Structural Instabilities in Factor Models for Time Series

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Outline

Generalised Dynamic Factor Models (GDFMs)

- Co-movements of (large) number of observed time series modelled in terms of small number of unobserved common factors.
- Literature on high-dimensional GDFMs focusses on case where parameters are stable; in particular, where factor loadings are time-invariant.

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- Roadmap: what the key players are saying

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- Debate: do structural instabilities matter?
- Model: with/without structural breaks
- Roadmap: what the key players are saying
- Extension: what I plan to do

I. What is the debate about?

Do structural instabilities matter?

NAY!

- Stock & Watson (JASA 2002)
- Bates, Plagborg-Moller, Stock & Watson (JoE 2013)

Claim:

Can consistently estimate the factor space despite structural instabilities

...provided certain conditions hold.

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- Breitung & Eickmeier (JoE 2011)
- Corradi & Swanson (Forthcoming, JoE)
- Chen, Dolado & Gonzalo (JoE 2014)
- Han & Inoue ('metric Theory 2014)
- Baltagi, Kao & Wang (Manuscript, 2015)
- Yamamoto & Tanaka (JoE 2015)

Claim:

Stock & Watson conditions not realistic.

II. The standard model, with no breaks

Standard model: $X_t = \chi_t + e_t = \Lambda_t F_t + e_t$

For $t = 1, ..., T$

- Observed series: X_t; $dim(X_t) = N$
- Common component: χ_t; $dim(\chi_t) = N$
- Idiosyncratic error: e_t; $dim(e_t) = N$, with Σ_e not necessarily diagonal
- Common factors: F_t; $dim(F_t) = r$, where $r << N$
- Factor loadings: Λ_t; an $(N \times r)$ matrix, with Λ_0 fixed

Notation alert:

$$\begin{cases} \text{“N”: total number of observed variables (sometimes denoted by “p”)} \\ \text{“T”: total number of time periods (sometimes denoted by “n”)} \end{cases}$$

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Question:

How can we estimate this model? Can we use PCA?

Can use PCA under certain conditions

Under certain assumptions (to be discussed), as $N \to \infty$, the following are equivalent:

- PCA - projection of X_t onto space spanned by the r first principal components;
- Factor Analysis - projection of X_t onto space spanned by the r factors F_t.

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Emphasis:

- Critical that $N \to \infty$ for equivalence between PCA and Factor Analysis to hold.
- Critical that certain assumptions are not violated. We discuss these now...

Under what conditions can we use PCA estimators? (1 of 2)

Main assumptions; Bai & Ng (Econometrica 2002)

1. Pervasiveness of the factors:
   - $\Lambda_0'\Lambda_0/N \rightarrow D$ as $N \rightarrow \infty$ for some positive definite $(r \times r)$ matrix D.

- Factors affect (most or) all of the observed series, even when the number of series grows very large.

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It is equivalent to assume that:

- Σ_χ has all r eigenvalues diverging as $N \rightarrow \infty$.

Under what conditions can we use PCA estimators? (2 of 2)

2. Mild cross-sectional correlation in idiosyncratic component:
   - $\frac{1}{N}\Sigma_{i,j=1}^{N}|Cov(e_{i,t}, e_{j,t})| \leq M$ for some $M \in (0, \infty)$.

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- We can assume that e_t is stationary. No restriction on temporal dependence.
- Heteroscedasticity in both dimensions is also allowed.

It is (almost) equivalent to assume that:

- Σ_e has all eigenvalues bounded for any N.

The assumptions on eigenvalues of Σ_χ and Σ_e imply...

- ...that the r^{th} largest eigenvalue of Σ_x diverges as $N \to \infty$; and
- the $(r+1)^{th}$ largest eigenvalue of Σ_x stays bounded for any N.

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As $N, T \to \infty$

$$||\hat{F}_t - H' F_t|| = O_p\left(max\left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right)$$

The rate of convergence is determined by the smaller of N or T, and thus depends on the panel structure.

III. The extended model, with breaks in factor loadings

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- Consider, for instance, a one-factor model with a structural break:

$$X_{it} = \begin{cases} \lambda_{1i} f_t + e_{it}, t \leq \kappa \\ \lambda_{2i} f_t + e_{it}, t > \kappa \end{cases}$$

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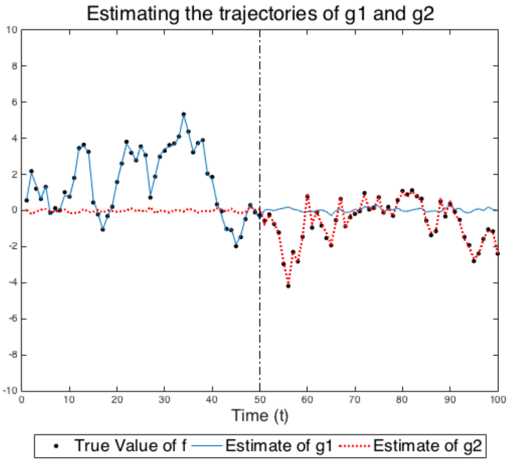
- Then, we have $X_{it} = \lambda_{1i} g_{1t} + \lambda_{2i} g_{2t} + e_{it}$, an equivalent stable model.

Graphical illustration with one-factor model



Potential problems

- Existing criteria to determine the number of factors, r, will fail.

$$\begin{cases} \text{Bai \& Ng (Econometrica 2002)} \\ \text{Onatski (RES 2010)} \\ \text{Alessi, Barigozzi \& Capasso (Stat. \& Prob. Letters 2010)} \\ \text{Ahn \& Horenstein (Econometrica 2013)} \end{cases}$$

- Existing methods to estimate the space spanned by the factors will fail.

$$\begin{cases} \text{PCA} \\ \text{Kalman Filter} \end{cases}$$

- Implications:

$$\begin{cases} \text{Pseudo-factors - not necessarily interpretable} \\ \text{Forecasting - performance may deteriorate} \\ \text{Structural analysis - ``impulse responses'' not recoverable} \end{cases}$$

IV. (Moderate) instabilities do not matter

Model with instability: $\begin{cases} X_t = \Lambda_t F_t + e_t \\ \Lambda_t = \Lambda_0 + \xi_t \end{cases}$

- ξ_t: a random process of dimension $N \times r$

For example, a single abrupt break at a common date κ:
Specifically, let $r = 1$, and $\Delta \in \mathbb{R}^N$ be a shift parameter.

We define:

$$\xi_t = \begin{cases} 0 \text{ for } t = 1, ..., \kappa \\ \Delta \text{ for } t = \kappa + 1, ..., T \end{cases}$$

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Question:

- Under what (extra) conditions is PCA estimator still consistent...
- ...with the same convergence rate as for stable model?

What additional assumptions do we need?

Bates et al. (JoE 2013):

- Factor loading innovations (breaks)

  1. Limited dependence between breaks and factors themselves; OR
  2. Full independence between breaks and factors, but limited dependence between breaks across series; OR
  3. Perfectly correlated breaks across series, but limited number of series undergoing breaks.

- Specifically, assume $|\Delta_i|$ for $i = 1, ..., N$ is bounded; and
- at most $O(N^{1/2})$ series undergo a break.

Averaging away the instabilities

For “moderate” sized breaks, we can treat the instabilities as simply another additive error term which satisfies the usual assumptions.

$$X_t = \Lambda_t F_t + e_t = (\Lambda_0 + \xi_t) F_t + e_t = \Lambda_0 F_t + e_t + \xi_t F_t$$

Assume a one-factor model with Λ_0 known. Consider $\hat{F}_t = ({\Lambda_0}'\Lambda_0)^{-1}{\Lambda_0}' X_t$.

$$\begin{aligned} \hat{F}_t - F_t &= ({\Lambda_0}'\Lambda_0)^{-1}{\Lambda_0}' e_t + ({\Lambda_0}'\Lambda_0)^{-1}{\Lambda_0}' \xi_t F_t \\ &\approx D^{-1}\left(({\Lambda_0}' e_t / N) + ({\Lambda_0}' \xi_t F_t / N) \right) \end{aligned}$$

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- When $t > \kappa$, we have ${\Lambda_0}'\xi_t/N = (\lambda_{10}\Delta_1 + ... + \lambda_{N0}\Delta_N)/N$.
- Thus, this term is $O_p(B/N)$ where B is the number of non-zero Δ_i terms in the summation.

Consistency under structural instability

As above, define B to be the number of series undergoing a break. Then:

$$\|\hat{F}_t - H' F_t\| = O_p\left(max \left\{ max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\}, \left(\frac{B/\sqrt{N}}{\sqrt{N}} \right) \right\} \right)$$

- To summarise, the PCA estimator can accomodate a certain degree of temporal instability in the factor loadings.
- Under the right conditions, effects of instabilities can be eliminated asymptotically by averaging across series.

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- Bates et al. (JoE 2013)
- This is where the disagreement arises.

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- Yamamoto (J o Bus. & Econ. Stats. 2016)
- Burgeoning literature on this side of the debate!

IV. Accounting for instabilities

Specifically, let us consider Baltagi, Kao & Wang (Manuscript, 2015)

- Paper is recent and its representation is fairly general; models in many other papers correspond to special cases.
- This paper tackles estimation of an approximate static factor model with a single abrupt break.
- Allows for changes in the number of factors and/or partial changes in factor loadings.

Model description for X (in $T \times N$ matrix form)

$$X = \begin{bmatrix} F^0_{bef}\Lambda^{0'} + F^1_{bef}\Lambda^{1'} \\ F^0_{aft}\Lambda^{0'} + F^1_{aft}\Lambda^{2'} \end{bmatrix} + E$$

Notation:

- pre-change $\begin{cases} F^0_{bef} = [f^0_1, ..., f^0_\kappa]'; & \kappa \times (r - q) \\ F^1_{bef} = [f^1_1, ..., f^1_\kappa]'; & \kappa \times q \end{cases}$

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- factor loadings $\begin{cases} \Lambda^0; & N \times (r-q) \\ \Lambda^1 \; \& \; \Lambda^2; & N \times q \end{cases}$
- define $\left\{ \Lambda_{bef} = \begin{bmatrix} \Lambda^0 & \Lambda^1 \end{bmatrix} \; \& \; \Lambda_{aft} = \begin{bmatrix} \Lambda^0 & \Lambda^2 \end{bmatrix}; \quad N \times r \right.$

Types of instability

$$X = \begin{bmatrix} F^0_{bef}\Lambda^{0'} + F^1_{bef}\Lambda^{1'} \\ F^0_{aft}\Lambda^{0'} + F^1_{aft}\Lambda^{2'} \end{bmatrix} + E$$

Factor loadings and/or number of factors may change:

- Structural change in the number of factors is incorporated as special case of structural change in factor loadings by allowing either Λ_{bef} or Λ_{aft} to have less than full column rank.
- $r = max\{r_{bef}, r_{aft}\}$
- Emerging factors: If $r_1 < r_2$, some columns in Λ_{bef} are zeros.
- Disappearing factors: If $r_1 > r_2$, some columns in Λ_{aft} are zeros.
- For now, assume both Λ_{bef} and Λ_{aft} have full column rank.

Equivalent model with stable loadings

$$X = G\Gamma' + E$$

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How do we define G and Γ?

$$G_{T\times(r+q)} = \begin{bmatrix} \begin{bmatrix} F^0_{bef} & F^1_{bef} \end{bmatrix} A' \\ \begin{bmatrix} F^0_{aft} & F^1_{aft} \end{bmatrix} B' \end{bmatrix}$$

where

$$A_{(r+q)\times r} = \begin{bmatrix} I_{(r-q)} & 0_{(r-q)\times q} \\ 0_{q\times(r-q)} & I_q \\ 0_{q\times(r-q)} & 0_q \end{bmatrix},$$

$$B_{(r+q)\times r} = \begin{bmatrix} I_{(r-q)} & 0_{(r-q)\times q} \\ 0_{q\times(r-q)} & 0_q \\ 0_{q\times(r-q)} & I_q \end{bmatrix},$$

and $\Gamma_{N\times(r+q)} = \begin{bmatrix} \Lambda^0 & \Lambda^1 & \Lambda^2 \end{bmatrix}$.

Suggested procedure

1. Obtain preliminary estimate, $\hat{r}$ of number of factors, r, ignoring structural change.
   - Use standard methods developed for stable models.
   - $\lim_{(N,T)\to\infty} P(\hat{r} = r + q) = 1$.

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2. Given $\hat{r}$, estimate the space spanned by the pseudo-factors, G_t.
   - Use PCA.
3. Estimate changepoint of factor loadings, κ.
   - Look for break in covariance matrix of pseudo-factors, Σ_G.
   - Note that $\Sigma_{G,bef} = A\Sigma_F A'$ and $\Sigma_{G,aft} = B\Sigma_F B'$.
   - Avoid potential infinite-dimensionality problem.

Estimating the changepoint, κ

- Split data into 2 subsamples for each "candidate" location of $\kappa \in [1, T-1]$.
- Pre-break subsample: $\hat{\Sigma}_{G,bef} = \sum_{t=1}^{\kappa} \hat{G}_t \hat{G}_t{}' / \kappa$
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- Then, define sum of squared residuals:

$$\hat{S}(\kappa) = \sum_{t=1}^{\kappa} [vec(\hat{G}_t \hat{G}_t{}' - \hat{\Sigma}_{G,bef})]' [vec(\hat{G}_t \hat{G}_t{}' - \hat{\Sigma}_{G,bef})] + \sum_{t=\kappa+1}^{T} [vec(\hat{G}_t \hat{G}_t{}' - \hat{\Sigma}_{G,aft})]' [vec(\hat{G}_t \hat{G}_t{}' - \hat{\Sigma}_{G,aft})]$$

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- Obtain LS estimator of changepoint: $\hat{\kappa} = \arg\min \hat{S}(\kappa)$.
- Estimator at most only "slightly misspecified": $\hat{\kappa} - \kappa = O_p(1)$:

Determining the number of factors

4. Identify the number of factors based on $\hat{\kappa}$.

   - Split the data into 2 subsamples based on $\hat{\kappa}$.
   - Obtain $\hat{r}_{bef}$ and $\hat{r}_{aft}$ using standard criteria.
   - $\hat{r}_{bef}$ and $\hat{r}_{aft}$ shown to be robust to "slightly misspecified" $\hat{\kappa}$:

Estimating the factor space

5. Estimate the factor space based on $\hat{\kappa}$ and $\hat{r}_{bef}, \hat{r}_{aft}$.
   - Use PCA on each of the subsamples.
   - Convergence rate is the same as in Bai & Ng (Econometrica 2002).

V. Sequential detection of structural instabilities in DFMs

Do structural instabilities matter?

NAY!

- Stock & Watson (JASA 2002)
- Bates, Plagborg-Moller, Stock & Watson (JoE 2013)

AYE!

- Breitung & Eickmeier (JoE 2011)
- Corradi & Swanson (Forthcoming, JoE)
- Chen, Dolado & Gonzalo (JoE 2014)
- Han & Inoue ('metric Theory 2014)
- Baltagi, Kao & Wang (Manuscript, 2015)
- Yamamoto & Tanaka (JoE 2015)
- Sabharwal (Journal of Optimism, 2016)

Thank you for your attention!

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