(y_1^3)}{3!}\right] + \left(\frac{t}{\sqrt{N}}\right)^2\left[\left(\frac{t}{\sqrt{N}}\right)^2\frac{E(y_1^4)}{4!}\right] + \dots$$

We see that: 1^{st} term is $\dot{O}\left((t/\sqrt{N})^0\right) = \dot{O}(1)$

2^{nd} ... 0 for any N

3^{rd} ... $\dot{O}\left((t/\sqrt{N})^2\right) = \dot{O}(t^2/N)$

4^{th} ... $\dot{O}\left((t/\sqrt{N})^3\right) = \dot{o}(t^2/N)$

5^{th} ... $\dot{O}\left((t/\sqrt{N})^4\right) = \dot{o}(t^2/N)$

$\vdots$

and so on.

IN CASE YOU ARE WONDERING:

The orange stuff is referring to Bachmann-Landau notation; Marcia talked about it in her last chapter of EC400 (in the context of stochastic orders of magnitude).

Step 4 So to summarise Steps 1-3, for some t,

$$\boxed{\lim_{N\to\infty} M_{\sqrt{N}(\bar{x}_N-\mu)/\sigma}(t)} = \lim_{N\to\infty} M_{\frac{1}{\sqrt{N}}\sum_{i=1}^{N} y_i}(t)$$

$$= \lim_{N\to\infty}\left(E\left[e^{(t/\sqrt{N})y_1}\right]\right)^N$$

$$= \lim_{N\to\infty}\left[1+\frac{1}{N}\left(\frac{t^2}{2}+N\,o(t^2/N)\right)\right]^N$$

KEY INTUITION

(just ignore the math if it's too hard; only look at what the yellow expressions are telling us.)

$$= \boxed{e^{\frac{1}{2}t^2}}$$

what does this walk/quack like?!

$\Big\{$ since $\lim_{N\to\infty}\left[1+\frac{a_N}{N}\right]^N = e^a$ where $\{a_1, a_2, \ldots\}$ is a sequence of numbers s.t. $\lim_{N\to\infty} a_N = a$.

(VII) "Ragvir, please just stop. You're a real PAIN IN THE ..."

FOLKS, THIS IS AN UNBELIEVABLY POWERFUL RESULT: WHEN I FIRST HEARD ABOUT THE CLT, MY COFFEE CAME OUT OF MY NOSE.

i.e. $\lim_{N\to\infty} M_{\sqrt{N}(\bar{X}_N-\mu)/\sigma}(t) = M_Z(t)$ where $Z \sim N(0,1)$.

By the UNIQUENESS property of MGFs

we can say:

$$\sqrt{N}\left(\bar{X}_N-\mu\right)/\sigma \xrightarrow{d} N(0,1) \text{ as } N\to\infty.$$

(Yes, this is the Lindeberg-Levy CLT we proved.)

(VIII) Two Useful Comments

1. Students often write "$\bar{x} \xrightarrow{d} N(\mu, \sigma^2/N)$ as $N \to \infty$"
Be careful because this statement is completely meaningless. That is, due to consistency of $\bar{x}_N$ for μ, the limiting distribution of $\bar{x}_N$ collapses to a spike on μ as $N \to \infty$. So how can it possibly be a (nice bell-shaped) Gaussian density?!
Please DO NOT write the highlighted part

... ever.

2. Let's answer now that question of why empirical economists like us should care about some strange abstract / theoretical result:

We use the above theoretical result as HAND-WAVY justification for the idea that

$$\bar{X}_N \overset{\text{approx}}{\sim} N(\mu, \sigma^2/N)$$

... and yes it's just a silly approximation (in view of my previous comment) but it is decent enough ...

for "large" N. You can write THIS highlighted stuff if you wish.

Thus, we have a mechanism for INFERENCE:

(i) Creating C.I.s that are at least asymptotically valid as $N \to \infty$.

(ii) Constructing test stats. whose distributions under the null are known at least asymptotically as $N \to \infty$.