"Ragvir, how do you apply the CLT to $T^{-1/2} X' \varepsilon$? Just the steps; no intuition!"

Prop. 1. Cramér-Wold device.

- Consider a sequence of $K \times 1$ random vectors $\{b_{(T)}\}$ and a $K \times 1$ random vector Z, where Z has joint distribution F_Z.
- If there exists some $a \in \mathbb{R}^K$ s.t. $\|a\|_2^2 = 1$ and

$$a' b_{(T)} \xrightarrow{d} a' Z \text{ as } T \to \infty,$$

then the limiting distribution of $b_{(T)}$ exists and equals F_Z.

Proof. Sorry, you are hitting the boundaries of my knowledge now.
[Can look at Rao (1973) "Linear Statistical Inference & its applications"] $\square$

PROP. 2. Lindeberg-Levy CLT.

- Let $\{Z_t\}$ be a sequence of IID random scalars with $E(Z_t) := \mu$ and $Var(Z_t) := \sigma^2$ for $0 < \sigma^2 < \infty$.
- Then, $$\sqrt{T}\,(\bar{Z}_{(T)} - \mu)/\sigma \xrightarrow{d} N(0,1) \text{ as } T \to \infty.$$

Proof. Can do using "characteristic functions". But a simpler version using moment-generating functions is explained in excruciating detail in my PAIN IN THE ASYMPTOTICS. $\square$

STUDENT:

Proving distributional results:

$$\hat{\beta}-\beta = (X'X)^{-1}X'\varepsilon$$ OK

$$\hat{\beta}-\beta = \left(\frac{X'X}{S}\right)^{-1}\left(\frac{X'\varepsilon}{S}\right)$$ OK

$$\sqrt{S}(\hat{\beta}-\beta) = \left(\frac{X'X}{S}\right)^{-1}\left(\frac{X'\varepsilon}{\sqrt{S}}\right)$$ OK

how to prove this converges in distribution to $N(0, \sigma^2_\varepsilon I_s)$?

$$X'\varepsilon = \begin{pmatrix} x_1' \\ \vdots \\ x_k' \end{pmatrix}_{k \times s} \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_s \end{pmatrix}_{s \times 1} = \begin{pmatrix} x_1'\varepsilon \\ \vdots \\ x_k'\varepsilon \end{pmatrix}$$

YOU ARE BASICALLY ON THE RIGHT TRACK. WELL DONE. MY NOTES BELOW MAY HELP YOU SEE HOW TO FORMALISE YOUR THINKING.

$$\frac{X'\varepsilon}{\sqrt{S}} = \begin{pmatrix} \frac{x_1'\varepsilon}{\sqrt{s}} \\ \vdots \\ \frac{x_k'\varepsilon}{\sqrt{s}} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{s} x_{i1}\varepsilon_i / \sqrt{s} \\ \vdots \\ \sum_{i=1}^{s} x_{ik}\varepsilon_i / \sqrt{s} \end{pmatrix}$$

Expected value $\Rightarrow$ $E(X'\varepsilon/\sqrt{s}) = \begin{pmatrix} \frac{1}{\sqrt{s}}[E(x_{11}\varepsilon_1) + \dots E(x_{s1}\varepsilon_1)] \\ \vdots \\ \frac{1}{\sqrt{s}} E[(x_{1k}\varepsilon_1) + \dots + E(x_{sk}\varepsilon_s)] \end{pmatrix}$

$$E(X'\varepsilon/\sqrt{s}) = \underset{\sim}{0} \quad \text{(by A3Rsru)}$$

Variance

$$\mathrm{Var}\left(\frac{X'\varepsilon}{\sqrt{s}}\right) = \begin{pmatrix} \frac{1}{s}\mathrm{Var}\left(\sum x_{i1}\varepsilon_i\right) \\ \vdots \\ \frac{1}{s}\mathrm{Var}\left(\sum x_{ik}\varepsilon_i\right) \end{pmatrix}$$

* we need to show that the variances are finite, so we can use the central limit theorem correct? Yes, see assumptions below

$$\frac{1}{s}\mathrm{Var}\left(\sum x_{i1}\varepsilon_i\right) = \frac{1}{s}\left[\mathrm{Var}(x_{11}\varepsilon_1) + \dots + \mathrm{Var}(x_{s1}\varepsilon_s)\right]$$

$$= \frac{1}{s}\left[x_{11}^2\mathrm{Var}(\varepsilon_1) + \dots + x_{s1}^2\mathrm{Var}(\varepsilon_s)\right]$$

$$= \frac{1}{s}\sigma^2_\varepsilon \sum_{i=1}^{s} x_{i1}^2$$

$$= ?$$

* when we have $\bar{X}$, then we show

$$E(\bar{x}) = \frac{1}{n}E\left(\sum x\right) \underset{\text{iid}}{=} \frac{n\,E(x)}{n} = \mu$$

$$\mathrm{Var}(\bar{x}) = \frac{1}{n}\sigma^2$$

so $\sqrt{n}(\bar{x}-\mu)$ has $E(.) = 0$ and $\mathrm{Var} = \sigma^2$

then we can apply CLT $\Rightarrow$ $\sqrt{n}(\bar{x}-\mu) \sim N(0, \sigma^2)$

Good. Details in "PAIN IN THE..."

But in this case we are not getting a result for $Var\left(\frac{X'\varepsilon}{\sqrt{s}}\right)$ so how do we get

$$\frac{X'\varepsilon}{\sqrt{s}} \xrightarrow{d} N(0, \sigma^2 \Sigma_{xx})$$

PAGUIR:

We need to assume the following:

A.1 $\{(X_t', \varepsilon_t)\}$ is an IID sequence

A.2.a $E(X_t \varepsilon_t) = 0$

A.2.b $E\left(\left(X_{tj}\varepsilon_t\right)^2\right) < \infty$, $j = 1, \ldots, K$

A.2.c $V := Var\left(T^{-1/2} X'\varepsilon\right)$ is P.D.

PROP. 3. $\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T} a' V^{-1/2} x_t \varepsilon_t\right] \xrightarrow{d} N(0,1)$ as $T \to \infty$.

Proof. Let $z_t := a' V^{-1/2} x_t \varepsilon_t$

- z_t is IID due to A.1.
- $E(z_t) = 0$ due to A.2.a
- $Var(z_t) = 1$ due to A.2.b and A.2.c, noting $\|a\|_2^2 = 1$

The required result follows from PROP. 2. $\square$

THEOREM 1. $\sqrt{T}\,\frac{1}{T}V^{-1/2}X'\varepsilon \xrightarrow{d} N(0, I_K)$ as $T \to \infty$.

Proof. Recognise that

$$\sqrt{T}\,\frac{1}{T}\sum_{t=1}^{T} a' V^{-1/2} x_t \varepsilon_t = a' \sqrt{T}\left(\frac{1}{T}V^{-1/2}X'\varepsilon\right);$$

let $b(T) := \sqrt{T}\,\frac{1}{T}V^{-1/2}X'\varepsilon$; and

the required result follows from PROP. 1. $\square$