EXTENSION OF THE ν -METRIC

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ABSTRACT. We extend the ν -metric introduced by Vinnicombe in robust control theory for rational plants to the case of infinite-dimensional systems/classes of nonrational transfer functions.

1. INTRODUCTION

The general stabilization problem in control theory is as follows. Suppose that R is a commutative integral domain with identity (thought of as the class of stable transfer functions) and let $\mathbb{F}(R)$ denote the field of fractions of R. The stabilization problem is:

Given $P \in (\mathbb{F}(R))^{p \times m}$ (an unstable plant transfer function), find $C \in (\mathbb{F}(R))^{m \times p}$ (a stabilizing controller transfer function), such that (the closed loop transfer function)

$$H(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix}$$

belongs to $R^{(p+m)\times(p+m)}$ (is stable).

Recipes for constructing such C is a central theme in control theory; see for example the book by Vidyasagar [24].

However, in the robust stabilization problem, one goes a step further. One knows that the plant is just an approximation of reality, and so one would really like the controller C to not only stabilize the nominal plant P_0 , but also all sufficiently close plants P to P_0 . The question of what one means by "closeness" of plants thus arises naturally. So one needs a function d defined on pairs of stabilizable plants such that

- (1) d is a metric on the set of all stabilizable plants,
- (2) d is amenable to computation, and
- (3) d has "good" properties in the robust stabilization problem.

Such a desirable metric, was introduced by Glenn Vinnicombe in [25] and is called the ν -metric. In that paper, essentially R was taken to be the rational functions without poles in the closed unit disk or, more generally, the disk algebra, and the most important results were that the ν -metric is indeed a metric on the set of stabilizable plants, and moreover, it has the following nice property in the context of the robust stabilization problem:

¹⁹⁹¹ Mathematics Subject Classification. Primary 93B36; Secondary 93D15, 46J15. Key words and phrases. ν -metric, robust control, Banach algebras.

(P): If the ν -metric between two stabilizable plants P_0 and P is less than the stability margin $\mu_{P_0,C}$ of P_0 and its stabilizing controller C, then C also stabilizes P.

The problem of what happens when R is some other ring of stable transfer functions of infinite-dimensional (that is, one time axis and infinitedimensional state space) or multidimensional systems (several "time" axes of evolution) was left open. This problem of extending the ν -metric from the rational case to transfer function classes of infinite-dimensional systems was also mentioned in article by Nicholas Young [26]. In this article, we address this issue of extending the ν -metric.

The starting point for our approach is abstract: we suppose that R is any commutative integral domain with identity which is a subset of a Banach algebra S satisfying certain assumptions, which we label (A1)-(A4). We then define an "abstract" ν -metric in this setup, and show that it does define a metric on the class of all stabilizable plants. We also show that it has the desired property (P) in the context of robust stabilization for an appropriate definition of stability margin $\mu_{P0,C}$.

Next we give several examples of integral domains R arising as natural classes of stable transfer functions of infinite-dimensional and multidimensional systems which satisfy the abstract assumptions (A1) to (A4). In particular, we cover the case of full subalgebras of the disk algebra, the causal almost periodic function classes, the class of measures on $[0, +\infty)$ without a singular nonatomic part, and the polydisk algebra.

The paper is organized as follows:

- (1) In Section 2, we give our general setup and assumptions, and define the abstract metric d_{ν} .
- (2) In Section 3, we will show that d_{ν} is a metric on the set of stabilizable plants.
- (3) In Section 4, we introduce a notion of stability margin $\mu_{P,C}$ and prove Theorem 4.6; this implies in particular that if the ν -metric between two stabilizable plants P_0 and P is less than the stability margin $\mu_{P_0,C}$ of P and its stabilizing controller C, then C also stabilizes P.
- (4) In Section 5, we specialize R to concrete rings of stable transfer functions of various types, and show that our abstract assumptions hold in these particular cases.
- (5) The final Section 6 mentions a loose end which is a direction for further work.

2. General setup and assumptions

Our setup is the following:

- (A1) R is commutative integral domain with identity.
- (A2) S is a unital commutative complex semisimple Banach algebra with an involution \cdot^* , such that $R \subset S$. We use inv S to denote the invertible elements of S.

- (A3) There exists a map ι : inv $S \to G$, where (G, +) is an Abelian group with identity denoted by \circ , and ι satisfies
 - (I1) $\iota(ab) = \iota(a) + \iota(b) \ (a, b \in \text{inv } S).$
 - (I2) $\iota(a^*) = -\iota(a) \ (a \in \text{inv } S).$
 - (I3) ι is locally constant, that is, ι continuous when G is equipped with the discrete topology.
- (A4) $x \in R \cap (\text{inv } S)$ is invertible as an element of R iff $\iota(x) = \circ$.

A consequence of (I3) is the following "homotopic invariance of the index", which we will use in the sequel.

Proposition 2.1. If $H : [0,1] \to \text{inv } S$ is a continuous map, then

$$\iota(H(0)) = \iota(H(1)).$$

Proof. The map h, given by $t \mapsto \iota(H(t)) : [0,1] \to G$ is continuous. Here [0,1] is equipped with usual topology from \mathbb{R} , while G is equipped with the discrete topology, given by the metric

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \quad (x,y \in G).$$

The image of the connected set [0, 1] under the continuous map h is connected. But the only connected subsets of G are the singleton sets, since G is carrying the discrete topology. Hence $\iota(H(0)) = \iota(H(1))$.

We recall the following standard definitions from the factorization approach to control theory.

Definition 2.2.

The notation $\mathbb{F}(R)$: $\mathbb{F}(R)$ denotes the field of fractions of R.

The notation F^* : If $F \in \mathbb{R}^{p \times m}$, then $F^* \in S^{m \times p}$ is the matrix with the entry in the *i*th row and *j*th column given by F_{ji}^* , for all $1 \le i \le p$, and all $1 \le j \le m$.

Right coprime/normalized coprime factorization: Given a matrix $P \in (\mathbb{F}(R))^{p \times m}$, a factorization $P = ND^{-1}$, where N, D are matrices with entries from R, is called a *right coprime factorization of* P if there exist matrices X, Y with entries from R such that $XN + YD = I_m$. If moreover there holds that $N^*N + D^*D = I_m$, then the right coprime factorization is referred to as a *normalized* right coprime factorization of P.

Left coprime/normalized coprime factorization: Similarly, a factorization $P = \tilde{D}^{-1}\tilde{N}$, where \tilde{N}, \tilde{D} are matrices with entries from R, is called a *left coprime factorization of* P if there exist matrices \tilde{X}, \tilde{Y} with entries from R such that $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_p$. If moreover there holds that $\tilde{N}\tilde{N}^* + \tilde{D}\tilde{D}^* = I_p$, then the left coprime factorization is referred to as a *normalized* left coprime factorization of P. We note that the existence of both a left and right normalized factorization $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ for P leads immediately to a normalized double coprime factorization of P, i.e., one has the identity

$$\begin{bmatrix} N^* & D^* \\ -\widetilde{D} & \widetilde{N} \end{bmatrix} \begin{bmatrix} N & -\widetilde{D}^* \\ D & \widetilde{N}^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$
 (2.1)

Since we are dealing with finite matrices over a commutative ring, (2.1) implies also the identity

$$\begin{bmatrix} N & -\widetilde{D}^* \\ D & \widetilde{N}^* \end{bmatrix} \begin{bmatrix} N^* & D^* \\ -\widetilde{D} & \widetilde{N} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$
 (2.2)

The notation $G, \tilde{G}, K, \tilde{K}$: Given $P \in (\mathbb{F}(R))^{p \times m}$ with normalized right and left factorizations $P = ND^{-1}$ and $P = \tilde{D}^{-1}\tilde{N}$, respectively, we introduce the following matrices with entries from R:

$$G = \begin{bmatrix} N \\ D \end{bmatrix}$$
 and $\widetilde{G} = \begin{bmatrix} -\widetilde{D} & \widetilde{N} \end{bmatrix}$.

In this notation the fact that the left and right coprime factorizations of ${\cal P}$ are normalized translates to

$$G^*G = I, \quad \widetilde{G}\widetilde{G}^* = I$$
 (2.3)

and the identity (2.2) assumes the form

$$GG^* + \tilde{G}^*\tilde{G} = I. \tag{2.4}$$

Similarly, given $C \in (\mathbb{F}(R))^{m \times p}$ with normalized right and left factorizations $C = N_C D_C^{-1}$ and $C = \widetilde{D}_C^{-1} \widetilde{N}_C$, respectively, we introduce the following matrices with entries from R:

$$K = \begin{bmatrix} D_C \\ N_C \end{bmatrix} \quad \text{and} \quad \widetilde{K} = \begin{bmatrix} -\widetilde{N}_C & \widetilde{D}_C \end{bmatrix}.$$

The notation $\mathbb{S}(R, p, m)$: We denote by $\mathbb{S}(R, p, m)$ the set of all elements $P \in (\mathbb{F}(R))^{p \times m}$ that posses a normalized right coprime factorization and a normalized left coprime factorization.

Remark 2.3. Given $P \in (\mathbb{F}(R))^{p \times m}$ and $C \in (\mathbb{F}(R))^{m \times p}$, define the closed loop transfer function

$$H(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \in (\mathbb{F}(R))^{(p+m) \times (p+m)}$$

It can be shown (see for example [24, Chapter 8]) that if $P \in S(R, p, m)$, then P is a stabilizable plant, that is,

$$\mathbb{S}(R,p,m) \subset \left\{ P \in (\mathbb{F}(R))^{p \times m} \middle| \begin{array}{l} \exists C \in (\mathbb{F}(R))^{m \times p} \text{ such that} \\ H(P,C) \in R^{(p+m) \times (p+m)} \end{array} \right\}.$$
(2.5)

It was shown by A. Quadrat [18, Theorem 6.3] that if the Banach algebra R is a projective-free ring, then every stabilizable plant admits a right coprime factorization and a left coprime factorization, that is, the reverse containment \supset and hence equality holds in (2.5).

We will need a couple of straightforward results on coprime factorizations, which we have listed below. The first lemma says that coprime factorizations are unique up to invertibles.

Lemma 2.4. Let $P \in (\mathbb{F}(R))^{p \times m}$.

- (1) If P has right coprime factorizations $P = N_1 D_1^{-1} = N_2 D_2^{-1}$, then there exist $V, \Lambda \in \mathbb{R}^{m \times m}$ such that $V\Lambda = \Lambda V = I_m$, $N_1 = N_2 V$ and $D_1 = D_2 V$.
- (2) If P has left coprime factorizations $P = \widetilde{D}_1^{-1}\widetilde{N}_1 = \widetilde{D}_2^{-1}\widetilde{N}_2$, then there exist $\widetilde{V}, \widetilde{\Lambda} \in \mathbb{R}^{p \times p}$ such that $\widetilde{V}\widetilde{\Lambda} = \widetilde{\Lambda}\widetilde{V} = I_p$, $\widetilde{N}_1 = \widetilde{V}\widetilde{N}_2$ and $\widetilde{D}_1 = \widetilde{V}\widetilde{D}_2$.

In the case of normalized coprime factorizations, the invertibles can be chosen to be unitary.

Lemma 2.5. Let $P \in (\mathbb{F}(R))^{p \times m}$.

- (1) If P has normalized right coprime factorizations $P = N_1 D_1^{-1} = N_2 D_2^{-1}$, then there exists a $U \in \mathbb{R}^{m \times m}$, which is invertible as an element of $\mathbb{R}^{m \times m}$, and such that $U^*U = UU^* = I_m$, $N_1 = N_2 U$ and $D_1 = D_2 U$.
- (2) If P has normalized left coprime factorizations $P = \widetilde{D}_1^{-1}\widetilde{N}_1 = \widetilde{D}_2^{-1}\widetilde{N}_2$, then there exists a $\widetilde{U} \in \mathbb{R}^{p \times p}$ which is invertible as an element of $\mathbb{R}^{p \times p}$, and such that $\widetilde{U}^*\widetilde{U} = \widetilde{U}\widetilde{U}^* = I_p$, $\widetilde{N}_1 = \widetilde{U}\widetilde{N}_2$ and $\widetilde{D}_1 = \widetilde{U}\widetilde{D}_2$.

Lemma 2.6. Suppose that $F \in \mathbb{R}^{m \times m}$, det $F \in \text{inv } S$ and $\iota(\det F) = \circ$. Then F is invertible as an element of $\mathbb{R}^{m \times m}$.

Proof. Since det $F \in \text{inv } S$ and $\iota(\det F) = \circ$, it follows from (A4) that det F is invertible as an element of R. The result then follows from Cramer's rule.

We now define the metric d_{ν} on $\mathbb{S}(R, p, m)$. But first we specify the norm we use for matrices with entries from S.

Definition 2.7 $(\|\cdot\|_{\infty})$. Let \mathfrak{M} denote the maximal ideal space of the Banach algebra S. For a matrix $M \in S^{p \times m}$, we set

$$\|M\|_{\infty} = \max_{\varphi \in \mathfrak{M}} \|\mathbf{M}(\varphi)\|.$$
(2.6)

Here **M** denotes the entry-wise Gelfand transform of M, and $|\cdot|$ denotes the induced operator norm from \mathbb{C}^m to \mathbb{C}^p . For the sake of concreteness, we fix the standard Euclidean norms on the vector spaces \mathbb{C}^m to \mathbb{C}^p .

The maximum in (2.6) exists since \mathfrak{M} is a compact space when it is equipped with Gelfand topology, that is, the weak-* topology induced from $\mathcal{L}(S;\mathbb{C})$. Since we have assumed S to be semisimple, the Gelfand transform

$$\widehat{\cdot} : S \to S \ (\subset C(\mathfrak{M}, \mathbb{C}))$$

is an isomorphism. If $M \in S^{1 \times 1} = S$, then we note that there are two norms available for M: the one as we have defined above, namely $||M||_{\infty}$, and the

norm $\|\cdot\|$ of M as an element of the Banach algebra S. But throughout this article, we will use the norm given by (2.6).

Definition 2.8 (Abstract ν -metric d_{ν}). For $P_1, P_2 \in \mathbb{S}(R, p, m)$, with the normalized left/right coprime factorizations

$$P_1 = N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1,$$

$$P_2 = N_2 D_2^{-1} = \tilde{D}_2^{-1} \tilde{N}_2,$$

we define

$$d_{\nu}(P_1, P_2) := \begin{cases} \|\widetilde{G}_2 G_1\|_{\infty} & \text{if } \det(G_1^* G_2) \in \text{inv } S \text{ and } \iota(\det(G_1^* G_2)) = \circ, \\ 1 & \text{otherwise.} \end{cases}$$

$$(2.7)$$

Normalized coprime factorizations are not unique for a given plant in $\mathbb{S}(R, p, m)$. But we have the following:

Lemma 2.9. d_{ν} given by (2.7) is well-defined.

Proof. This follows from Lemma 2.5.

Lemma 2.10. d_{ν} given by (2.7) is bounded above by 1.

Proof. We have $\|\widetilde{G}_2G_1\|_{\infty} \leq \|\widetilde{G}_2\|_{\infty}\|G_1\|_{\infty}$. As $G_1^*G_1 = I_m$ and $\widetilde{G}_2\widetilde{G}_2^* = I_p$, we see that $\|G_1\|_{\infty} = 1 = \|\widetilde{G}_2\|_{\infty}$.

In Section 3, we will also prove the following.

Theorem 2.11. d_{ν} given by (2.7) is a metric on $\mathbb{S}(R, p, m)$.

We recall the definition of singular values of a square matrix, and a few properties which will be needed in the sequel.

Definition 2.12. If $M \in \mathbb{C}^{k \times k}$, then the set of eigenvalues of MM^* and M^*M are equal and the eigenvalues are real. The square roots of these eigenvalues are called the *singular values of* M, and the largest of these is denoted by $\overline{\sigma}(M)$, while the smallest of these is denoted by $\underline{\sigma}(M)$.

Proposition 2.13. The following hold for $P, Q \in \mathbb{C}^{k \times k}$.

- (S1) $||P|| = \overline{\sigma}(P).$
- (S2) If P is invertible, then $\underline{\sigma}(P) > 0$, and $||P^{-1}|| = (\underline{\sigma}(P))^{-1}$.
- (S3) $|\underline{\sigma}(P+Q) \underline{\sigma}(P)| \leq \overline{\sigma}(Q).$
- $(S4) \ \overline{\sigma}(PQ) \le \overline{\sigma}(P) \cdot \overline{\sigma}(Q).$
- (S5) $\underline{\sigma}(PQ) \ge \underline{\sigma}(P) \cdot \underline{\sigma}(Q).$
- (S6) $\overline{\sigma}(PQ) = \overline{\sigma}((P^*P)^{\frac{1}{2}}Q) = \overline{\sigma}(P(QQ^*)^{\frac{1}{2}}).$

Proof. (S1), (S2) follow from the spectral theorem. (S3), (S4), (S5) are given in [1, Proposition 9.6.8, Corollary 9.6.6]. (S6) can be verified directly using the definition of $\overline{\sigma}$.

Lemma 2.14. Suppose that $A, B \in \mathbb{C}^{p \times m}$ and that $A^*A + B^*B = I$. Then $(\underline{\sigma}(A))^2 + (\overline{\sigma}(B))^2 = 1$.

Proof. This follows from the spectral theorem. Indeed, $A^*A = I - B^*B$, and so for all $x \in \mathbb{C}^m$ with unit norm, we have

$$\langle A^*Ax, x \rangle = \langle x, x \rangle - \langle B^*Bx, x \rangle = 1 - \langle B^*Bx, x \rangle.$$

Thus
$$(\overline{\sigma}(A))^2 = \max_{\|x\|=1} \langle A^*Ax, x \rangle = 1 - \min_{\|x\|=1} \langle B^*Bx, x \rangle = 1 - (\underline{\sigma}(B))^2.$$

In particular, we have the following consequence as an application of this lemma. (In this article, we often suppress the argument of the Gelfand transforms of matrices with S-entries.)

Lemma 2.15. If $P_1, P_2 \in \mathbb{S}(R, p, m)$, then $\left(\underline{\sigma}(\mathbf{G}_2^*\mathbf{G}_1)\right)^2 + \left(\overline{\sigma}(\widetilde{\mathbf{G}}_2\mathbf{G}_1)\right)^2 = 1$ pointwise on \mathfrak{M} .

Proof. Observing from (2.3) and (2.4) that $G_1^*G_1 = I$ and $G_2G_2^* + \widetilde{G}_2^*\widetilde{G}_2 = I$, we obtain

 $\mathbf{G}_{1}^{*}\mathbf{G}_{2}\mathbf{G}_{2}^{*}\mathbf{G}_{1} + \mathbf{G}_{1}^{*}\widetilde{\mathbf{G}}_{2}^{*}\widetilde{\mathbf{G}}_{2}\mathbf{G}_{1} = I$

pointwise on \mathfrak{M} . An application of Lemma 2.14 now yields the result. \Box

3. d_{ν} is a metric

In this section, we will prove Theorem 2.11.

Proof (of Theorem 2.11).

3.1. **Positivity.** If $P_1, P_2 \in \mathbb{S}(R, p, m)$, then clearly $d_{\nu}(P_1, P_2) \ge 0$. Also, if $d_{\nu}(P_1, P_2) = 0$, then $\|\widetilde{G}_2G_1\|_{\infty} = 0$, and so $\widetilde{G}_2G_1 = 0$. But

$$\widetilde{G}_2 G_1 = \widetilde{D}_2 (P_2 - P_1) D_1.$$

Thus $P_1 = P_2$. Finally, for $P \in \mathbb{S}(R, p, m)$, it is clear that $d_{\nu}(P, P) = 0$.

3.2. Symmetry. Let $P_1, P_2 \in S(R, p, m)$. Since $G_1^*G_2 = (G_2^*G_1)^*$, it follows that $\det(G_1^*G_2)$ is invertible as an element of S iff $\det(G_2^*G_1)$ is invertible as an element of S. Using (I2), we see that $\iota(\det(G_1^*G_2)) = \circ$ iff $\iota(\det(G_2^*G_1)) = \circ$. Hence $d_{\nu}(P_1, P_2) = d_{\nu}(P_2, P_1)$.

3.3. The triangle inequality. Suppose that $P_1, P_2, P_0 \in S(R, p, m)$. We want to show that $d_{\nu}(P_1, P_2) \leq d_{\nu}(P_1, P_0) + d_{\nu}(P_0, P_2)$. Since d_{ν} is bounded above by 1, this inequality is trivially satisfied if either $d_{\nu}(P_1, P_0) = 1$ or $d_{\nu}(P_0, P_2) = 1$. So in the rest of this subsection, we will assume that $d_{\nu}(P_1, P_0) < 1$ and $d_{\nu}(P_0, P_2) < 1$. This means that

- (1) det $(G_1^*G_0)$ is invertible in S and $\iota(\det(G_1^*G_0)) = \circ$.
- (2) det $(G_0^*G_2)$ is invertible in S and $\iota(\det(G_0^*G_2)) = \circ$.

We will consider separately the following two possible cases:

<u>1</u>° det($G_1^*G_2$) ∈ inv S and $\iota(\det(G_1^*G_2)) = \circ$. Then $d_\nu(P_1, P_2) = \|\widetilde{G}_2G_1\|_{\infty}$. <u>2</u>° ¬[det($G_1^*G_2$) ∈ inv S and $\iota(\det(G_1^*G_2)) = \circ$]. Then $d_\nu(P_1, P_2) = 1$.

First, using the fact (2.4) that $G_0 G_0^* + \widetilde{G}_0^* \widetilde{G}_0 = I$, we obtain that

$$G_1^* G_2 = G_1^* G_0 G_0^* G_2 + G_1^* \widetilde{G}_0^* \widetilde{G}_0 G_2.$$
(3.1)

<u>1</u>° Suppose that $\det(G_1^*G_2) \in \text{inv } S$ and $\iota(\det(G_1^*G_2)) = \circ$. In this case, $d_{\nu}(P_1, P_2) = \|\widetilde{G}_2G_1\|_{\infty}$. Using (S3) from Proposition 2.13, with

$$P := \mathbf{G}_1^* \mathbf{G}_0 \mathbf{G}_0^* \mathbf{G}_2,$$
$$Q := \mathbf{G}_1^* \widetilde{\mathbf{G}}_0^* \widetilde{\mathbf{G}}_0 \mathbf{G}_2.$$

and (3.1), we have $\underline{\sigma}(\mathbf{G}_{1}^{*}\mathbf{G}_{0}\mathbf{G}_{0}^{*}\mathbf{G}_{2}) - \underline{\sigma}(\mathbf{G}_{1}^{*}\mathbf{G}_{2}) \leq \overline{\sigma}(\mathbf{G}_{1}^{*}\widetilde{\mathbf{G}}_{0}^{*}\widetilde{\mathbf{G}}_{0}\mathbf{G}_{2})$ pointwise on \mathfrak{M} . Furthermore, using (S4) and (S5), and rearranging, we obtain

$$\underline{\sigma}(\mathbf{G}_{1}^{*}\mathbf{G}_{2}) \geq \underline{\sigma}(\mathbf{G}_{1}^{*}\mathbf{G}_{0}) \cdot \underline{\sigma}(\mathbf{G}_{0}^{*}\mathbf{G}_{2}) - \overline{\sigma}(\widetilde{\mathbf{G}}_{0}\mathbf{G}_{1}) \cdot \overline{\sigma}(\widetilde{\mathbf{G}}_{0}\mathbf{G}_{2})$$
(3.2)

pointwise on \mathfrak{M} . Since $d_{\nu}(P_1, P_0)$ and $d_{\nu}(P_0, P_2)$ are both in [0, 1], we can find α, β (which are maps from \mathfrak{M} to $[0, \frac{\pi}{2}]$) such that

$$\begin{aligned} \sin \alpha &= \overline{\sigma}(\mathbf{G_0}\mathbf{G_1}), \\ \sin \beta &= \overline{\sigma}(\widetilde{\mathbf{G}_0}\mathbf{G_2}), \end{aligned}$$

pointwise on \mathfrak{M} . Then using Lemma 2.15, it follows from (3.2) that

$$\underline{\sigma}(\mathbf{G}_{1}^{*}\mathbf{G}_{2}) \ge (\cos\alpha) \cdot (\cos\beta) - (\sin\alpha) \cdot (\sin\beta) = \cos(\alpha + \beta)$$
(3.3)

pointwise on \mathfrak{M} . Similarly, define $\gamma : \mathfrak{M} \to [0, \frac{\pi}{2}]$ by $\overline{\sigma}(\mathbf{G_1}\mathbf{G_2}) = \sin \gamma$, then $\underline{\sigma}(\mathbf{G_1^*G_2}) = \cos \gamma$ pointwise on \mathfrak{M} . The inequality (3.3) now says that $\cos \gamma \geq \cos(\alpha + \beta)$ pointwise on \mathfrak{M} . Hence

$$\sin\gamma \leq \sin(\alpha + \beta) = (\sin\alpha) \cdot (\cos\beta) + (\sin\beta) \cdot (\cos\alpha) \leq (\sin\alpha) \cdot 1 + (\sin\beta) \cdot 1,$$

that is, $\overline{\sigma}(\widetilde{\mathbf{G}}_{1}\mathbf{G}_{2}) \leq \overline{\sigma}(\widetilde{\mathbf{G}}_{0}\mathbf{G}_{1}) + \overline{\sigma}(\widetilde{\mathbf{G}}_{0}\mathbf{G}_{2}) \leq d_{\nu}(P_{1}, P_{0}) + d_{\nu}(P_{0}, P_{2})$ pointwise on \mathfrak{M} . Consequently, $d_{\nu}(P_{1}, P_{2}) = \|\widetilde{G}_{1}G_{2}\|_{\infty} \leq d_{\nu}(P_{1}, P_{0}) + d_{\nu}(P_{0}, P_{2}).$

 $\underline{2}^{\circ} \neg [\det(G_1^*G_2) \in \text{inv } S \text{ and } \iota(\det(G_1^*G_2)) = \circ].$ In this case $d_{\nu}(P_1, P_2) = 1$. Let

$$A := G_1^* G_0 G_0^* G_2, \text{ and}$$
$$B := G_1^* \widetilde{G}_0^* \widetilde{G}_0 G_2.$$

Using the fact that $G_1^*G_0$ and $G_0^*G_2$ are invertible in $S^{m \times m}$, it follows also that A is invertible in $S^{m \times m}$.

Suppose that $||A^{-1}B||_{\infty} < 1$. Then it follows from (3.1) that

$$G_1^*G_2 = A + B = A(I + A^{-1}B)$$

and so $G_1^*G_2$ is also invertible in $S^{m \times m}$. Consider the map $H : [0, 1] \to \text{inv } S$, given by $H(t) = \det(A(I + tA^{-1}B)), t \in [0, 1]$. By Proposition 2.1,

$$\circ = \circ + \circ = \iota(G_1^*G_0) + \iota(G_0^*G_2) = \iota(\det A) = \iota(H(0)) \\ = \iota(H(1)) = \iota(\det(G_1^*G_2)).$$

But then we have that $\det(G_1^*G_2) \in \text{inv } S$ and $\iota(\det(G_1^*G_2)) = \circ$, which is a contradiction.

So our assumption that $||A^{-1}B||_{\infty} < 1$ cannot be true. From the compactness of \mathfrak{M} and the definition of the norm on $\mathbb{C}^{m \times m}$, it follows that there is a $\varphi \in \mathfrak{M}$ such that $\overline{\sigma}((\mathbf{A}^{-1}\mathbf{B})(\varphi)) \geq 1$. But then we have that

$$1 \le \overline{\sigma}((\mathbf{A}^{-1}\mathbf{B})(\varphi)) \le \overline{\sigma}((\mathbf{A}(\varphi))^{-1}) \cdot \overline{\sigma}(\mathbf{B}(\varphi)),$$

and so

$$\underline{\sigma}(\mathbf{A}(\varphi)) \le \overline{\sigma}(\mathbf{B}(\varphi)). \tag{3.4}$$

Thus

$$(1 - (\overline{\sigma}((\mathbf{G_0}\mathbf{G_1})(\varphi)))^2) \cdot (1 - (\overline{\sigma}((\mathbf{G_0}\mathbf{G_2})(\varphi)))^2)$$

$$= (\underline{\sigma}((\mathbf{G_1^*}\mathbf{G_0})(\varphi))^2 \cdot (\underline{\sigma}((\mathbf{G_0^*}\mathbf{G_2})(\varphi))^2 \text{ by Lemma 2.15})$$

$$\leq \underline{\sigma}((G_1^*G_0G_0^*G_2)(\varphi))^2 \text{ by (S5) in Proposition 2.13}$$

$$\leq \overline{\sigma}((G_1^*\widetilde{G_0}G_0G_2(\varphi))^2 \text{ by (3.4)})$$

$$\leq (\overline{\sigma}((\widetilde{\mathbf{G_0}}\mathbf{G_1})(\varphi)))^2 \cdot (\overline{\sigma}((\widetilde{\mathbf{G_0}}\mathbf{G_2})(\varphi)))^2 \text{ by (S4) in Proposition 2.13}.$$

With

$$\begin{aligned} x &:= \overline{\sigma}((\widetilde{\mathbf{G}}_{\mathbf{0}}\mathbf{G}_{\mathbf{1}})(\varphi)), \text{ and} \\ y &:= \overline{\sigma}((\widetilde{\mathbf{G}}_{\mathbf{0}}\mathbf{G}_{\mathbf{2}})(\varphi)), \end{aligned}$$

the above says that $(1 - x^2) \cdot (1 - y^2) \le x^2 y^2$, and so $1 \le x^2 + y^2$. Thus

$$1 \le (\overline{\sigma}((\widetilde{\mathbf{G}}_{\mathbf{0}}\mathbf{G}_{\mathbf{1}})(\varphi)))^2 + (\overline{\sigma}((\widetilde{\mathbf{G}}_{\mathbf{0}}\mathbf{G}_{\mathbf{2}})(\varphi)))^2 \le (d_{\nu}(P_0, P_1))^2 + (d_{\nu}(P_0, P_2))^2.$$

Consequently,

$$(d_{\nu}(P_0, P_1) + d_{\nu}(P_0, P_2))^2 \ge (d_{\nu}(P_0, P_1))^2 + (d_{\nu}(P_0, P_2))^2 \ge 1 = (d_{\nu}(P_1, P_2))^2.$$

Taking square roots, we obtain the desired conclusion.

This completes the proof of the triangle inequality, and also the proof of Theorem 2.11. $\hfill \Box$

4. Robust stability theorem

In this section we prove Theorem 4.6.

Definition 4.1. Given $P \in (\mathbb{F}(R))^{p \times m}$ and $C \in (\mathbb{F}(R))^{m \times p}$, we define the *stability margin* of the pair (P, C) by

$$\mu_{P,C} = \begin{cases} \|H(P,C)\|_{\infty}^{-1} & \text{if } P \text{ is stabilized by } C, \\ 0 & \text{otherwise.} \end{cases}$$

The number $\mu_{P,C}$ can be interpreted as a measure of the performance of the closed loop system comprising P and C: larger values of $\mu_{P,C}$ correspond to better performance, with $\mu_{P,C} > 0$ if C stabilizes P.

Proposition 4.2. If *P* is stabilized by *C*, then
$$\mu_{P,C} = \inf_{\varphi \in \mathfrak{M}} \underline{\sigma}(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi)).$$

Proof. We now write $P = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$ for a normalized left/right coprime factorization of P and $C = N_c M_c^{-1} = \widetilde{M}_c^{-1} \widetilde{N}_c$ for a normalized left/right coprime factorization of C and we set

$$G = \begin{bmatrix} N \\ M \end{bmatrix}, \quad \widetilde{G} = \begin{bmatrix} -\widetilde{M} & \widetilde{N} \end{bmatrix}, \quad K = \begin{bmatrix} N_c \\ M_c \end{bmatrix}, \quad \widetilde{K} = \begin{bmatrix} -\widetilde{N}_c & \widetilde{M}_c \end{bmatrix}.$$
Then we have

Then we have

$$H(P,C) = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix}$$

$$= \begin{bmatrix} NM^{-1} \\ I \end{bmatrix} (I - \widetilde{M}_c^{-1}\widetilde{N}_c N M^{-1})^{-1} \begin{bmatrix} -\widetilde{M}_c^{-1}\widetilde{N}_c & I \end{bmatrix}$$

$$= \begin{bmatrix} N \\ M \end{bmatrix} (-\widetilde{N}_c N + \widetilde{M}_c M)^{-1} \begin{bmatrix} -\widetilde{N}_c & \widetilde{M}_c \end{bmatrix}$$

$$= G(\widetilde{K}G)^{-1}\widetilde{K}.$$

Also, $G^*G = \begin{bmatrix} N^* & M^* \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix} = N^*N + M^*M = I$. Similarly, $\widetilde{K}\widetilde{K}^* = \begin{bmatrix} -\widetilde{N}_c & \widetilde{M}_c \end{bmatrix} \begin{bmatrix} -\widetilde{N}_c^* \\ \widetilde{M}_c^* \end{bmatrix} = \widetilde{N}_c\widetilde{N}_c^* + \widetilde{M}_c\widetilde{M}_c^* = I$.

Using (S6) of Proposition 2.13, we obtain for each $\varphi \in \mathfrak{M}$ that

$$\overline{\sigma}(\mathbf{G}(\varphi)(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi))^{-1}\widetilde{\mathbf{K}}(\varphi)) = \overline{\sigma}((\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi))^{-1}) = \frac{1}{\underline{\sigma}(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi))}.$$

Thus

and

$$\frac{1}{\mu_{P,C}} = \sup_{\varphi \in \mathfrak{M}} \overline{\sigma}(\mathbf{G}(\varphi)(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi))^{-1}\widetilde{\mathbf{K}}(\varphi)) = \sup_{\varphi \in \mathfrak{M}} \frac{1}{\underline{\sigma}(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi))},$$

so $\mu_{P,C} = \inf_{\varphi \in \mathfrak{M}} \underline{\sigma}(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi)).$

Remark 4.3. It is useful to note that

$$\mu_{P,C} < 1 \tag{4.1}$$

for any P and C as above. One way to see this is to note that H(P,C) is idempotent $H(P,C) \cdot H(P,C) = H(P,C)$; this forces $||H(P,C)||_{\infty} \ge 1$. Another way to see (4.1) is to use the formula for $\mu_{P,C}$ in Proposition 4.2 as follows. Since $G^*G = I$ and $\widetilde{K}\widetilde{K}^* = i$, it follows that $\overline{\sigma}(G) = 1$ and $\overline{\sigma}(\widetilde{K}) = 1$. Then it follows from various of the properties singular values listed in Proposition 2.13 that

$$\underline{\sigma}(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi)) \leq \overline{\sigma}(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi)) \leq \overline{\sigma}(\widetilde{\mathbf{K}}(\varphi)) \cdot \overline{\sigma}(\mathbf{G}(\varphi)) = 1.$$

Proposition 4.4. The following are equivalent:

- (1) C stabilizes P.
- (2) $\det(\widetilde{\mathbf{K}}(\varphi)\mathbf{G}(\varphi)) \neq 0$ for all $\varphi \in \mathfrak{M}$ and $\iota(\det(\widetilde{K}G)) = \circ$.

Proof. Suppose that C stabilizes P. Then from the calculation done above in the proof of Proposition 4.2, we have

$$H(P,C) = G(\widetilde{K}G)^{-1}\widetilde{K}.$$
(4.2)

But we know that G is left invertible and \widetilde{K} is right invertible as matrices with entries from R. So from the above, we see that $\widetilde{K}G \in \mathbb{R}^{m \times m}$ is invertible as an element of $\mathbb{R}^{m \times m}$. In particular det $(\widetilde{K}G)$ is invertible as an element of R and so det $(\widetilde{K}(\varphi)\mathbf{G}(\varphi)) \neq 0$ for all $\varphi \in \mathfrak{M}$. Also from (A4), it follows that $\iota(\det(\widetilde{K}G)) = \circ$.

Now suppose that $\det(\mathbf{K}(\varphi)\mathbf{G}(\varphi)) \neq 0$ for all $\varphi \in \mathfrak{M}$ and $\iota(\det(KG)) = \circ$. Then $\widetilde{K}G \in R \cap \text{inv } S$. From (A4), we obtain that $\det(\widetilde{K}G)$ is invertible as an element of R, and so we see from (4.2) that H(P,C) has entries from R. So P is stabilized by C.

Proposition 4.5. $\mu_{P,C} = \mu_{C,P}$.

Proof. It is not hard to see that C stabilizes P iff P stabilizes C. We have

$$\begin{split} &\widetilde{\mathbf{K}}\widetilde{\mathbf{G}}^*\widetilde{\mathbf{G}}\widetilde{\mathbf{K}}^* + \widetilde{\mathbf{K}}\mathbf{G}\mathbf{G}^*\widetilde{\mathbf{K}}^* = I, \\ &\widetilde{\mathbf{G}}\widetilde{\mathbf{K}}^*\widetilde{\mathbf{K}}\widetilde{\mathbf{G}}^* + \widetilde{\mathbf{G}}\mathbf{K}\mathbf{K}^*\widetilde{\mathbf{G}}^* = I \end{split}$$

pointwise on \mathfrak{M} . So it follows from Lemma 2.14 that

$$\underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}))^2 = 1 - (\overline{\sigma}(\widetilde{\mathbf{K}}\widetilde{\mathbf{G}}^*))^2 = (\underline{\sigma}(\widetilde{\mathbf{G}}\mathbf{K}))^2.$$
(4.3)
he proof.

This completes the proof.

Theorem 4.6. If $P_0, P_1 \in \mathbb{S}(P, p, m)$ and $C \in \mathbb{S}(R, m, p)$, then $\sin^{-1} \mu_{P_1, C} \geq \sin^{-1} \mu_{P_0, C} - \sin^{-1}(d_{\nu}(P_0, P_1)).$

Proof. If $d_{\nu}(P_0, P_1) \geq \mu_{P_0,C}$, then $\sin^{-1}(d_{\nu}(P_0, P_1)) \geq \sin^{-1}\mu_{P_0,C}$ and so $\sin^{-1}\mu_{P_0,C} - \sin^{-1}(d_{\nu}(P_0, P_1)) \leq 0$. The claimed inequality in the statement of the theorem now follows trivially since $\mu_{P_1,C} \geq 0$.

We therefore assume in the rest of the proof that $d_{\nu}(P_0, P_1) < \mu_{P_0,C}$. As noted in Remark4.3, $\mu_{P_0,C} \leq 1$: hence we must have $d_{\nu}(P_0, P_1) < 1$. Also $\mu_{P_0,C} = 0$ implies that $d_{\nu}(P_0, P_1) < 0$, a contradiction to the fact that d_{ν} is a metric. Hence $\mu_{P_0,C} > 0$, that is, C stabilizes P_0 . Now

$$d_{\nu}(P_0, P_1) = \sup_{\varphi \in \mathfrak{M}} \overline{\sigma}((\widetilde{\mathbf{G}}_0 \mathbf{G}_1)(\varphi)) < \inf_{\varphi \in \mathfrak{M}} \underline{\sigma}((\widetilde{\mathbf{K}} \mathbf{G}_0)(\varphi)) = \mu_{P_0, C},$$

and so pointwise on \mathfrak{M} , there holds that $\overline{\sigma}(\mathbf{G}_0\mathbf{G}_1) < \underline{\sigma}(\mathbf{K}\mathbf{G}_0)$. But for numbers $a, b \in (0, 1)$,

$$a < b$$
 iff $\frac{a^2}{1 - a^2} < \frac{b^2}{1 - b^2}$,

and so we have

$$\frac{(\overline{\sigma}(\widetilde{\mathbf{G}}_{\mathbf{0}}\mathbf{G}_{\mathbf{1}}))^2}{1-(\overline{\sigma}(\widetilde{\mathbf{G}}_{\mathbf{0}}\mathbf{G}_{\mathbf{1}}))^2} < \frac{(\underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_{\mathbf{0}}))^2}{1-(\underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_{\mathbf{0}}))^2}.$$

Using Lemma 2.15 and (4.3), we obtain $\frac{\overline{\sigma}(\widetilde{\mathbf{G}}_{\mathbf{0}}\mathbf{G}_{\mathbf{1}})}{\underline{\sigma}(\mathbf{G}_{\mathbf{0}}^{*}\mathbf{G}_{\mathbf{1}})} < \frac{\underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_{\mathbf{0}})}{\overline{\sigma}(\widetilde{\mathbf{K}}\widetilde{\mathbf{G}}_{\mathbf{0}}^{*})}$. Thus

$$\overline{\sigma}(\widetilde{\mathbf{K}}\widetilde{\mathbf{G}}_{\mathbf{0}}^{*}\widetilde{\mathbf{G}}_{\mathbf{0}}\mathbf{G}_{\mathbf{1}}) < \underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_{\mathbf{0}}\mathbf{G}_{\mathbf{0}}^{*}\mathbf{G}_{\mathbf{1}}).$$

$$(4.4)$$

But

$$\widetilde{K}G_1 = \widetilde{K}G_0G_0^*G_1 + \widetilde{K}\widetilde{G}_0^*\widetilde{G}_0G_1.$$
(4.5)

Let $A := \widetilde{K}G_0G_0^*G_1$, and $B := \widetilde{K}\widetilde{G}_0^*\widetilde{G}_0G_1$. Using the fact that $\widetilde{K}G_0$ and $G_0^*G_1$ are invertible in $S^{m\times m}$, it follows also that A is invertible in $S^{m\times m}$. Also, from (4.4), it follows that $||A^{-1}B||_{\infty} < 1$. Then it follows from (4.5) that $\widetilde{K}G_1 = A + B = A(I + A^{-1}B)$ and so $\widetilde{K}G_1$ is also invertible in $S^{m\times m}$. Consider the map $H : [0, 1] \to \text{inv } S$, defined by $H(t) = \det(A(I + tA^{-1}B)), t \in [0, 1]$. By Proposition 2.1, it follows that H(0) = H(1), that is,

$$\iota(\det(\widetilde{K}G_1)) = \iota(\det(\widetilde{K}G_0G_0^*G_1)) = \iota(\det(\widetilde{K}G_0)) + \iota(\det(G_0^*G_1)) = \circ + \circ = \circ.$$

But $\det(\widetilde{K}G_1) \in R$. By (A4) it follows that $\det(\widetilde{K}G_1)$ is invertible as an element of R. Consequently C stabilizes P_1 and

$$\mu_{P_1,C} = \inf_{\varphi \in \mathfrak{M}} \underline{\sigma}((\mathbf{K}\mathbf{G_1})(\varphi)).$$

From (4.5), we have

$$\begin{array}{lll} \underline{\sigma}(\mathbf{K}\mathbf{G}_{1}) &=& \underline{\sigma}(\mathbf{K}\mathbf{G}_{0}\mathbf{G}_{0}^{*}\mathbf{G}_{1} + \mathbf{K}\mathbf{G}_{0}^{*}\mathbf{G}_{0}\mathbf{G}_{1}) \\ &\geq & \underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_{0}\mathbf{G}_{0}^{*}\mathbf{G}_{1}) - \overline{\sigma}(\widetilde{\mathbf{K}}\widetilde{\mathbf{G}}_{0}^{*}\widetilde{\mathbf{G}}_{0}\mathbf{G}_{1}) \\ &\geq & \underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_{0})\underline{\sigma}(\mathbf{G}_{0}^{*}\mathbf{G}_{1}) - \overline{\sigma}(\widetilde{\mathbf{K}}\widetilde{\mathbf{G}}_{0}^{*})\overline{\sigma}(\widetilde{\mathbf{G}}_{0}\mathbf{G}_{1}) \\ &= & \sin(\sin^{-1}\underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_{0}) - \sin^{-1}\overline{\sigma}(\widetilde{\mathbf{G}}_{0}\mathbf{G}_{1})). \end{array}$$

Since $\sin^{-1}: [-1,1] \to [-\frac{\pi}{2},\frac{\pi}{2}]$ is an increasing function, it now follows that

$$\sin^{-1}\underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_1) \ge \sin^{-1}\underline{\sigma}(\widetilde{\mathbf{K}}\mathbf{G}_0) - \sin^{-1}\overline{\sigma}(\widetilde{\mathbf{G}}_0\mathbf{G}_1).$$

Consequently, $\sin^{-1} \mu_{P_1,C} \ge \sin^{-1} \mu_{P_0,C} - \sin^{-1}(d_{\nu}(P_0,P_1)).$

Corollary 4.7. If $P_0, P \in \mathbb{S}(R, p, m)$, then

 $\mu_{P,C} \ge \mu_{P_0,C} - d_{\nu}(P_0,P).$

Proof. For $x, y, z \in [0, 1]$, if $\sin^{-1} x \leq \sin^{-1} y + \sin^{-1} z$. By taking the cosine of both sides and using that the cos is a decreasing function on $[0, \frac{\pi}{2}]$, we then get $\sqrt{1-x^2} \geq \sqrt{1-y^2}\sqrt{1-z^2} - yz$, which in turn implies that

$$(\sqrt{1-x^2}+yz)^2 \ge (1-y^2)(1-z^2).$$

Hence $x^2 \leq y^2 + z^2 + 2yz\sqrt{1-x^2} \leq y^2 + z^2 + 2yz \cdot 1 = (y+z)^2$, which gives finally that $x \leq y+z$. The claimed inequality now follows immediately from the inequality in Theorem 4.6 upon setting $x = \mu_{P_0,C}$, $y = d_{\nu}(P_0, P)$ and $z = \mu_{P,C}$.

The above result says that if the controller C performs sufficiently well with the nominal plant P_0 , and the distance $d_{\nu}(P_0, P)$ between the plant Pand P_0 is sufficiently small, then C is guaranteed to achieve a certain level of performance with the plant P. So if P and P_0 represent alternate models of the system (one which is "true" and one which is our nominal model) and if $d_{\nu}(P_0, P)$ is small, then the difference between P and P_0 can be ignored for the purposes of designing a stabilizing controller.

Another way of stating the result in Theorem 4.6 is that if C stabilizes P_0 with a stability margin $\mu_{P,C} > m$, and P is another plant which is close to P_0 in the sense that $d_{\nu}(P, P_0) \leq m$, then C is also guaranteed to stabilize P. Furthermore, if C satisfies the stronger condition $\mu_{P,C} > M > m$ for a number M, then C is also guaranteed to stabilize P with a stability margin $\mu_{P,C}$ which satisfies $\mu_{P,C} \geq \sin^{-1} M - \sin^{-1} m$.

5. Applications

Now we specialize R to several classes of stable transfer functions and obtain various extensions of the ν -metric. Some of the verifications of the properties (A1)-(A4) are similar to the section on examples from [22].

5.1. The disk algebra. Let

$$\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}, \quad \overline{\mathbb{D}} := \{ z \in \mathbb{C} : |z| \le 1 \}, \quad \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}.$$

The disk algebra $A(\mathbb{D})$ is the set of all functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ such that f is holomorphic in \mathbb{D} and continuous on $\overline{\mathbb{D}}$. Let $C(\mathbb{T})$ denote the set of complexvalued continuous functions on the unit circle \mathbb{T} . For each $f \in \text{inv } C(\mathbb{T})$, we can define the winding number $w(f) \in \mathbb{Z}$ of f as follows:

$$\mathbf{w}(f) = \frac{1}{2\pi} (\Theta(2\pi) - \Theta(0)),$$

where $\Theta : [0, 2\pi] \to \mathbb{R}$ is a continuous function such that

$$f(e^{it}) = |f(e^{it})|e^{i\Theta(t)}, \quad t \in [0, 2\pi].$$

The existence of such a Θ can be proved; see [23, Lemma 4.6]. Also, it can be checked that **w** is well-defined and integer-valued. Geometrically, $\mathbf{w}(f)$ is the number of times the curve $t \mapsto f(e^{it}) : [0, 2\pi] \to \mathbb{C}$ winds around the origin in a counterclockwise direction.

Recall the definition of a full subring.

Definition 5.1. Let R_1, R_2 be commutative unital rings, and let R_1 be a subring of R_2 . Then R_1 is said to be a *full* subring of R_2 if for every $x \in R_1$ such that x is invertible in R_2 , it holds that x is invertible in R_1 .

Lemma 5.2. Let

 $\begin{aligned} R &= \text{ a unital full subring of } A(\mathbb{D}), \\ S &:= C(\mathbb{T}), \\ G &:= \mathbb{Z}, \\ \iota &:= \mathbb{w}. \end{aligned}$

Then (A1)-(A4) are satisfied.

Proof. (A1) is clear. The involution \cdot^* in (A2) is defined by $f^*(z) = \overline{f(z)}$ ($z \in \mathbb{T}$) for $f \in C(\mathbb{T})$. (A3)(I1) and (A3)(I2) are evident from the definition of w. Also, the map $w : \text{inv } C(\mathbb{T}) \to \mathbb{Z}$ is locally constant (that is, it is continuous when \mathbb{Z} is equipped with the discrete topology and $C(\mathbb{T})$ is equipped with the usual sup-norm); see [23, Lemma 4.6.(ii)]. So (A3)(I3) holds as well. Finally, we will show below that (A4) holds.

Suppose that $f \in R \cap (\text{inv } C(\mathbb{T}))$ is invertible as an element of R. Then obviously f is also invertible as an element of $A(\mathbb{D})$. Hence it has no zeros or poles in $\overline{\mathbb{D}}$. For $r \in (0, 1)$, define $f_r \in A(\mathbb{D})$ by $f_r(z) = f(rz)$ $(z \in \overline{\mathbb{D}})$. Then f_r also has no zeros or poles in $\overline{\mathbb{D}}$, and has a holomorphic extension across \mathbb{T} . From the Argument Principle (applied to f_r), it follows that $\mathbf{w}(f_r) = 0$. But $||f_r - f||_{\infty} \to 0$ as $r \nearrow 1$. Hence $\mathbf{w}(f) = \lim_{r \to 1} \mathbf{w}(f_r) = \lim_{r \to 1} 0 = 0$.

Suppose, conversely, that $f \in R \cap (\text{inv } C(\mathbb{T}))$ is such that $\mathbf{w}(f) = 0$. For all $r \in (0, 1)$ sufficiently close to 1, we have that $f_r \in \text{inv } C(\mathbb{T})$. Also, by the local constancy of \mathbf{w} , for r sufficiently close to 1, $\mathbf{w}(f_r) = \mathbf{w}(f) = 0$. By the Argument principle, it then follows that f_r has no zeros in $\overline{\mathbb{D}}$. Equivalently, f has no zeros in $r\overline{\mathbb{D}}$. But letting $r \nearrow 1$, we see that f has no zeros in \mathbb{D} . Moreover, f has no zeros on \mathbb{T} either, since $f \in \text{inv } C(\mathbb{T})$. Thus f has no zeros in $\overline{\mathbb{D}}$. Consequently, we conclude that f is invertible as an element of $A(\mathbb{D})$. (Indeed, f is invertible as an element of $C(\overline{\mathbb{D}})$, and it is also then clear that this inverse is holomorphic in \mathbb{D} .) Finally, since R is a full subring of $A(\mathbb{D})$, we can conclude that f is invertible also as an element of R. \Box

Besides $A(\mathbb{D})$ itself, some other examples of such R are:

- (1) $RH^{\infty}(\mathbb{D})$, the set of all rational functions without poles in $\overline{\mathbb{D}}$.
- (2) The Wiener algebra $W^+(\mathbb{D})$ of all functions $f \in A(\mathbb{D})$ that have an absolutely convergent Taylor series about the origin:

$$\sum_{n=0}^{\infty} |f_n| < +\infty, \text{ where } f(z) = \sum_{n=0}^{\infty} f_n z^n \ (z \in \mathbb{D}).$$

(3) $\partial^{-n} H^{\infty}(\mathbb{D})$, the set of $f : \mathbb{D} \to \mathbb{C}$ such that $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ belong to $H^{\infty}(\mathbb{D})$. Here $H^{\infty}(\mathbb{D})$ denotes the Hardy algebra of all bounded and holomorphic functions on \mathbb{D} .

In the definition of the ν -metric given in Definition 2.8 corresponding to Lemma 5.2, the $\|\cdot\|_{\infty}$ now means the following: if $F \in (C(\mathbb{T}))^{p \times m}$, then

$$||F||_{\infty} = \max_{z \in \mathbb{T}} |F(z)|.$$

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This follows from (2.6), since the maximal ideal space \mathfrak{M} of $S = C(\mathbb{T})$ can be identified with the unit circle as a topological space; see [20, Example 11.13.(a)].

Remark 5.3. $RH^{\infty}(\mathbb{D})$ is a projective free ring since it is a Bézout domain. Also $A(\mathbb{D})$, $W^+(\mathbb{D})$, or $\partial^{-n}H^{\infty}(\mathbb{D})$ are projective free rings, since their maximal ideal space is $\overline{\mathbb{D}}$, which is contractible; see [2]. Thus if R is one of $RH^{\infty}(\mathbb{D})$, $A(\mathbb{D})$, $W^+(\mathbb{D})$ or $\partial^{-n}H^{\infty}(\mathbb{D})$, then the set $\mathbb{S}(R, p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

5.2. Almost periodic functions. The algebra AP of complex valued (uniformly) almost periodic functions is the smallest closed subalgebra of $L^{\infty}(\mathbb{R})$ that contains all the functions $e_{\lambda} := e^{i\lambda y}$. Here the parameter λ belongs to \mathbb{R} . For any $f \in AP$, its Bohr-Fourier series is defined by the formal sum

$$\sum_{\lambda} f_{\lambda} e^{i\lambda y}, \quad y \in \mathbb{R},$$
(5.1)

where

$$f_{\lambda} := \lim_{N \to \infty} \frac{1}{2N} \int_{[-N,N]} e^{-i\lambda y} f(y) dy, \quad \lambda \in \mathbb{R},$$

and the sum in (5.1) is taken over the set $\sigma(f) := \{\lambda \in \mathbb{R} \mid f_{\lambda} \neq 0\}$, called the *Bohr-Fourier spectrum* of f. The Bohr-Fourier spectrum of every $f \in AP$ is at most a countable set.

The almost periodic Wiener algebra APW is defined as the set of all AP such that the Bohr-Fourier series (5.1) of f converges absolutely. The almost periodic Wiener algebra is a Banach algebra with pointwise operations and the norm $||f|| := \sum_{\lambda \in \mathbb{R}} |f_{\lambda}|$. Set

$$AP^+ = \{ f \in AP \mid \sigma(f) \subset [0,\infty) \}$$
$$APW^+ = \{ f \in APW \mid \sigma(f) \subset [0,\infty) \}.$$

Then AP^+ (respectively APW^+) is a Banach subalgebra of AP (respectively APW). For each $f \in inv AP$, we can define the *average winding number* $w(f) \in \mathbb{R}$ of f as follows:

$$w(f) = \lim_{T \to \infty} \frac{1}{2T} \bigg(\arg(f(T)) - \arg(f(-T)) \bigg).$$

See [15, Theorem 1, p. 167].

Lemma 5.4. Let

$$R := a \text{ unital full subring of } AP^+$$

$$S := AP,$$

$$G := \mathbb{R},$$

$$\iota := w.$$

Then (A1)-(A4) are satisfied.

Proof. (A1) is clear. The involution \cdot^* used in (A2) is defined by

$$f^*(y) = \overline{f(y)}, \quad y \in \mathbb{R},$$

for $f \in AP$. (A3)(I1) and (A3)(I2) follow from the definition of w. (A3)(I3) follows for example from [16, Theorem 2.6 and Example 2.10], where it is shown that w is a *topological index* on AP, and hence in particular, it is locally constant.

Finally, (A4) follows from [7, Theorem 1, p.776] which says that $f \in AP^+$ satisfies

$$\inf_{\mathrm{Im}(s)>0} |f(s)| > 0 \tag{5.2}$$

iff $\inf_{y \in \mathbb{R}} |f(y)| > 0$ and w(f) = 0. But

$$\inf_{y \in \mathbb{R}} |f(y)| > 0$$

is equivalent to f being an invertible element of AP by the corona theorem for AP (see for example [11, Exercise 18, p.24]). Also the equivalence of (5.2) with that of the invertibility of f as an element of AP^+ follows from the Arens-Singer corona theorem for AP^+ (see for example [3, Theorems 3.1, 4.3]). Finally, the invertibility of $f \in R$ in R is equivalent to the invertibility of f as an element of AP^+ since R is a full subring of AP^+ . \Box

Remark 5.5. Specific examples of such R are AP^+ and APW^+ . More generally, let $\Sigma \subset [0, +\infty)$ be an *additive semigroup* (if $\lambda, \mu \in \Sigma$, then $\lambda + \mu \in \Sigma$) and suppose $0 \in \Sigma$. Denote

$$AP_{\Sigma} = \{ f \in AP \mid \sigma(f) \subset \Sigma \}$$

$$APW_{\Sigma} = \{ f \in APW \mid \sigma(f) \subset \Sigma \}.$$

Then AP_{Σ} (respectively APW_{Σ}) is a unital Banach subalgebra of AP^+ (respectively APW^+). Let $\overline{Y_{\Sigma}}$ denote the set of all maps $\theta : \Sigma \to [0, +\infty]$ such that $\theta(0) = 0$ and $\theta(\lambda + \mu) = \theta(\lambda) + \theta(\mu)$ for all $\lambda, \mu \in \Sigma$. Examples of such maps θ are the following. If $y \in [0, +\infty)$, then θ_y , defined by $\theta_y(\lambda) = \lambda y, \lambda \in \Sigma$, belongs to $\overline{Y_{\Sigma}}$. Another example is θ_{∞} , defined as follows:

$$\theta_{\infty}(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0, \\ +\infty & \text{if } \lambda \neq 0. \end{cases}$$

So in this way we can consider $[0, +\infty]$ as a subset of $\overline{Y_{\Sigma}}$.

The results [3, Proposition 4.2, Theorem 4.3] say that if $\overline{Y_{\Sigma}} \subset [0, +\infty]$, and f belongs to AP_{Σ} (respectively to APW_{Σ}), then f belongs to inv AP_{Σ} (respectively to inv APW_{Σ}) iff (5.2) holds. So in this case AP_{Σ} and APW_{Σ} are unital full subalgebras of AP^+ .

In the definition of the ν -metric given in Definition 2.8 corresponding to Lemma 5.4, the $\|\cdot\|_{\infty}$ now means the following: if $F \in (AP)^{p \times m}$, then

$$\|F\|_{\infty} = \sup_{y \in \mathbb{R}} \|F(y)\|.$$

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This follows from (2.6), since \mathbb{R} is dense in the maximal ideal space \mathfrak{M} (which is the Bohr compactification \mathbb{R}_B of \mathbb{R}) of the Banach algebra S = AP; see [11, Exercise 18, p.24].

Remark 5.6. It was shown in [2] that AP^+ and APW^+ are projective free rings. Thus if $R = AP^+$ or APW^+ , then the set $\mathbb{S}(R, p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

5.3. Algebras of Laplace transforms of measures without a singular nonatomic part. Let $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 0\}$ and let \mathcal{A}^+ denote the Banach algebra

$$\mathcal{A}^+ = \left\{ s(\in \mathbb{C}_+) \mapsto \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \left| \begin{array}{c} f_a \in L^1(0,\infty), \ (f_k)_{k\geq 0} \in \ell^1, \\ 0 = t_0 < t_1, t_2, t_3, \dots \end{array} \right\} \right\}$$

equipped with pointwise operations and the norm:

$$||F|| = ||f_a||_{L^1} + ||(f_k)_{k \ge 0}||_{\ell^1}, \ F(s) = \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \ (s \in \mathbb{C}_+).$$

Here f_a denotes the Laplace transform of f_a , given by

$$\widehat{f}_a(s) = \int_0^\infty e^{-st} f_a(t) dt, \quad s \in \mathbb{C}_+.$$

Similarly, define the Banach algebra \mathcal{A} as follows ([13]):

$$\mathcal{A} = \left\{ iy(\in i\mathbb{R}) \mapsto \widehat{f_a}(iy) + \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \middle| \begin{array}{c} f_a \in L^1(\mathbb{R}), \ (f_k)_{k\in\mathbb{Z}} \in \ell^1, \\ \dots, t_{-2}, t_{-1} < 0 = t_0 < t_1, t_2, \dots \end{array} \right\}$$
equipped with pointwise operations and the norm:

$$\|F\| = \|f_a\|_{L^1} + \|(f_k)_{k \in \mathbb{Z}}\|_{\ell^1}, \quad F(iy) := \widehat{f_a}(iy) + \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \quad (y \in \mathbb{R}).$$

Here $\widehat{f_a}$ is the Fourier transform of f_a , $\widehat{f_a}(iy) = \int_{-\infty}^{\infty} e^{-iyt} f_a(t) dt, \ (y \in \mathbb{R}).$

It can be shown that
$$L^1(\mathbb{R})$$
 is an ideal of \mathcal{A} .
For $F = \hat{f}_a + \sum_{k=-\infty}^{\infty} f_k e^{-i \cdot t_k} \in \mathcal{A}$, we set $F_{AP}(iy) = \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \ (y \in \mathbb{R})$.

If $F = \hat{f}_a + F_{AP} \in \text{inv } \mathcal{A}$, then it can be shown that $F_{AP}(i \cdot) \in \text{inv } AP$ as follows. First of all, the maximal ideal space of \mathcal{A} contains a copy of the maximal ideal space of APW in the following manner: if $\varphi \in M(APW)$, then the map $\Phi : \mathcal{A} \to \mathbb{C}$ defined by $\Phi(F) = \Phi(\hat{f}_a + F_{AP}) = \varphi(F_{AP}(i \cdot))$, $(F \in \mathcal{A})$, belongs to $M(\mathcal{A})$. So if F is invertible in \mathcal{A} , in particular for every Φ of the type describe above, $0 \neq \Phi(F) = \varphi(F_{AP}(i \cdot))$. Thus by the elementary theory of Banach algebras, $F_{AP}(i \cdot)$ is an invertible element of AP.

Moreover, since $\widehat{L^1(\mathbb{R})}$ is an ideal in \mathcal{A} , $F_{AP}^{-1}\widehat{f_a}$ is the Fourier transform of a function in $L^1(\mathbb{R})$, and so the map $y \mapsto 1 + (F_{AP}(iy))^{-1}\widehat{f_a}(iy) = \frac{F(iy)}{F_{AP}(iy)}$

has a well-defined winding number w around 0. Define $W : \text{inv } \mathcal{A} \to \mathbb{R} \times \mathbb{Z}$ by

$$W(F) = (w(F_{AP}), \mathbf{w}(1 + F_{AP}^{-1}\hat{f}_a)),$$
(5.3)

where $F = \hat{f}_a + F_{AP} \in \text{inv } \mathcal{A}$, and

$$w(F_{AP}) := \lim_{R \to \infty} \frac{1}{2R} \bigg(\arg \big(F_{AP}(iR) \big) - \arg \big(F_{AP}(-iR) \big) \bigg),$$

$$w(1 + F_{AP}^{-1} \widehat{f}_a) := \frac{1}{2\pi} \bigg(\arg \big(1 + (F_{AP}(iy))^{-1} \widehat{f}_a(iy) \big) \bigg|_{y=-\infty}^{y=+\infty} \bigg).$$

Lemma 5.7. $F = \hat{f}_a + F_{AP} \in \mathcal{A}$ is invertible iff for all $y \in \mathbb{R}$, $F(iy) \neq 0$ and $\inf_{y \in \mathbb{R}} |F_{AP}(iy)| > 0$.

Proof. The 'only if' part is clear. We simply show the 'if' part below. Let $F = \hat{f}_a + F_{AP} \in \mathcal{A}$ be such that $F(iy) \neq 0$ for all $y \in \mathbb{R}$ and

$$\inf_{y\in\mathbb{R}}|F_{AP}(iy)|>0.$$

Thus $F_{AP}(i \cdot)$ is invertible as an element of AP. Hence $F = F_{AP}(1 + \hat{f}_a F_{AP}^{-1})$ and so it follows that $(1 + \hat{f}_a F_{AP}^{-1})(iy) \neq 0$ for all $y \in \mathbb{R}$. But by the corona theorem for

$$\mathcal{W} := \widehat{L^1(\mathbb{R})} + \mathbb{C}$$

(see [12, Corollary 1, p.109]), it follows that $1 + \hat{f}_a F_{AP}^{-1}$ is invertible as an element of \mathcal{W} and in particular, also as an element of \mathcal{A} . This completes the proof.

Lemma 5.8. Let

$$R := a \text{ unital full subring of } \mathcal{A}^+,$$

$$S := \mathcal{A},$$

$$G := \mathbb{R} \times \mathbb{Z},$$

$$\iota := W.$$

Then (A1)-(A4) are satisfied.

Proof. (A1) is clear. The involution \cdot^* in (A2) is defined by

$$F^*(iy) = \overline{F(iy)}, \quad y \in \mathbb{R},$$

for $F \in \mathcal{A}$. (A3)(I2) is now easy to see from the definition of W. Also, (A3)(I1) follows from the definition of W as follows. Let $F = \hat{f}_a + F_{AP}$ and $G = \hat{g}_a + G_{AP}$. Then we have

$$w(F_{AP}G_{AP}) = w(F_{AP}) + w(G_{AP})$$

from the definition of w. Thus

$$\begin{split} W(FG) &= W((f_a + F_{AP})(\hat{g}_a + G_{AP})) \\ &= W(\hat{f}_a \hat{g}_a + \hat{f}_a G_{AP} + \hat{g}_a F_{AP} + F_{AP} G_{AP}) \\ &= (\texttt{w}(1 + (F_{AP} G_{AP})^{-1}(\hat{f}_a \hat{g}_a + \hat{f}_a G_{AP} + \hat{g}_a F_{AP}), w(F_{AP} G_{AP})) \\ &= (\texttt{w}((1 + F_{AP}^{-1} \hat{f}_a)(1 + G_{AP}^{-1} \hat{g}_a)), w(F_{AP}) + w(G_{AP})) \\ &= (\texttt{w}(1 + F_{AP}^{-1} \hat{f}_a) + \texttt{w}(1 + G_{AP}^{-1} \hat{g}_a), w(F_{AP}) + w(G_{AP})) \\ &= W(\hat{f}_a + F_{AP}) + W(\hat{g}_a + G_{AP}). \end{split}$$

So (A3)(I2) holds.

The local constancy of W demanded in (A3)(I3) can be seen in the following manner. We have already noted that w is locally constant on inv APand w is locally constant on inv $C(\mathbb{T})$. Note that $w(1+F_{AP}^{-1}\hat{f}_a)$ defined above is just $w(\varphi)$ where

$$\varphi(\theta) = (1 + F_{AP}^{-1}\widehat{f}_a)(iy), \text{ where } iy = \frac{1 + e^{i\theta}}{1 - e^{i\theta}}, \quad \theta \in (0, 2\pi).$$

Hence (A3)(I3) follows.

Finally we check that (A4) holds. Suppose that $F = \hat{f}_a + F_{AP}$ belonging to $\mathcal{A}^+ \cap (\text{inv } \mathcal{A})$, is such that W(F) = 0. Since F is invertible in \mathcal{A} , it follows that $F_{AP}(i \cdot)$ is invertible as an element of AP. But $w(F_{AP}) = 0$, and so $F_{AP}(i \cdot) \in AP^+$ is invertible as an element of AP^+ . But this implies that $1 + F_{AP}^{-1}\hat{f}_a$ belongs to the Banach algebra

$$\mathcal{W}^+ := L^{\widehat{1}(0,\infty)} + \mathbb{C}.$$

Moreover, it is bounded away from 0 on $i\mathbb{R}$ since

$$1 + F_{AP}^{-1}\widehat{f}_a = \frac{F}{F_{AP}},$$

and F is bounded away from zero on $i\mathbb{R}$. Moreover $\mathbf{w}(1 + F_{AP}^{-1}\hat{f}_a) = 0$, and so it follows that $1 + F_{AP}^{-1}\hat{f}_a$ is invertible as an element of \mathcal{W}^+ , and in particular in \mathcal{A}^+ . Since $F = (1 + F_{AP}^{-1}\hat{f}_a)F_{AP}$ and we have shown that both $(1 + F_{AP}^{-1}\hat{f}_a)$ as well as F_{AP} are invertible as elements of \mathcal{A}^+ , it follows that F is invertible in \mathcal{A}^+ . \Box

An example of such a R (besides \mathcal{A}^+) is the algebra

$$L^{1}(0,+\infty) + APW_{\Sigma}(i\cdot) := \{\widehat{f}_{a} + F_{AP} : f_{a} \in L^{1}(0,+\infty), \ F_{AP}(i\cdot) \in APW_{\Sigma}\},$$

where Σ is as described in Remark 5.5.

In the definition of the ν -metric given in Definition 2.8 corresponding to Lemma 5.8, the $\|\cdot\|_{\infty}$ now means the following: if $F \in \mathcal{A}^{p \times m}$, then

$$||F||_{\infty} = \sup_{y \in \mathbb{R}} |F(iy)|.$$

This follows from (2.6), since \mathbb{R} is dense in the maximal ideal space \mathfrak{M} of the Banach algebra $S = \mathcal{A}$; see [14, Theorems 4.20.1 and 4.20.4].

Remark 5.9. It was shown in [2] that \mathcal{A}^+ is a projective free ring. Thus the set $\mathbb{S}(\mathcal{A}^+, p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

5.4. The polydisk algebra. Let

$$\mathbb{D}^n := \{ (z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1 \text{ for } i = 1, \dots, n \}, \\ \overline{\mathbb{D}^n} := \{ (z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| \le 1 \text{ for } i = 1, \dots, n \}, \\ \mathbb{T}^n := \{ (z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| = 1 \text{ for } i = 1, \dots, n \}.$$

The polydisk algebra $A(\mathbb{D}^n)$ is the set of all functions $f: \overline{\mathbb{D}^n} \to \mathbb{C}$ such that f is holomorphic in \mathbb{D}^n and continuous on $\overline{\mathbb{D}^n}$.

If $f \in A(\mathbb{D}^n)$, then the function f_d defined by $z \mapsto f(z, \ldots, z) : \overline{\mathbb{D}} \to \mathbb{C}$ belongs to the disk algebra $A(\mathbb{D})$, and in particular also to $C(\mathbb{T})$. The map

$$f \mapsto (f|_{\mathbb{T}^n}, f_d) : A(\mathbb{D}^n) \to C(\mathbb{T}^n) \times C(\mathbb{T})$$

is a ring homomorphism. This map is also injective, and this is an immediate consequence of Cauchy's formula; see [19, p.4-5]. We recall the following result; see [19, Theorem 4.7.2, p.87].

Proposition 5.10. Suppose that $\Psi = (\psi_1, \ldots, \psi_n)$ is a continuous map from $\overline{\mathbb{D}}$ into $\overline{\mathbb{D}^n}$, which carries \mathbb{T} into \mathbb{T}^n and the winding number of each ψ_i is positive. Then for every $f \in A(\mathbb{D}^n)$, $f(\Psi(\overline{\mathbb{D}}) \cup \mathbb{T}^n) = f(\overline{\mathbb{D}^n})$.

Lemma 5.11. Let

$$R = \text{a unital full subring of } A(\mathbb{D}^n)$$

$$S := C(\mathbb{T}^n) \times C(\mathbb{T}),$$

$$G := \mathbb{Z},$$

$$\iota := ((g, h) \mapsto \mathbf{w}(h)).$$

Then (A1)-(A4) are satisfied.

Proof. (A1) is clear. The involution \cdot^* in (A2) is defined as follows: if $(f,g) \in C(\mathbb{T}^n) \times C(\mathbb{T})$, then $(f,g)^* := (f^*,g^*)$, where

$$f^*(z_1, \dots, z_n) = \overline{f(z_1, \dots, z_n)}, \quad (z_1, \dots, z_n) \in \mathbb{T}^n,$$
$$g^*(z) = \overline{g(z)}, \quad z \in \mathbb{T}.$$

(A3) was proved earlier in Subsection 5.1. Finally, we will show below that (A4) holds, following [10].

Suppose that $f \in A(\mathbb{D}^n)$ is such that $f|_{\mathbb{T}^n} \in \text{inv } C(\mathbb{T}^n)$, $f_d \in \text{inv } C(\mathbb{T})$ and that $w(f_d) = 0$. We use Proposition 5.10, with $\Psi(z) := (z, \ldots, z)$ $(z \in \overline{\mathbb{D}})$. Then we know that f will have no zeros in $\overline{\mathbb{D}^n}$ if $f(\Psi(\overline{\mathbb{D}}))$ does not contain 0. But since $f_d \in \text{inv } C(\mathbb{T})$ and $w(f_d) = 0$, it follows that f_d is invertible as an element of $A(\mathbb{D})$ by the result in Subsection 5.1. But this implies that $f(\Psi(\overline{\mathbb{D}}))$ does not contain 0. Now suppose that $f \in A(\mathbb{D}^n)$ with $f|_{\mathbb{T}^n} \in \text{inv } C(\mathbb{T}^n)$, $f_d \in \text{inv } C(\mathbb{T})$, and that it is invertible as an element of $A(\mathbb{D}^n)$. But then in particular, f_d is an invertible element of $A(\mathbb{D})$, and so again by the result in Subsection 5.1, it follows that $w(f_d) = 0$.

Besides $A(\mathbb{D}^n)$ itself, another example of such an R is $RH^{\infty}(\mathbb{D}^n)$, the set of all rational functions without poles in $\overline{\mathbb{D}^n}$.

In the definition of the ν -metric given in Definition 2.8 corresponding to Lemma 5.11, the $\|\cdot\|_{\infty}$ now means the following: if $F = (G, H) \in (C(\mathbb{T}^n) \times C(\mathbb{T}))^{p \times m}$, then

$$||F||_{\infty} = \max\left\{\max_{z\in\mathbb{T}^n} |G(z)|, \max_{w\in\mathbb{T}} |H(w)|\right\}$$

This follows from (2.6), since the maximal ideal space \mathfrak{M} of the Banach algebra $S = C(\mathbb{T}^n) \times C(\mathbb{T})$ can be identified with $\mathbb{T}^n \cup \mathbb{T}$.

Remark 5.12. By [2], it follows that $A(\mathbb{D}^n)$ is a projective free ring, since its maximal ideal space the polydisk $\overline{\mathbb{D}^n}$ is contractible. Thus the set $\mathbb{S}(A(\mathbb{D}^n), p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

Remark 5.13. Roughly, the index function ι : inv $S \to G$ in all the examples given above (Sections 5.1-5.4) can be viewed as generalizations of the winding number for a continuous nonvanishing function on the unit circle. Another important application of such index functions, apart from robust control theory as presented here, is to the Fredholm theory of various classes of operators (e.g., Toeplitz, Wiener-Hopf, convolution) associated with the function. In this context we mention that Murphy [16] has given an abstract quantized C^* -algebra setting which, among other things, unifies the connection between analytic index and Fredholm index for the $C(\mathbb{T})$ -setting of Section 5.1 and the AP-setting of Section 5.2. There has also been a substantial amount of other work (see the books [6, 4]) where the analytic index has been extended to more general classes of functions (e.g. piecewisecontinuous) in order to develop the Fredholm theory for more general classes of Toeplitz operators. On the other hand, the index theory for semi-almost periodic symbols (a version of the Callier-Desoer class where \hat{f}_a is only required to be continuous on the extended imaginary line and where $f - f_a$ is required only to be AP rather than APW) follows a different more complicated path rather than making use of the index function W as in (5.3). Similarly, the Fredholm theory for Toeplitz operators on the quarter plane (associated with continuous functions on the bitorus \mathbb{T}^2) (see [6, Chapter 8]) makes use of the \mathbb{Z}^2 -valued index associated with the winding number of a function f on \mathbb{T}^2 taken with respect to each variable separately, rather than with the index ι as in Lemma 5.11.

6. FURTHER DIRECTIONS

It was shown in [25] that when R comprised rational functions without poles in the closed unit disk, then the bound established in Theorem 4.6 is the best possible one in the following sense:

(P'): C satisfying $\mu_{P_0,C} > m$ stabilizes P only if $d_{\nu}(P,P_0) \leq m$.

Since this property of d_{ν} already holds in the rational case, we expect the same to hold also in the specific examples considered in the previous section. We leave the question of investigation of whether the property (P') always holds in our abstract setup for future work.

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