EC402 (23|24)

- COMMON QUESTION

"RAGVIR, WHAT THE FRISCH...?!"

MODEL: $y = X\beta + \varepsilon$, $\varepsilon \sim N(0, \sigma_\varepsilon^2 I_N)$

where $0 < \sigma_\varepsilon^2 < \infty$.

REGRESSORS: $X = [X_1 \;\; X_2]$ where $\begin{cases} X \text{ is } N \times (K_1 + K_2) \\ X_1 \text{ is } N \times K_1 \\ X_2 \text{ is } N \times K_2 \end{cases}$

and $\beta := (\beta_1', \beta_2')'$

ASSUMPTIONS: $\text{rank}(X) = K_1 + K_2$; Vassilis' "A3F"

HEART'S DESIRE: To ignore X_2 and just use

$\hat{\beta}_{1,\text{SILLY}} := (X_1'X_1)^{-1}X_1'y$ to estimate β_1.

PROBLEM : $\hat{\beta}_{1,SILLY} = (X_1'X_1)^{-1}X_1'y$

just plug in for y $\leftarrow$ $= \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2 + (X_1'X_1)^{-1}X_1'\varepsilon$

HEARTBREAK!

SOLUTION #1: Well, if X_1 and X_2 are orthogonal i.e. $(X_1'X_2) = 0$, and/or say $\beta_2 = 0$, then $\hat{\beta}_{1,SILLY}$ is not so silly anymore!

Note: this is not some weird special case. Ever wonder why in your econometric life you've never included say "Planet Jupiter's night-time temperature" in your regressions? Why are we not worried about OVB? Answer: indeed, it is likely to be quite unrelated to any earthly social science phenomena, so ignoring it doesn't create any problems for us.

i.e. $(\beta_2 = 0$ and/or $X_1'X_2 = 0)$

Solution #2: Recognise the source of the problem; let's think about it together...

Remember the aim is to tease out the impact PURELY of X_1 on Y
i.e. to estimate β_1.

The problem:

$$
\begin{tikzcd}
X_1 \arrow[r] & y \\
& X_2 \arrow[ul] \arrow[u]
\end{tikzcd}
$$

Say we have 2 students and their exam marks over time are denoted Y and X_1.

Say one student's grades fall from UG to PG: eg: X_{11} is 90
X_{12} is 30

Say the other student's grades also fall: eg Y_1 is 87
Y_2 is 31

One might conclude that the first student's performance had a massive impact on the second student's performance (e.g. cheating by copying the first student's answers)

... but behind the scenes is $X_2 = \begin{cases} 1, & \text{Ragvir is their teacher} \\ 0, & \text{o/w} \end{cases}$

i.e. both students are failing because Ragvir started teaching them (not because one is failing and the other just copied)

WOULDN'T IT BE NICE IF SOMEHOW WE COULD ELIMINATE RAGVIR, THE SO-CALLED "CONFOUNDER"?

LET'S EXAMINE HOW WE DO THIS...

Recap Model: $y = X_1\beta_1 + X_2\beta_2 + \varepsilon$

$y \rightarrow$ strip out effect of X_2 $\rightarrow M_{X_2}y$

$X_1 \rightarrow$ strip out effect of X_2 $\rightarrow M_{X_2}X_1$

$X_2 \rightarrow$ strip out effect of X_2 $\rightarrow M_{X_2}X_2$

$M_{X_2}y$ and $M_{X_2}X_1$ are the residuals in y and X_1 once the variation due to X_2 has been removed from each variable respectively. By design, $M_{X_2}X_1$ is orthogonal to X_2 so it's pretty much exactly like taking us back to the "Solution 1" scenario.

That is, "it is possible to simply regress

(#1) $\tilde{y} := M_{X_2}y$ on $\tilde{X}_1 := M_{X_2}X_1$ to obtain $\hat{\beta}_{1,OLS}$."

This is the FWL result

Above was an intuitive explanation. Let's see some additional explanations below.

MATH explanation

$$y = X_1\beta_1 + X_2\beta_2 + \varepsilon$$

Pre-multiply by M_{X_2} on both sides $\Rightarrow$

$$\tilde{y} = \tilde{X}_1\beta_1 + \eta \quad , \text{ where } \tilde{y} := M_{X_2}y, \ \tilde{X}_1 := M_{X_2}X_1$$

and η is some error.

[Test yourself: Where did $\tilde{X}_2$ go?]

FWL Theorem

$$\hat{\beta}_{1\,OLS} = (\tilde{X}_1'\tilde{X}_1)^{-1}\tilde{X}_1'\tilde{y}$$

$$= (X_1'M_{X_2}'M_{X_2}X_1)^{-1}X_1'M_{X_2}'M_{X_2}y = (X_1'M_{X_2}X_1)^{-1}X_1'M_{X_2}y.$$

REGULAR-ENGLISH explanation

If you estimate $\beta := (\beta_1', \beta_2')'$ using $\hat{\beta}_{OLS} := (X'X)^{-1}X'y$ and then you look at the first K_1 elements in $\hat{\beta}_{OLS} = \begin{pmatrix} \hat{\beta}_{1\,OLS} \\ \hat{\beta}_{2\,OLS} \end{pmatrix}$, then, you get exactly the same (FWL Theorem) K_1 estimates as if you attempted a LS regression of $M_{X_2}y$ on $M_{X_2}X_1$ and then looked at whatever $(K_1 \times 1)$ vector of estimates you obtain.

CONCISE-ENGLISH explanation

PART of my estimates (K_1 of K_1+K_2) from a full regression will be identical (FWL Theorem) to ALL of my estimates from a (suitably executed) PARTitioned regression