Technical Appendix to "Dynamic Selection: An Idea Flows Theory of Entry, Trade and Growth"

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This Technical Appendix accompanies the paper "Dynamic Selection: An Idea Flows Theory of Entry, Trade and Growth". The Technical Appendix: (i) provides further details on the derivation of selected equations from the paper; (ii) analyzes the growth rate when all firms are exporters and under free trade, and; (iii) shows that the symmetric balanced growth path studied in the paper is locally stable to asymmetric perturbations of the initial conditions.

1 Derivation of equations (14) and (15)

Start by observing that since the exit cut-off in terms of relative productivity is $\phi = 1$, we have $H_t(1) = 0$ and equation (13) can be written as:

$$\frac{1}{M_t}\frac{M_{t+\Delta} - M_t}{\Delta} = -\frac{H_t\left(\frac{\theta^*_{t+\Delta}}{\theta^*_t}\right) - H_t\left(\frac{\theta^*_t}{\theta^*_t}\right)}{\Delta} + \left[1 - F\left(\frac{\theta^*_{t+\Delta}}{x_t}\right)\right]\frac{R_t}{M_t}. \tag{TA1}$$

Now define $K_t(s) = H_t\left(\frac{\theta^*_s}{\theta^*_t}\right)$. Then we have:

$$\frac{H_t\left(\frac{\theta^*_{t+\Delta}}{\theta^*_t}\right) - H_t\left(\frac{\theta^*_t}{\theta^*_t}\right)}{\Delta} = \frac{K_t(t+\Delta) - K_t(t)}{\Delta},$$

Taking the limit of this expression as $\Delta \to 0$ gives:

$$\lim_{\Delta \to 0}\frac{H_t\left(\frac{\theta^*_{t+\Delta}}{\theta^*_t}\right) - H_t\left(\frac{\theta^*_t}{\theta^*_t}\right)}{\Delta} = K'_t(t) = H'_t(1)\frac{\dot{\theta}^*_t}{\theta^*_t},$$

where the second equality follows from the chain rule and the definition of K_t. Now using this result and taking the limit of equation (TA1) as $\Delta \to 0$ gives equation (14) in the paper.

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Using equation (13) to substitute for $M_{t+\Delta}$ in equation (12) and rearranging gives:

$$\begin{aligned}\frac{H_{t+\Delta}(\phi)-H_t(\phi)}{\Delta} &= \frac{H_t\left(\frac{\theta^*_{t+\Delta}}{\theta^*_t}\phi\right)-H_t\left(\frac{\theta^*_t}{\theta^*_t}\phi\right)}{\Delta}-\frac{H_t\left(\frac{\theta^*_{t+\Delta}}{\theta^*_t}\right)-H_t\left(\frac{\theta^*_t}{\theta^*_t}\right)}{\Delta}\left[1-H_{t+\Delta}(\phi)\right] \\ &\quad +\left\{F\left(\frac{\phi\theta^*_{t+\Delta}}{x_t}\right)-F\left(\frac{\theta^*_{t+\Delta}}{x_t}\right)-H_{t+\Delta}(\phi)\left[1-F\left(\frac{\theta^*_{t+\Delta}}{x_t}\right)\right]\right\}\frac{R_t}{M_t}.\end{aligned}$$

Taking the limit as $\Delta\to 0$ and applying the chain rule as needed then gives equation (15) in the paper.

2 Derivation of equations (22) and (23)

Substituting (9) into (7) implies:

$$\begin{aligned}W_t(\phi_t) &= \int_t^{t+\frac{\log\phi_t}{g}} f w_t e^{(q-r)(v-t)}\left[\left(\phi_t e^{-g(v-t)}\right)^{\sigma-1}-1\right]dv \\ &\quad +JI\left[\phi_t\geq\tilde{\phi}\right]\int_t^{t+\frac{\log\left(\phi_t/\tilde{\phi}\right)}{g}} f\tau^{1-\sigma}w_t e^{(q-r)(v-t)}\left[\left(\phi_t e^{-g(v-t)}\right)^{\sigma-1}-\tilde{\phi}^{\sigma-1}\right]dv.\end{aligned}$$

Computing the integrals on the right hand side of this expression and using $\tilde{\phi}^{\sigma-1}=\frac{f_x}{f}\tau^{\sigma-1}$ gives equation (22) in the paper.

The free entry condition (11) can be written as:

$$f_e w_t=\int_1^\infty W_t(\phi)dH\left(\frac{\phi}{\lambda}\right).$$

Noting that on the balanced growth path $dH\left(\frac{\phi}{\lambda}\right)=k\lambda^k\phi^{-k-1}d\phi$ it is straightforward to obtain equation (23) in the paper by solving the integral above and simplifying the resulting expression.

3 Growth when all firms export

Proposition 1 in the paper is derived assuming trade costs satisfy $\tau^{\sigma-1}f_x>f$ which is a necessary and sufficient condition to ensure some, but not all, firms select into exporting. When $0<\tau^{\sigma-1}f_x\leq f$ all firms export and on the balanced growth path consumption per capita grows at rate:

$$q=\frac{\gamma}{1+\gamma(k-1)}\left[\frac{\sigma-1}{k+1-\sigma}\left(\frac{k}{k-1}\psi_{\min}\right)^k\frac{1}{f_e}\left(f+Jf_x\right)+\frac{kn}{\sigma-1}-\rho\right].$$

In this case, growth is independent of τ and strictly increasing in f_x. By contrast, when not all firms export, Proposition 1 shows growth is strictly decreasing in both τ and f_x. Thus, the effects of changes in trade costs on growth are qualitatively different when all firms export than when there is selection into exporting. This demonstrates how allowing for selection into exporting is central to understanding the relationship between trade and growth in the dynamic selection model.

When there are no trade costs (i.e. $\tau = 1$, $f_x = 0$) all firms export and the growth rate on the balanced growth path is:

$$q = \frac{\gamma}{1+\gamma(k-1)}\left[\frac{\sigma-1}{k+1-\sigma}\left(\frac{k}{k-1}\psi_{\min}\right)^k\frac{f}{f_e}+\frac{kn}{\sigma-1}-\rho\right].$$

Comparing this expression with Proposition 1 when $J = 0$ shows that the free trade growth rate is the same as the autarky growth rate. The reason free trade does not affect growth is that, in the absence of fixed costs of exporting, the competition effect from imports exactly offsets the size effect of access to foreign markets. An increase in population size leaves the growth rate unchanged for the same reason; the size effect from access to a larger market is exactly offset by increased competition. Thus, it is the absence of a scale effect that means free trade does not affect the growth rate.

4 Stability

Suppose countries differ in their initial conditions. In equilibrium, does the global economy converge to the symmetric balanced growth path described in Proposition 1? To address this question, I assume that at time zero the productivity distribution of potential producers $\hat{G}_0(\theta)$ is Pareto with shape parameter k in all countries,[1] but the mass of potential producers and the scale parameter of the productivity distribution vary across countries. I assume at time zero country j has $\hat{M}_0^j$ potential producers with productivity distribution $\hat{G}_0^j = 1 - \left(\frac{\theta}{\hat{\theta}_0^{*j}}\right)^{-k}$ for $j = 0, 1, \ldots, J$. To be specific, I consider the case where countries $j = 1, \ldots, J$ are symmetric, but there is an asymmetry between country 0 and the rest of the world. For example, country 0 may have more potential producers or higher initial average productivity.

The local stability of the symmetric balanced growth path to asymmetric shocks to the initial conditions can be analyzed by solving for the equilibrium of the global economy allowing for asymmetric outcomes across countries and then differentiating these equilibrium conditions about the symmetric balanced growth path equilibrium. I will use j superscripts to denote countries and q_y to denote the growth rate of any variable y.

Let P^j be the price of the consumption good in country j. The household budget constraint implies:

$$q_{at}^j = \frac{w_t^j}{a_t^j} + r_t^j - \frac{c_t^j P_t^j}{a_t^j} - n, \tag{TA2}$$

and intertemporal optimization gives the Euler equation:

$$q_{ct}^j = \gamma(r_t^j - q_{Pt}^j - \rho). \tag{TA3}$$

[1]The reasoning used in Proposition 2 implies that in any country where Assumptions 1 and 3 hold and the exit cut-off grows without bound over time, the productivity distribution converges to a Pareto distribution with shape parameter k as $t \to \infty$.

Firms select into markets where variable profits exceed fixed production costs. Maintaining the assumption that in all countries the export threshold for entering any foreign market is greater than the exit cut-off for serving the domestic market[2] the productivity threshold $\tilde{\theta}_t^{ji}$ above which firms in country j sell in market i is:

$$\tilde{\theta}_t^{ji} = \frac{\sigma^{\frac{\sigma}{\sigma-1}}}{\sigma-1}\tau^{ji}\left[\frac{f^{ji}}{c_t^i L_t}\left(\frac{w_t^j}{P_t^i}\right)^{\sigma}\right]^{\frac{1}{\sigma-1}}, \tag{TA4}$$

where τ^{ji} equals 1 if $j = i$ and equals τ otherwise, while f^{ji} equals f if $j = i$ and equals f_x otherwise. Differentiating the above equation with respect to time gives:

$$g_t^{ji} = \frac{\sigma}{\sigma-1}(q_{wt}^j - q_{Pt}^i) - \frac{1}{\sigma-1}(q_t^i + n), \tag{TA5}$$

where g_t^{ji} is the growth rate of $\tilde{\theta}_t^{ji}$. Using (TA4) we can write the profits from selling to country i as:

$$\pi_t^{ji}(\phi) = f^{ji}w_t^j\left[\left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}}\right)^{1-\sigma}\phi^{\sigma-1} - 1\right]. \tag{TA6}$$

Allowing for asymmetric initial conditions does not affect the knowledge spillover process or the equations that determine the evolution of the productivity distribution. Therefore, λ is unchanged from the symmetric countries model and is constant across countries and since each country starts from a Pareto productivity distribution, the relative productivity distribution in every country is Pareto with scale parameter 1 and shape parameter k in all periods. In addition, the growth rate of the mass of firms in country j is:

$$q_{Mt}^j = -kg_t^{jj} + \lambda^k\frac{R_t^j}{M_t^j}. \tag{TA7}$$

Knowing that the productivity distribution is Pareto, it is straightforward to calculate the consumption price index:

$$P_t^j = \frac{\sigma}{\sigma-1}\left[\frac{k}{k+1-\sigma}\sum_{i=0}^{J} M_t^i\left(\frac{\tau^{ij}w_t^i}{\tilde{\theta}_t^{ij}}\right)^{1-\sigma}\left(\frac{\tilde{\theta}_t^{ij}}{\tilde{\theta}_t^{ii}}\right)^{-k}\right]^{\frac{1}{1-\sigma}}, \tag{TA8}$$

and by differentiating this expression we obtain the growth rate of the price index:

$$(1-\sigma)q_{Pt}^j = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}\frac{k}{k+1-\sigma}\sum_{i=0}^{J}\frac{M_t^i\left(\frac{\tau^{ij}w_t^i}{\tilde{\theta}_t^{ij}}\right)^{1-\sigma}\left(\frac{\tilde{\theta}_t^{ij}}{\tilde{\theta}_t^{ii}}\right)^{-k}}{\left(P_t^j\right)^{1-\sigma}}$$
$$* \left[q_{Mt}^i - (\sigma-1)q_{wt}^i - (k+1-\sigma)g_t^{ij} + kg_t^{ii}\right]. \tag{TA9}$$

[2]Since I am interested in local perturbations of the symmetric equilibrium the restriction $\tau^{\sigma-1}f_x > f$ guarantees this assumption holds.

Using the functional form of the productivity distribution also implies that the labor market clearing condition is:

$$L_t = R_t^j f_e + \frac{k\sigma + 1 - \sigma}{k + 1 - \sigma} M_t^j \sum_{i=0}^{J} f^{ji} \left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}} \right)^{-k}. \tag{TA10}$$

Since asset markets are assumed to operate at the national level, trade is balanced every period implying:

$$\sum_{i=0, i\neq j}^{J} M_t^j w_t^j \left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}} \right)^{-k} = \sum_{i=0, i\neq j}^{J} M_t^i w_t^i \left(\frac{\tilde{\theta}_t^{ij}}{\tilde{\theta}_t^{ii}} \right)^{-k}, \tag{TA11}$$

with time derivative:

$$\begin{aligned} &\sum_{i=0, i\neq j}^{J} \left[q_{Mt}^j + q_{wt}^j - k\left(g_t^{ji} - g_t^{jj} \right) \right] M_t^j w_t^j \left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}} \right)^{-k} \\ &\qquad = \sum_{i=0, i\neq j}^{J} \left[q_{Mt}^i + q_{wt}^i - k\left(g_t^{ij} - g_t^{ii} \right) \right] M_t^i w_t^i \left(\frac{\tilde{\theta}_t^{ij}}{\tilde{\theta}_t^{ii}} \right)^{-k}. \end{aligned} \tag{TA12}$$

Free entry requires the R&D cost equals the expected value of starting a new firm. Note that entrants at time t draw productivity from a Pareto distribution $\tilde{G}_t(\theta)$ with shape parameter k and scale parameter $\lambda\tilde{\theta}_t^{jj}$. Therefore, using the profit function (TA6) we can write the free entry condition as:

$$\begin{aligned} f_e &= \sum_{i=0}^{J} f^{ji} \int_t^{\infty} e^{-\int_t^v \left(r_s^j - q_{ws}^j \right) ds} \int_{\tilde{\theta}_v^{ji}}^{\infty} \left[\left(\frac{\theta}{\tilde{\theta}_v^{ji}} \right)^{\sigma - 1} - 1 \right] d\tilde{G}_t(\theta) dv, \\ &= \frac{\sigma - 1}{k + 1 - \sigma} \lambda^k \sum_{i=0}^{J} f^{ji} \left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}} \right)^{-k} \int_t^{\infty} e^{-\int_t^v \left(k g_s^{ji} + r_s^j - q_{ws}^j \right) ds} dv, \end{aligned} \tag{TA13}$$

and differentiating the free entry condition with respect to time we obtain:

$$\sum_{i=0}^{J} f^{ji} \left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}} \right)^{-k} = \left(k g_t^{jj} + r_t^j - q_{wt}^j \right) \sum_{i=0}^{J} f^{ji} \left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}} \right)^{-k} \int_t^{\infty} e^{-\int_t^v \left(k g_s^{ji} + r_s^j - q_{ws}^j \right) ds} dv. \tag{TA14}$$

Given that the free entry condition holds, the asset market clearing condition is:

$$a_t^j L_t = \frac{f_e}{\lambda^k} M_t^j w_t^j, \tag{TA15}$$

and taking the time derivative of the asset market clearing condition gives:

$$q_{at}^j + n = q_{Mt}^j + q_{wt}^j, \tag{TA16}$$

Finally, the initial conditions imply that the mass of firms that produce at time zero satisfies:

$$M_0^j = \left(\frac{\tilde{\theta}_0^{jj}}{\hat{\theta}_0^{*j}} \right)^{-k} \hat{M}_0^j. \tag{TA17}$$

All the equilibrium conditions above hold for all $i, j = 0, 1, \ldots, J$ and for all $t \geq 0$.

The next step is to totally differentiate the equilibrium conditions about the symmetric balanced growth path allowing for asymmetric shocks to the initial conditions. Taking the total derivative of equations (TA2)-(TA5) and (TA7)-(TA16) gives:

$$dq_{at}^j = \left(\frac{dw_t^j}{w_t} - \frac{da_t^j}{a_t} \right) \frac{w_t}{a_t} + dr_t^j - \left(\frac{dc_t^j}{c_t} + \frac{dP_t^j}{P_t} - \frac{da_t^j}{a_t} \right) \frac{c_t}{a_t}, \tag{TA18}$$

$$dq_{ct}^j = \gamma (dr_t^j - dq_{Pt}^j), \tag{TA19}$$

$$(\sigma - 1) \frac{d\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{ji}} = \sigma \left(\frac{dw_t^j}{w_t} - \frac{dP_t^i}{P_t} \right) - \frac{dc_t^i}{c_t}, \tag{TA20}$$

$$(\sigma - 1) dg_t^{ji} = \sigma \left(dq_{wt}^j - dq_{Pt}^i \right) - dq_{ct}^i, \tag{TA21}$$

$$dq_{Mt}^j + k dg_t^{jj} = \lambda^k \frac{R_t}{M_t} \left(\frac{dR_t^j}{R_t} - \frac{dM_t^j}{M_t} \right), \tag{TA22}$$

$$(1 - \sigma) \frac{dP_t^j}{P_t} = \frac{\sum_{i=0}^J \left[\frac{dM_t^i}{M_t} - (\sigma - 1) \frac{dw_t^i}{w_t} + k \frac{d\tilde{\theta}_t^{ii}}{\tilde{\theta}_t^{ii}} - (k + 1 - \sigma) \frac{d\tilde{\theta}_t^{ij}}{\tilde{\theta}_t^{ij}} \right] \left(\tau^{ij} \right)^{1-\sigma} \left(\tilde{\theta}_t^{ij} \right)^{\sigma - k - 1}}{\sum_{i=0}^J \left(\tau^{ij} \right)^{1-\sigma} \left(\tilde{\theta}_t^{ij} \right)^{\sigma - k - 1}}, \tag{TA23}$$

$$(1 - \sigma) dq_{Pt}^j = \frac{\sum_{i=0}^J \left[dq_{Mt}^i - (\sigma - 1) dq_{wt}^i + k dg_t^{ii} - (k + 1 - \sigma) dg_t^{ij} \right] \left(\tau^{ij} \right)^{1-\sigma} \left(\tilde{\theta}_t^{ij} \right)^{\sigma - k - 1}}{\sum_{i=0}^J \left(\tau^{ij} \right)^{1-\sigma} \left(\tilde{\theta}_t^{ij} \right)^{\sigma - k - 1}}, \tag{TA24}$$

$$f_e \frac{R_t}{M_t} \frac{dR_t^j}{R_t} + \frac{k\sigma + 1 - \sigma}{k + 1 - \sigma} \left[\frac{dM_t^j}{M_t} \sum_{i=0}^J f^{ji} \left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}} \right)^{-k} - k \sum_{i=0}^J f^{ji} \left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}} \right)^{-k} \left(\frac{d\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{ji}} - \frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}} \right) \right] = 0, \tag{TA25}$$

$$J \frac{dM_t^j}{M_t} + J \frac{dw_t^j}{w_t} - k \sum_{i=0, i \neq j}^J \left(\frac{d\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{ji}} - \frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}} \right) = \sum_{i=0, i \neq j}^J \left[\frac{dM_t^i}{M_t} + \frac{dw_t^i}{w_t} - k \left(\frac{d\tilde{\theta}_t^{ij}}{\tilde{\theta}_t^{ij}} - \frac{d\tilde{\theta}_t^{ii}}{\tilde{\theta}_t^{ii}} \right) \right], \tag{TA26}$$

$$Jdq_{Mt}^{j}+Jdq_{wt}^{j}-k\sum_{i=0,i\neq j}^{J}\left(dg_t^{ji}-dg_t^{jj}\right)=\sum_{i=0,i\neq j}^{J}\left[dq_{Mt}^{i}+dq_{wt}^{i}-k\left(dg_t^{ij}-dg_t^{ii}\right)\right],\tag{TA27}$$

$$\frac{k}{kg+r-q}\sum_{i=0}^{J}f^{ji}\left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}}\right)^{-k}\left(\frac{d\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{ji}}-\frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}}\right)$$
$$+\sum_{i=0}^{J}f^{ji}\left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}}\right)^{-k}\int_t^{\infty}e^{-(kg+r-q)(v-t)}\int_t^{v}\left(kdg_s^{ji}+dr_s^{j}-dq_{ws}^{j}\right)dsdv=0,\tag{TA28}$$

$$\frac{kdg_t^{jj}+dr_t^{j}-dq_{wt}^{j}}{kg+r-q}\sum_{i=0}^{J}f^{ji}\left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}}\right)^{-k}+k\sum_{i=0}^{J}f^{ji}\left(\frac{\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{jj}}\right)^{-k}\left(\frac{d\tilde{\theta}_t^{ji}}{\tilde{\theta}_t^{ji}}-\frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}}\right)=0,\tag{TA29}$$

$$\frac{da_t^j}{a_t}=\frac{dM_t^j}{M_t}+\frac{dw_t^j}{w_t},\tag{TA30}$$

$$dq_{at}^{j}=dq_{Mt}^{j}+dq_{wt}^{j}.\tag{TA31}$$

All variables in these equations except for the differentials take their symmetric balanced growth path values and are constant across countries. In particular, note that $\tilde{\theta}_t^{ji}$ equals θ_t^* if $j=i$ and equals $\tilde{\phi}\theta_t^*$ otherwise. Finally, if we define $\eta_t^j\equiv M_t^j\left(\tilde{\theta}_t^{jj}\right)^k$ then:

$$\frac{d\eta_t^j}{\eta_t}=\frac{dM_t^j}{M_t}+k\frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}},\tag{TA32}$$

and from (TA17) we have:

$$\frac{d\eta_0^j}{\eta_0}=\frac{d\hat{M}_0^j}{\hat{M}_0}+k\frac{d\hat{\theta}_0^{*j}}{\hat{\theta}_0^{*}}.\tag{TA33}$$

From this expression we see that variation in the initial conditions affect the equilibrium only through $d\eta_0^j$ for $j=0,1,\ldots,J$. Moreover, it follows from equations (TA18)-(TA32) that if $d\eta_t^j$ is constant across countries then the equilibrium is a symmetric balanced growth path from time t onwards. Therefore, to prove the symmetric balanced growth path is stable we need to show that cross-country variation in $d\eta_t^j$ dissipates over time, which requires $\lim_{t\to\infty}\left(d\eta_t^j-d\eta_t^i\right)=0$ for all $i,j=0,1,\ldots,J$.

Now remember that countries $j=1,\ldots,J$ are symmetric. Symmetry implies if y is any country specific variable then $dy_t^i=dy_t^j$ for all $i,j=1,\ldots,J$. In addition, (TA20) and (TA21) imply $\frac{d\tilde{\theta}_t^{ij}}{\tilde{\theta}_t^{ij}}$ and dg_t^{ij} are constant for all pairs i,j with $i\neq 0$ and $j\neq 0$. Consequently, we can simplify equations (TA18)-(TA32) by taking differences between pairs of equations for country 0 and country $j\neq 0$.

Start by observing that (TA20) implies:

$$\frac{d\tilde{\theta}_t^{00}}{\tilde{\theta}_t^{00}} - \frac{d\tilde{\theta}_t^{j0}}{\tilde{\theta}_t^{j0}} = \frac{d\tilde{\theta}_t^{0j}}{\tilde{\theta}_t^{0j}} - \frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}}, \tag{TA34}$$

and (TA21) implies:

$$dg_t^{00} - dg_t^{j0} = dg_t^{0j} - dg_t^{jj}. \tag{TA35}$$

Now, substituting (TA22), (TA32) and (TA34) into (TA25) gives:

$$\begin{aligned} 0 = & \left[\left(dq_{Mt}^0 + kdg_t^{00}\right) - \left(dq_{Mt}^j + kdg_t^{jj}\right)\right] + \frac{\lambda^k}{f_e}\frac{L_t}{M_t}\left[\left(\frac{d\eta_t^0}{\eta_t} - \frac{d\eta_t^j}{\eta_t}\right) - k\left(\frac{d\tilde{\theta}_t^{00}}{\tilde{\theta}_t^{00}} - \frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}}\right)\right] \\ & - \frac{\lambda^k}{f_e}\frac{k\sigma + 1 - \sigma}{k + 1 - \sigma}(J+1) f_x \tilde{\phi}^{-k} k \left(\frac{d\tilde{\theta}_t^{0j}}{\tilde{\theta}_t^{0j}} - \frac{d\tilde{\theta}_t^{00}}{\tilde{\theta}_t^{00}}\right). \end{aligned} \tag{TA36}$$

Substituting (TA20), (TA32) and (TA34) into (TA26) gives:

$$0 = \left(\frac{d\eta_t^0}{\eta_t} - \frac{d\eta_t^j}{\eta_t}\right) - \frac{k\sigma + 1 - \sigma}{\sigma}\left(\frac{d\tilde{\theta}_t^{00}}{\tilde{\theta}_t^{00}} - \frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}}\right) - \frac{2k\sigma + 1 - \sigma}{\sigma}\left(\frac{d\tilde{\theta}_t^{0j}}{\tilde{\theta}_t^{0j}} - \frac{d\tilde{\theta}_t^{00}}{\tilde{\theta}_t^{00}}\right). \tag{TA37}$$

Substituting (TA19), (TA24) and (TA35) into (TA21) gives:

$$\begin{aligned} 0 = & -\frac{\gamma(\sigma-1)}{\sigma-\gamma}\left(1 + J\frac{f_x}{f}\tilde{\phi}^{-k}\right)\left[\left(kdg_t^{00} + dr_t^0 - dq_{wt}^0\right) - \left(kdg_t^{jj} + dr_t^j - dq_{wt}^j\right)\right] \\ & + \left(1 - \frac{f_x}{f}\tilde{\phi}^{-k}\right)\left[\left(dq_{Mt}^0 + kdg_t^{00}\right) - \left(dq_{Mt}^j + kdg_t^{jj}\right)\right] + \frac{k\sigma + 1 - \sigma}{\sigma}(1+J)\frac{f_x}{f}\tilde{\phi}^{-k}\left(dg_t^{0j} - dg_t^{00}\right) \\ & + \frac{k\sigma + 1 - \sigma}{\sigma - \gamma}\frac{1}{f}\left[(\gamma - 1)\left(f - f_x\tilde{\phi}^{-k}\right) + \frac{\gamma(\sigma-1)}{\sigma}(1+J) f_x \tilde{\phi}^{-k}\right]\left(dg_t^{00} - dg_t^{jj}\right). \end{aligned} \tag{TA38}$$

Substituting (TA34) into (TA29) gives:

$$\begin{aligned} 0 = & \frac{f + J f_x \tilde{\phi}^{-k}}{kg + r - q}\left[\left(kdg_t^{00} + dr_t^0 - dq_{wt}^0\right) - \left(kdg_t^{jj} + dr_t^j - dq_{wt}^j\right)\right] \\ & + (1+J) f_x \tilde{\phi}^{-k} k \left(\frac{d\tilde{\theta}_t^{0j}}{\tilde{\theta}_t^{0j}} - \frac{d\tilde{\theta}_t^{00}}{\tilde{\theta}_t^{00}}\right), \end{aligned} \tag{TA39}$$

and substituting (TA21) and (TA35) into (TA27) gives:

$$0 = \left[\left(dq_{Mt}^{0} + kdg_t^{00}\right) - \left(dq_{Mt}^{j} + kdg_t^{jj}\right)\right] - \frac{k\sigma + 1 - \sigma}{\sigma}\left(dg_t^{00} - dg_t^{jj}\right)$$
$$- \frac{2k\sigma + 1 - \sigma}{\sigma}\left(dg_t^{0j} - dg_t^{00}\right). \quad \text{(TA40)}$$

To simplify notation define:

$$\begin{aligned}
Q_{0t} &\equiv \frac{d\eta_t^0}{\eta_t} - \frac{d\eta_t^j}{\eta_t}, \\
Q_{1t} &\equiv \left(dq_{Mt}^{0} + kdg_t^{00}\right) - \left(dq_{Mt}^{j} + kdg_t^{jj}\right), \\
Q_{2t} &\equiv \left(kdg_t^{00} + dr_t^0 - dq_{wt}^0\right) - \left(kdg_t^{jj} + dr_t^j - dq_{wt}^j\right), \\
Q_{3t} &\equiv \frac{d\tilde{\theta}_t^{00}}{\tilde{\theta}_t^{00}} - \frac{d\tilde{\theta}_t^{jj}}{\tilde{\theta}_t^{jj}}, \\
Q_{4t} &\equiv \frac{d\tilde{\theta}_t^{0j}}{\tilde{\theta}_t^{0j}} - \frac{d\tilde{\theta}_t^{00}}{\tilde{\theta}_t^{00}}, \\
Q_{5t} &\equiv dg_t^{00} - dg_t^{jj}, \\
Q_{6t} &\equiv dg_t^{0j} - dg_t^{00}.
\end{aligned}$$

Given Q_{0t}, (TA36)–(TA40) are five linear equations in six unknowns Q_{1t}–Q_{6t}. Observe also that:

$$Q_{0t} = Q_{00} + \int_0^t Q_{1v} dv.$$

or, equivalently:

$$\dot{Q}_{0t} = Q_{1t}. \quad \text{(TA41)}$$

A sixth equation comes from substituting (TA34) and (TA35) into (TA28) to obtain:

$$0 = \frac{k(1+J)f_x\tilde{\phi}^{-k}}{kg + r - q} Q_{4t} + \int_t^\infty \int_t^v \left[\left(f + Jf_x\tilde{\phi}^{-k}\right) Q_{2s} + k(1+J)f_x\tilde{\phi}^{-k} Q_{6s}\right] ds\, e^{-(kg+r-q)(v-t)} dv,$$

and integrating by parts this implies:

$$0 = k(1+J)f_x\tilde{\phi}^{-k} Q_{4t} + \int_t^\infty \left[\left(f + Jf_x\tilde{\phi}^{-k}\right) Q_{2v} + k(1+J)f_x\tilde{\phi}^{-k} Q_{6v}\right] e^{-(kg+r-q)(v-t)} dv. \quad \text{(TA42)}$$

Using (TA36)–(TA40) we can rewrite (TA42) as an equation in Q_{0t} and Q_{1t}. This gives:

$$0 = Q_{1t} + \beta_2 Q_{0t} + \int_t^\infty \left[\beta_3 Q_{1v} + \beta_4 Q_{0v}\right] e^{-(kg+r-q)(v-t)} dv, \quad \text{(TA43)}$$

where:

$$
\begin{aligned}
\beta_2 &\equiv -\frac{\sigma-1}{k\sigma+1-\sigma}\frac{L_t}{M_t}\frac{\lambda^k}{f_e}, \\
\beta_3 &\equiv -(kg+r-q)+\frac{1}{\tilde{\beta}}\frac{k\gamma}{\sigma-\gamma}\frac{\sigma-1}{\sigma}(k\sigma+1-\sigma)\frac{\lambda^k}{f_e}\left(1+J\frac{f_x}{f}\tilde{\phi}^{-k}\right) \\
&\quad\times\left[\frac{2k\sigma+1-\sigma}{k\sigma+1-\sigma}\frac{L_t}{M_t}-\frac{k\sigma}{k+1-\sigma}(1+J)f_x\tilde{\phi}^{-k}\right], \\
\beta_4 &\equiv \frac{\sigma-1}{\sigma}\frac{L_t}{M_t}\frac{\lambda^k}{f_e}(kg+r-q)\left[\frac{\sigma}{k\sigma+1-\sigma}+\frac{1}{\tilde{\beta}}\frac{k\gamma(\sigma-1)}{\sigma-\gamma}(1+J)\frac{f_x}{f}\tilde{\phi}^{-k}\right], \\
\tilde{\beta} &\equiv \frac{k\sigma+1-\sigma}{\sigma(\sigma-\gamma)f}\left[(2k\sigma+1-\sigma)(1-\gamma)\left(f+Jf_x\tilde{\phi}^{-k}\right)-k(\sigma-\gamma)(1+J)f_x\tilde{\phi}^{-k}\right].
\end{aligned}
$$

Now differentiating (TA43) and using (TA41) to substitute for $\dot{Q}_{0t}$ gives:

$$\dot{Q}_{1t}=\beta_0 Q_{0t}+\beta_1 Q_{1t}, \tag{TA44}$$

where:

$$
\begin{aligned}
\beta_0 &\equiv (kg+r-q)\beta_2+\beta_4, \\
\beta_1 &\equiv (kg+r-q)-\beta_2+\beta_3.
\end{aligned}
$$

Equations (TA41) and (TA44) are a system of two, linear, first order differential equations.

Asymmetries in the initial conditions affect the equilibrium only through Q_{0t} and a necessary and sufficient condition for stability is: $\lim_{t\to\infty} Q_{0t}=0$. Since $Q_{0t}=Q_{1t}=0$ is the unique steady state of (TA41) and (TA44), the symmetric balanced growth path is locally stable if and only if the system of differential equations (TA41) and (TA44) is stable. Analyzing (TA41) and (TA44), I find that $\gamma\leq 1$ is a sufficient condition for stability.[3] Proposition TA 1 summarizes this result.

Proposition TA 1. *The symmetric balanced growth path characterized in Proposition 1 is locally stable to asymmetries in the initial conditions across countries when the intertemporal elasticity of substitution* $\gamma\leq 1$.

Proof. Equations (TA41) and (TA44) can be written as:

$$\begin{pmatrix}\dot{Q}_{0t}\\ \dot{Q}_{1t}\end{pmatrix}=B\begin{pmatrix}Q_{0t}\\ Q_{1t}\end{pmatrix}, \qquad \text{where } B=\begin{pmatrix}0 & 1\\ \beta_0 & \beta_1\end{pmatrix}.$$

This system is globally stable if both eigenvalues of B have negative real parts and saddle path stable if one eigenvalue of B is negative. From the characteristic equation of B it then follows that if both $\beta_0<0$ and $\beta_1<0$ the system is globally stable, while if $\beta_0>0$ the system is saddle path stable.

[3]Alternatively, $k\geq 2$ is sufficient for stability for any value of γ. See the proof of Proposition TA 1.

We have:

$$\begin{aligned}
\beta_0 &= \frac{1}{\tilde{\beta}}\frac{(\sigma-1)^2}{\sigma}\frac{k\gamma}{\sigma-\gamma}\frac{L_t}{M_t}\frac{\lambda^k}{f_e}(1+J)\frac{f_x}{f}\tilde{\phi}^{-k}(kg+r-q), \\
\beta_1 &= \frac{\sigma-1}{k\sigma+1-\sigma}\frac{L_t}{M_t}\frac{\lambda^k}{f_e}+\frac{1}{\tilde{\beta}}\frac{k\gamma}{\sigma-\gamma}\frac{\sigma-1}{\sigma}(k\sigma+1-\sigma)\frac{\lambda^k}{f_e}\left(1+J\frac{f_x}{f}\tilde{\phi}^{-k}\right) \\
&\quad \times\left[\frac{2k\sigma+1-\sigma}{k\sigma+1-\sigma}\frac{L_t}{M_t}-\frac{k\sigma}{k+1-\sigma}(1+J)f_x\tilde{\phi}^{-k}\right],
\end{aligned}$$

where:

$$\tilde{\beta} \equiv \frac{k\sigma+1-\sigma}{\sigma(\sigma-\gamma)f}\left[(2k\sigma+1-\sigma)(1-\gamma)\left(f+Jf_x\tilde{\phi}^{-k}\right)-k(\sigma-\gamma)(1+J)f_x\tilde{\phi}^{-k}\right].$$

Note that $\tilde{\beta}>0$ if and only if $\gamma<\gamma_0\equiv\frac{(2k\sigma+1-\sigma)\left(f+Jf_x\tilde{\phi}^{-k}\right)-k\sigma(1+J)f_x\tilde{\phi}^{-k}}{(2k\sigma+1-\sigma)\left(f+Jf_x\tilde{\phi}^{-k}\right)-k(1+J)f_x\tilde{\phi}^{-k}}<1$ or $\gamma>\sigma$ meaning that $\beta_0>0$ if and only if $\gamma<\gamma_0$.

Using equation (27) to substitute for $\frac{L_t}{M_t}$ gives:

$$\begin{aligned}
\beta_1 &= \frac{1}{\tilde{\beta}}\frac{\sigma-1}{\sigma(\sigma-\gamma)f} \\
&\quad \Big\{(n+gk)\Big[(2k\sigma+1-\sigma)(k\gamma+1-\gamma)(f-f_x\tilde{\phi}^{-k})+(k\sigma+1-\sigma)(2k\gamma+1-\gamma)(1+J)f_x\tilde{\phi}^{-k}\Big] \\
&\quad +\frac{k\sigma+1-\sigma}{k+1-\sigma}\frac{\lambda^k}{f_e}\left(f+Jf_x\tilde{\phi}^{-k}\right)\Big[(2k\sigma+1-\sigma)(k\gamma+1-\gamma)(f-f_x\tilde{\phi}^{-k}) \\
&\quad +\left[(k+1-\sigma)(k\gamma+1-\gamma)+k(\sigma-1)\left[(k-2)\gamma+1\right]\right](1+J)f_x\tilde{\phi}^{-k}\Big]\Big\}.
\end{aligned}$$

Now note that $\frac{f_x}{f}\tilde{\phi}^{-k}=\tau^{1-\sigma}\left(\frac{\tau^{\sigma-1}f_x}{f}\right)^{-\frac{k+1-\sigma}{\sigma-1}}<1$ since by assumption $\tau\geq 1$, $\tau^{\sigma-1}f_x>f$ and $k>\sigma-1$. Careful inspection of the above expression then shows that if either $\gamma\leq 1$ or $k\geq 2$ then $\beta_1<0$ if and only if $\gamma>\gamma_0$. It follows that either $\gamma\leq 1$ or $k\geq 2$ is sufficient to ensure the system of differential equations (TA41) and (TA44) is stable. □

To understand why the symmetric balanced growth path is stable consider a shock to the symmetric equilibrium that either increases the initial mass of producers in country zero or shifts the initial productivity distribution in country zero upwards. This shock raises aggregate productivity and real wages in country zero relative to the rest of the world. Following the shock aggregate demand in country zero is higher than in the other countries, making country zero the most attractive export market. Consequently, the ratio of the export threshold to the exit cut-off in country zero is higher than in the rest of the world, implying that a smaller fraction of country zero firms choose to export. Effectively, country zero is less open to trade than the rest of the world, where openness is measured by the ratio of exports to domestic sales.[4] Since

[4] For a related result in a static economy see Demidova and Rodríguez-Clare (2013) who introduce asymmetries in country size

the free entry condition mandates that openness increases growth, the dynamic selection effect is weaker in country zero than elsewhere. Therefore, country zero grows more slowly than the rest of the world and the world economy converges to the symmetric balanced growth path. Intuitively, since trade raises growth and trade is relatively less important in larger markets, market size differences resulting from variation in initial conditions dissipate over time.

References

Melitz, Marc J., "The Impact of Trade on Intra-industry Reallocations and Aggregate Industry Productivity," *Econometrica*, 71 (2003), 1695-1725.

Demidova, Svetlana, and Andres Rodríguez-Clare, "The Simple Analytics of the Melitz Model in a Small Economy," *Journal of International Economics*, 90 (2013): 266-272.

into Melitz (2003) and show that as the size difference between countries increases the larger country approaches autarky, while the smaller country continues to trade.