EC402 (2023/24)

RAGVIR'S SPEAKING NOTES FOR CLASS

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INTUITION FOR IV

Say $y_t = x_t' \beta + \varepsilon_t$ and $E[x_t \varepsilon_t] \neq 0$ for $t = 1, \dots, T$.

Find a consistent estimator for β.

QUESTION 1

(Q1) $y = X_A\beta_A + X_B\beta_B + \varepsilon$

where X_A is $T \times (K-1)$ and

X_B is $T \times 1$ and $E[X_B'\varepsilon] \neq 0$.

We have A1, A2, A3Rmi.X_A, A4Ω, A5N.

(a) $\hat{\beta}_{OLS} := (X'X)^{-1}X'y$ where $X := [X_A \; X_B]$.

$\hat{\beta}_{IGLS} := (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$.

let $\hat{\beta} := (\beta_A', \beta_B)'$, $\hat{\beta}_{OLS} := (\hat{\beta}_{A_{OLS}}', \hat{\beta}_{B_{OLS}})'$ and

$\hat{\beta}_{IGLS} := (\hat{\beta}_{A_{IGLS}}', \hat{\beta}_{B_{IGLS}})'$.

"Explain carefully... unbiasedness"

Consider $(\hat{\beta}_{A_{OLS}} - \beta_A) = (X_A' M_B X_A)^{-1} X_A' M_B \varepsilon$ where $M_B := [I_T - X_B(X_B' X_B)^{-1} X_B']$.

Now, $E[\hat{\beta}_{A_{OLS}} - \beta_A] \overset{LIE}{=} E[(X_A' M_B X_A)^{-1} X_A' M_B E[\varepsilon | X]] \overset{A3.X_B}{\neq} 0$, even under A3Rmi.$X_A$

Similarly,

$(\hat{\beta}_{A_{IGLS}} - \hat{\beta}_A) = (\tilde{X}_A' \tilde{M}_B \tilde{X}_A)^{-1} \tilde{X}_A' \tilde{M}_B \tilde{\varepsilon}$ where

$$\tilde{X}_A := \Omega^{-1/2} X_A$$
$$\tilde{\varepsilon} := \Omega^{-1/2} \varepsilon$$
$$\tilde{M}_B := [I_T - \Omega^{-1/2} X_B((\Omega^{-1/2} X_B)'(\Omega^{-1/2} X_B))^{-1}(\Omega^{-1/2} X_B)'].$$

So $E[\hat{\beta}_{A_{IGLS}} - \beta_A] \overset{LIE}{=} E[(\tilde{X}_A' \tilde{M}_B \tilde{X}_A)^{-1} \tilde{X}_A' \tilde{M}_B \Omega^{-1/2} E[\varepsilon | X]] \overset{A3.X_B}{\neq} 0$

even under A3Rmi.X_A.

"Explain carefully... Consistency."

For some K-1 by K-1 finite P.D. matrix J,

$$\underset{T\to\infty}{\text{plim}}\left(\hat{\beta}_{A_{OLS}} - \beta_A\right) \overset{ST}{=} \left[\underset{T\to\infty}{\text{plim}}\left(T^{-1} X_A' M_B X_A\right)\right]^{-1} \underset{T\to\infty}{\text{plim}}\, T^{-1} X_A' M_B \varepsilon$$

$$= J \cdot \underset{T\to\infty}{\text{plim}}\; T^{-1}\left[X_A'\varepsilon - X_A' X_B \left(X_B' X_B\right)^{-1} X_B' \varepsilon\right]$$

$\neq 0$ even if $\underset{T\to\infty}{\text{plim}}\left(T^{-1} X_A' \varepsilon\right) \overset{A3Rmi.XA}{=} 0$ since

$\underset{T\to\infty}{\text{plim}}\left(T^{-1} X_B' \varepsilon\right) \overset{A3.XB}{\neq} 0$ unless $\underset{T\to\infty}{\text{plim}} \frac{1}{T}\left(X_A' X_B\right) = 0$.

Similarly, since $\underset{T\to\infty}{\text{plim}}\, T^{-1}\left(\Omega^{-1/2} X_B\right)' \Omega^{-1/2} \varepsilon \neq 0$, we also have that $\underset{T\to\infty}{\text{plim}}\left(\hat{\beta}_{A_{GLS}} - \beta_A\right) \neq 0$.

ASK YOURSELF: What's the key message of these slides?

A1, A2, A3Rmi.X_A, A4GM

(b) Consider $\hat{\beta}_{IV} := (W'X)^{-1} W'y$ where

W is a $T \times K$ matrix of instruments s.t. $\operatorname{rank}(W'X) = K$

and (i) $\operatorname{plim}_{T\to\infty} T^{-1} W'\varepsilon = 0$; and

(ii) $\operatorname{plim}_{T\to\infty} T^{-1} W'X = \Sigma_{WX}$, where Σ_{WX} is a finite non-singular $K \times K$ matrix.

[Which is validity and which is relevance?]

In the over-identified case (such as if $K_Z > 1$),

$W'X$ is $[K_Z + (K-1)] > K$ by K matrix.

[So what can we do? Throw instruments away? Nope...]

Consider the following scheme:

1. Define $Q := [X_A \; Z]$, a T by $(K-1)+K_Z$ matrix

   where $\text{rank}(Q'X) = K$ and

   (i) $\underset{T\to\infty}{\text{plim}}\, T^{-1} Q'\varepsilon = 0$

   (ii) $\underset{T\to\infty}{\text{plim}}\, T^{-1} Q'X = \Sigma_{QX}$, a full column rank matrix.

2. Choose $W := Q(Q'Q)^{-1}Q'X$

3. Obtain $\hat{\beta}_{2SLS} := (W'W)^{-1}W'y$ [careful of "auto" SE's]

$$= [X'Q(Q'Q)^{-1}Q'Q(Q'Q)^{-1}Q'X]^{-1}X'Q(Q'Q)^{-1}Q'y$$

$$\equiv \hat{\beta}_{GIV} := [X'Q(Q'Q)^{-1}Q'X]^{-1}X'Q(Q'Q)^{-1}Q'y$$

$$\equiv \hat{\beta}_{IV} := (W'X)^{-1}W'y.$$

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- Asymptotics (as $T \to \infty$):

$$\sqrt{T}\left(\hat{\beta}_{IV} - \beta\right) \xrightarrow{d} MVN\left(0, \sigma^2\left[\Sigma_{XW}\Sigma_{WW}^{-1}\Sigma_{WX}\right]^{-1}\right) \text{ as } T \to \infty$$

where $\Sigma_{XW} := \operatorname{plim}_{T\to\infty} \frac{1}{T} X'W$,

$\Sigma_{WW} := \operatorname{plim}_{T\to\infty} \frac{1}{T} W'W$, and

$\Sigma_{WX} := \Sigma_{XW}'$ are all finite and invertible matrices.

(Also, σ^2 is as defined in the question)

- Estimation of variance:

$$\hat{\operatorname{var}}(\hat{\beta}_{IV}) := \hat{\sigma}^2\left[(X'W)(W'W)^{-1}(W'X)\right]^{-1}$$

where $\hat{\sigma}^2 := \frac{1}{T-k}\left(y - X\hat{\beta}_{IV}\right)'\left(y - X\hat{\beta}_{IV}\right)$