CONTRACTIBILITY OF THE MAXIMAL IDEAL SPACE OF ALGEBRAS OF MEASURES IN A HALF-SPACE

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Abstract. Let $\mathbb{H}^{[n]}$ be the canonical half space in $\mathbb{R}^n$, that is,

$$\mathbb{H}^{[n]} = \{(t_1, \dots, t_n) \in \mathbb{R}^n \setminus \{0\} \mid \forall j,\ [t_j \neq 0 \text{ and } t_1 = t_2 = \dots = t_{j-1} = 0] \Rightarrow t_j > 0\} \cup \{0\}.$$

Let $\mathcal{M}(\mathbb{H}^{[n]})$ denote the Banach algebra of all complex Borel measures with support contained in $\mathbb{H}^{[n]}$, with the usual addition and scalar multiplication, and with convolution $*$, and the norm being the total variation of μ. It is shown that the maximal ideal space $X(\mathcal{M}(\mathbb{H}^{[n]}))$ of $\mathcal{M}(\mathbb{H}^{[n]})$, equipped with the Gelfand topology, is contractible as a topological space. In particular, it follows that $\mathcal{M}(\mathbb{H}^{[n]})$ is a projective free ring. In fact, for all subalgebras R of $\mathcal{M}(\mathbb{H}^{[n]})$ that satisfy a certain condition, it is shown that the maximal ideal space $X(R)$ of R is contractible. Several examples of such subalgebras are also given. We also show that this condition, although sufficient, is not necessary for the contractibility of unital subalgebras of $\mathcal{M}(\mathbb{H}^{[n]})$.

1. Introduction

The aim of this paper is to show that the maximal ideal space $X(R)$ of some Banach subalgebras (possessing a certain property) of the convolution algebra $\mathcal{M}(\mathbb{H}^{[n]})$ of all complex Borel measures with support in the half space $\mathbb{H}^{[n]}$, is contractible. It follows then that such Banach algebras are projective free rings. All the notation and precise definitions are explained below.

In particular, our result can be viewed as a two-fold generalization:

(1) of the result in [12], from the *one* dimensional case (of the half space $[0, +\infty)$ of $\mathbb{R}$) to the n-dimensional case (the half space $\mathbb{H}^{[n]}$ of $\mathbb{R}^n$).

(2) of the result in [10], from the *specific* subalgebra of almost periodic measures of $\mathcal{M}(\mathbb{H}^{[n]})$ to all subalgebras of $\mathcal{M}(\mathbb{H}^{[n]})$ satisfying a certain condition. (The result in [10] was in turn a generalization of a *one*-dimensional result of A. Brudnyi [2] to the *multi*-dimensional setting.)

Although the current result is a generalization of the result from the conference paper [12], it does not follow automatically.

1.1. Preliminary definitions and notation.

Definition 1.1. Let $\mathbb{H}^{[n]} \subset \mathbb{R}^n$ be the *canonical half space* defined by

$$\mathbb{H}^{[n]} = \{(t_1, \dots, t_n) \in \mathbb{R}^n \setminus \{0\} \mid \forall j,\ [t_1 = t_2 = \dots = t_{j-1} = 0,\ t_j \neq 0] \Rightarrow t_j > 0\} \cup \{0\}.$$

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$\mathcal{M}(\mathbb{H}^{[n]})$ denotes the set of all complex Borel measures with support contained in $\mathbb{H}^{[n]}$. Then $\mathcal{M}(\mathbb{H}^{[n]})$ is a complex vector space with addition and scalar multiplication defined in the pointwise manner as usual. The space $\mathcal{M}(\mathbb{H}^{[n]})$ becomes a complex algebra if convolution of measures (denoted henceforth by $*$) is taken as the operation of multiplication in the algebra. With the norm of μ taken as the total variation of μ, $\mathcal{M}(\mathbb{H}^{[n]})$ is a Banach algebra. Recall that the *total variation* $\|\mu\|$ *of* μ is defined by

$$\|\mu\| = \sup \sum_{k=1}^{\infty} |\mu(E_k)|,$$

the supremum being taken over all *partitions of* $\mathbb{H}^{[n]}$, that is over all countable collections $(E_k)_{k\in\mathbb{N}}$ of Borel subsets of $\mathbb{H}^{[n]}$ such that $E_k \bigcap E_m = \emptyset$ whenever $m \neq k$ and $\bigcup_{k\in\mathbb{N}} E_k = \mathbb{H}^{[n]}$. The identity with respect to convolution in $\mathcal{M}(\mathbb{H}^{[n]})$ is the *Dirac measure* δ_0^n in $\mathbb{R}^n$ supported at 0, given by

$$\delta_0^n(E) = \begin{cases} 1 & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E, \end{cases}$$

where E is any Borel subset of $\mathbb{H}^{[n]}$.

Definition 1.2.

(1) ($\mu^{[\bullet]}$) For $\mu \in \mathcal{M}(\mathbb{H}^{[n]})$, define the measures $\mu^{[k]} \in \mathcal{M}(\mathbb{H}^{[k]})$, $k = n, n-1, \dots, 2, 1$, inductively as follows. Set

$$\mu^{[n]} = \mu.$$

Suppose $\mu^{[k]} \in \mathcal{M}(\mathbb{H}^{[k]})$ has been defined. Then $\mu^{[k-1]} \in \mathcal{M}(\mathbb{H}^{[k-1]})$ is defined by

$$\mu^{[k-1]}(E) = \mu(\{0\} \times E),$$

where E is any Borel subset of $\mathbb{H}^{[k-1]}$.

(2) ($\mu_\bullet$) Given $\theta \in [0, 1)$ and $\mu \in \mathcal{M}(\mathbb{H}^{[k]})$, the measure $\mu_\theta \in \mathcal{M}(\mathbb{H}^{[k]})$ is defined by

$$\mu_\theta(E) = \int_E (1-\theta)^{t_1} d\mu(t), \tag{1}$$

where E is any Borel subset of $\mathbb{H}^{[k]}$. If $\theta = 1$, then for $\mu \in \mathcal{M}(\mathbb{H}^{[k]})$, μ_1 is defined as follows:

$$\mu_1 := \begin{cases} \delta_0^1 \otimes \mu^{[k-1]} & \text{if } k > 1, \\ \mu(\{0\})\delta_0^1 & \text{if } k = 1. \end{cases}$$

Notation 1.3. If R is a complex commutative unital Banach algebra, then $X(R)$ denotes the maximal ideal space of R. Thus $X(R)$ is the set of all nonzero complex homomorphisms from R to $\mathbb{C}$. $X(R)$ is endowed with the *Gelfand topology*, that is, the weak-$\star$ topology induced from the dual space $\mathcal{L}(R; \mathbb{C})$ of the Banach space R.

If R is any Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ which satisfies an assumption, namely Property (P) in Theorem 1.5 below, then we will show that $X(R)$ is contractible. The notion of contractibility of a topological space is recalled below.

Definition 1.4. A topological space X is said to be *contractible* if there exists a continuous map $H : X \times [0, 1] \to X$ and an $x_0 \in X$ such that

$$\text{for all } x \in X, \quad H(x, 0) = x \text{ and } H(x, 1) = x_0.$$

1.2. Main result.

Our main result is the following:

Theorem 1.5. *Suppose that R is a unital Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ satisfying:*

$$(\mathrm{P})\quad \boxed{\begin{array}{l}\text{For all } \mu \in R \text{ and all } \theta \in [0,1),\\ \quad \left.\begin{array}{l}\mu_\theta\\ \delta_0^1 \otimes \mu_\theta^{[n-1]}\\ \vdots\\ \delta_0^{n-1} \otimes \mu_\theta^{[1]}\end{array}\right\} \in R.\end{array}}$$

Then the maximal ideal space $X(R)$ equipped with the Gelfand topology is contractible.

The *Laplace transform* $\widehat{\mu}$ of $\mu \in \mathcal{M}(\mathbb{H}^{[n]})$ is defined by

$$\widehat{\mu}(s) = \int_{\mathbb{H}^{[n]}} e^{-\langle s,t\rangle} d\mu(t), \quad s \in \mathbb{H}^{[n]}.$$

Then $\widehat{\mu}$ is a holomorphic function of the variable s in the interior $(\mathbb{H}^{[n]})^\circ$ of $\mathbb{H}^{[n]}$, and $\widehat{\mu}$ is continuous on $\mathbb{H}^{[n]}$. Let $\widehat{\mathcal{M}}(\mathbb{H}^{[n]})$ denote the set of all Laplace transforms of elements of $\mathcal{M}(\mathbb{H}^{[n]})$. Then $\widehat{\mathcal{M}}(\mathbb{H}^{[n]})$ is a complex vector space with addition and multiplication defined pointwise, and it is a complex algebra if we define multiplication also in a pointwise manner. With the norm of $\widehat{\mu} \in \widehat{\mathcal{M}}(\mathbb{H}^{[n]})$ defined to be the norm of $\mu \in \mathcal{M}(\mathbb{H}^{[n]})$, $\widehat{\mathcal{M}}(\mathbb{H}^{[n]})$ becomes a Banach algebra, which is isometrically isomorphic to $\mathcal{M}(\mathbb{H}^{[n]})$. Then our main result (Theorem 1.5) yields the following:

Theorem 1.6. *Suppose that R is a unital Banach subalgebra of $\widehat{\mathcal{M}}(\mathbb{H}^{[n]})$ satisfying:*

$$(\widehat{\mathrm{P}})\quad \boxed{\begin{array}{l}\text{For all } \widehat{\mu} \in R \text{ and all } \theta \in [0,1), \text{ the maps}\\ \quad \begin{array}{rcl}\mathbb{H}^{[n]} \ni (s_1,\dots,s_n) & \mapsto & \widehat{\mu}(s_1 - \log(1-\theta), s_2, \dots, s_n),\\ \mathbb{H}^{[n]} \ni (s_1,\dots,s_n) & \mapsto & \widehat{\mu^{[n-1]}}(s_2 - \log(1-\theta), s_3, \dots, s_n),\\ & \vdots & \\ \mathbb{H}^{[n]} \ni (s_1,\dots,s_n) & \mapsto & \widehat{\mu^{[1]}}(s_n - \log(1-\theta)),\end{array}\\ \text{belong to } R.\end{array}}$$

Then the maximal ideal space $X(R)$ equipped with the Gelfand topology is contractible.

1.3. Corollaries of the main result.

By a result proved in [3], our main result from Theorem 1.5 implies that R is a projective free ring. The definition of a projective free ring is given below.

Definition 1.7. A commutative ring R with identity is said to be *projective free* if every finitely generated projective R-module is free. Recall that if M is an R-module, then

(1) M is *free* if $M \cong R^d$ for some integer $d \geq 0$;
(2) M is *projective* if there is an R-module N and an integer $d \geq 0$ such that $M \oplus N \cong R^d$.

In terms of matrices (with entries from R), the ring R is projective free iff for every square matrix P satisfying $P^2 = P$, there exists an invertible matrix G such that

$$GPG^{-1} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix};$$

see [4, Proposition 2.6].

For example, it can be seen from the matricial definition that any field $\mathbb{F}$ is projective free, since matrices P satisfying $P^2 = P$ are diagonalizable over $\mathbb{F}$. Quillen and Suslin independently proved, that the polynomial ring over a projective free ring is again projective free (see [7]), and so in particular, the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$ is projective free, settling Serre's conjecture from 1955. In the context of Banach algebras, the following result was shown recently [3, Corollary 1.4.(1)]:

Proposition 1.8. *Let R be a semisimple complex commutative unital Banach algebra. If the maximal ideal space $X(R)$ (equipped with the Gelfand topology) of the Banach algebra R is contractible, then R is a projective free ring.*

Recall that a commutative unital Banach algebra is said to be *semisimple* if its *radical* (that is, the intersection of all maximal ideals) is 0.

Proposition 1.9. *Every Banach subalgebra R of $\mathcal{M}(\mathbb{H}^{[n]})$ is semisimple.*

This will be proved at the end of Section 2. In light of Proposition 1.8, the main result given in Theorem 1.5 then implies the following.

Corollary 1.10. *Let R be a Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ satisfying the property* (P) *from Theorem 1.5. Then R is projective free.*

At the end of this article, we give examples of subalgebras of $\mathcal{M}(\mathbb{H}^{[n]})$ (respectively $\widehat{\mathcal{M}}(\mathbb{H}^{[n]})$) which satisfy the property (P) (respectively $\widehat{\mathrm{P}}$), which include several well-known classical convolution algebras of measures (and classes of almost periodic functions). Thus we have (with the notation explained in Section 4):

Corollary 1.11. *Let R be one of the Banach algebras $L^1(\mathbb{H}^{[n]}) + \mathbb{C}\delta_0^n$, $\mathcal{A}(\mathbb{H}^{[n]})$, APW^n_Σ or AP^n_Σ. Then the maximal ideal space $X(R)$ is contractible. In particular, R is projective free.*

The motivation for investigating whether or not convolution algebras of measures are projective free rings also arises from control theory, in the problem of stabilization of linear systems, since if R is a projective free ring, then every stabilizable plant with a transfer function over the field of fractions of R has a doubly coprime factorization. The reader is referred to [9], [3] for details.

The proof of Theorem 1.5 is given in Section 3, while examples are given in Section 4. In Subsection 3.2, we also show that the condition (P) is sufficient but not necessary for the contractibility of the maximal ideal space of the unital Banach subalgebra R of $\mathcal{M}(\mathbb{H}^{[n]})$. But first, a few technical results used in the sequel are proved in Section 2.

2. Preliminaries

In this section, we show a few auxiliary facts needed to prove the main result.

Lemma 2.1. *Let $k \in \{1, \dots, n\}$ and $\mu \in \mathcal{M}(\mathbb{H}^{[k]})$. Then for all $\theta \in [0, 1]$,*

(1) $\mu_\theta \in \mathcal{M}(\mathbb{H}^{[k]})$.
(2) $\|\mu_\theta\| \le \|\mu\|$.
(3) $(\delta_0^k)_\theta = \delta_0^k$.

Proof. (1) and (3) follow immediately from the definitions. The inequality in (2) is a straightforward verification when $\theta = 1$. We give a proof below when $\theta \in [0, 1)$. Given a $\mu \in \mathcal{M}(\mathbb{H}^{[k]})$, there exists a Borel measurable function w such that $d|\mu|(t) = e^{iw(t)}d\mu(t)$. Note that $\|\mu_\theta\| = \sup \sum |\mu_\theta(E_i)|$, the supremum being taken over all partitions $(E_i)_{i\in\mathbb{N}}$ of $\mathbb{H}^{[k]}$. So

$$\begin{aligned}|\mu_\theta(E_i)| &= \left|\int_{E_i}(1-\theta)^{t_1}d\mu(t)\right| = \left|\int_{E_i}e^{-iw(t)}(1-\theta)^{t_1}e^{iw(t)}d\mu(t)\right| \\ &= \left|\int_{E_i}e^{-iw(t)}(1-\theta)^{t_1}d|\mu|(t)\right| \leq \int_{E_i}1d|\mu|(t) = |\mu|(E_i).\end{aligned}$$

Hence $\sum |\mu_\theta(E_i)| \leq \sum |\mu|(E_i) = |\mu|(\mathbb{H}^{[k]}) = \|\mu\|$. □

Lemma 2.2. *If $\mu,\nu \in \mathcal{M}(\mathbb{H}^{[k+1]})$ and $k \geq 1$, then $(\mu*\nu)^{[k]} = \mu^{[k]} * \nu^{[k]}$.*

Proof. Let $E \subset \mathbb{H}^{[k]}$ be a Borel set. Then

$$\begin{aligned}(\mu*\nu)^{[k]}(E) &= (\mu*\nu)(\{0\}\times E) = \int_{\{0\}\times E}\mu((\{0\}\times E)-t)d\nu(t) \\ &= \int_{\{0\}\times E}\mu(\{0\}\times(E-\tau))d\nu^{[k]}(\tau) \\ &= \int_E \mu^{[k]}(E-\tau)d\nu^{[k]}(\tau) = (\mu^{[k]}*\nu^{[k]})(E).\end{aligned}$$

This completes the proof. □

Lemma 2.3. *If $\mu,\nu \in \mathcal{M}(\mathbb{H}^{[k+1]})$ where $k \geq 1$, then $(\delta_0^1\otimes\mu^{[k]})*(\delta_0^1\otimes\nu^{[k]}) = \delta_0^1\otimes(\mu^{[k]}*\nu^{[k]})$.*

Proof. (The notation $\mathcal{F}\mu$ is used for the Fourier transform of μ: $(\mathcal{F}\mu)(w) = \int e^{iwt}d\mu(t)$, $w \in \mathbb{R}$). For $w_1 \in \mathbb{R}$ and $\omega \in \mathbb{R}^k$,

$$\begin{aligned}\mathcal{F}((\delta_0^1\otimes\mu^{[k]})*(\delta_0^1\otimes\nu^{[k]}))(w_1,\omega) &= (\mathcal{F}(\delta_0^1\otimes\mu^{[k]}))(w_1,\omega)\cdot(\mathcal{F}(\delta_0^1\otimes\nu^{[k]}))(w_1,\omega) \\ &= (\mathcal{F}\delta_0^1)(w_1)\cdot(\mathcal{F}\mu^{[k]})(\omega)\cdot(\mathcal{F}\delta_0^1)(w_1)\cdot(\mathcal{F}\nu^{[k]})(\omega) \\ &= 1\cdot(\mathcal{F}\mu^{[k]})(\omega)\cdot 1\cdot(\mathcal{F}\nu^{[k]})(\omega) = (\mathcal{F}\mu^{[k]})(\omega)\cdot(\mathcal{F}\nu^{[k]})(\omega) \\ &= (\mathcal{F}(\mu^{[k]}*\nu^{[k]}))(\omega) = 1\cdot(\mathcal{F}(\mu^{[k]}*\nu^{[k]}))(\omega) \\ &= (\mathcal{F}\delta_0^1)(w_1)\cdot(\mathcal{F}(\mu^{[k]}*\nu^{[k]}))(\omega) \\ &= (\mathcal{F}(\delta_0^1\otimes(\mu^{[k]}*\nu^{[k]})))(w_1,\omega).\end{aligned}$$

Taking the inverse Fourier transform, the claim follows. □

Proposition 2.4. *If $\mu,\nu \in \mathcal{M}(\mathbb{H}^{[k]})$, then for all $\theta \in [0,1]$, $(\mu*\nu)_\theta = \mu_\theta * \nu_\theta$.*

Proof. Let us first suppose that $\theta \in [0,1)$. If E is a Borel subset of $\mathbb{H}$, then

$$(\mu*\nu)_\theta(E) = \int_E (1-\theta)^{t_1}d(\mu*\nu)(t) = \iint_{\substack{\sigma+\tau\in E \\ \sigma,\tau\in\mathbb{H}^{[k]}}}(1-\theta)^{\sigma_1+\tau_1}d\mu(\sigma)d\nu(\tau).$$

On the other hand,

$$\begin{aligned}(\mu_\theta*\nu_\theta)(E) &= \int_{\tau\in\mathbb{H}^{[k]}}\mu_\theta(E-\tau)d\nu_\theta(\tau) = \int_{\tau\in\mathbb{H}^{[k]}}\left(\int_{\substack{\sigma\in E-\tau \\ \sigma\in\mathbb{H}^{[k]}}}(1-\theta)^{\sigma_1}d\mu(\sigma)\right)d\nu_\theta(\tau) \\ &= \iint_{\substack{\sigma+\tau\in E \\ \sigma,\tau\in\mathbb{H}^{[k]}}}(1-\theta)^{\sigma_1+\tau_1}d\mu(\sigma)d\nu(\tau).\end{aligned}$$

Now consider the case when $\theta = 1$. If $k = 1$, the claim follows immediately, since

$$(\mu * \nu)_1 = (\mu * \nu)(\{0\})\delta_0^1 = \mu(\{0\}) \cdot \nu(\{0\})\delta_0^1 = (\mu(\{0\})\delta_0^1) * (\nu(\{0\})\delta_0^1) = \mu_1 * \nu_1.$$

If $k > 1$, then

$$\mu_1 * \nu_1 = (\delta_0^1 \otimes \mu^{[k-1]}) * (\delta_0^1 \otimes \nu^{[k-1]}) = \delta_0^1 \otimes (\mu^{[k-1]} * \nu^{[k-1]}) = \delta_0^1 \otimes (\mu * \nu)^{[k-1]} = (\mu * \nu)_1.$$

This completes the proof. □

The following result says that for a fixed $\mu \in \mathcal{M}(\mathbb{H}^{[k]})$, the map $\theta \mapsto \mu_\theta : [0,1] \to \mathcal{M}(\mathbb{H}^{[k]})$ is continuous.

Proposition 2.5. *If $\mu \in \mathcal{M}(\mathbb{H}^{[k]})$ and $\theta_0 \in [0,1]$, then $\lim_{\theta \to \theta_0} \mu_\theta = \mu_{\theta_0}$ in $\mathcal{M}(\mathbb{H}^{[k]})$.*

Proof. 1° Consider first the case when $\theta_0 \in [0,1)$. Let $\theta \in [0,1)$. There exists a Borel measurable function w such that $d(\mu_\theta - \mu_{\theta_0})(t) = e^{-iw(t)} d|\mu_\theta - \mu_{\theta_0}|(t)$. Thus

$$\begin{aligned} \|\mu_\theta - \mu_{\theta_0}\| &= |\mu_\theta - \mu_{\theta_0}|(\mathbb{H}^{[k]}) = \int_{\mathbb{H}^{[k]}} e^{iw(t)} d(\mu_\theta - \mu_{\theta_0})(t) \\ &= \left| \int_{\mathbb{H}^{[k]}} e^{iw(t)} d(\mu_\theta - \mu_{\theta_0})(t) \right| = \left| \int_{\mathbb{H}^{[k]}} e^{iw(t)} ((1-\theta)^{t_1} - (1-\theta_0)^{t_1}) d\mu(t) \right|. \end{aligned}$$

Given an $\epsilon > 0$, first choose an $R > 0$ large enough so that $|\mu|(B) < \epsilon$, where

$$B = \{t \in \mathbb{H}^{[k]} \mid \|t\|_2 \leq R\}.$$

Hence

$$\begin{aligned} \|\mu_\theta - \mu_{\theta_0}\| &\leq \left| \int_B e^{iw(t)} ((1-\theta)^{t_1} - (1-\theta_0)^{t_1}) d\mu(t) \right| + \left| \int_{\mathbb{H}^{[k]} \setminus B} e^{iw(t)} ((1-\theta)^{t_1} - (1-\theta_0)^{t_1}) d\mu(t) \right| \\ &\leq \left(\max_{t \in B} \left| (1-\theta)^{t_1} - (1-\theta_0)^{t_1} \right| \right) |\mu|(B) + 2|\mu|(\mathbb{H}^{[k]} \setminus B) \\ &\leq \left(\max_{t \in B} \left| (1-\theta)^{t_1} - (1-\theta_0)^{t_1} \right| \right) |\mu|(\mathbb{H}^{[k]}) + 2\epsilon. \end{aligned}$$

But by the mean value theorem applied to the function $\theta \mapsto (1-\theta)^{t_1}$,

$$(1-\theta)^{t_1} - (1-\theta_0)^{t_1} = (\theta - \theta_0) \cdot t_1 \cdot (1-c)^{t_1 - 1} = (\theta - \theta_0) \cdot t_1 \cdot \frac{(1-c)^{t_1}}{1-c},$$

for some c (depending on $t = t_1$, θ and θ_0) in between θ and θ_0. Since c lies between θ and θ_0, and since both θ and θ_0 lie in $[0,1)$, and $0 \leq t_1 \leq R$, it follows that $(1-c)^{t_1} \leq 1$ and

$$\frac{1}{1-c} \leq \max\left\{ \frac{1}{1-\theta}, \frac{1}{1-\theta_0} \right\}.$$

Thus using the above, and the fact that $0 \leq t_1 \leq R$,

$$\begin{aligned} \max_{t \in B} \left| (1-\theta)^{t_1} - (1-\theta_0)^{t_1} \right| &= \max_{t \in B} |\theta - \theta_0| \cdot |t_1| \cdot |(1-c)^{t_1}| \cdot \frac{1}{|1-c|} \\ &\leq |\theta - \theta_0| \cdot R \cdot 1 \cdot \max\left\{ \frac{1}{1-\theta}, \frac{1}{1-\theta_0} \right\}. \end{aligned}$$

Hence

$$\begin{aligned}\limsup_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\| &\le \limsup_{\theta\to\theta_0}\left(\left(\max_{t\in B}\left|(1-\theta)^{t_1}-(1-\theta_0)^{t_1}\right|\right)|\mu|(\mathbb{H}^{[k]})+2\epsilon\right)\\ &\le \limsup_{\theta\to\theta_0}\left(|\theta-\theta_0|\cdot R\cdot\max\left\{\frac{1}{1-\theta},\frac{1}{1-\theta_0}\right\}\cdot|\mu|(\mathbb{H}^{[k]})\right)+2\epsilon\\ &= 0\cdot R\cdot\frac{1}{1-\theta_0}|\mu|(\mathbb{H}^{[k]})+2\epsilon=0+2\epsilon=2\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, it follows that $\limsup_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|=0$. Also $\|\mu_\theta-\mu_{\theta_0}\|\ge 0$, and so $\lim_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|=0$.

2° Now consider the case when $\theta_0=1$. Assume first that $k>1$ and $\mu^{[k-1]}=0$. We will show that $\lim_{\theta\to 1}\mu_\theta=0$ in $\mathcal{M}(\mathbb{H}^{[k]})$. Given an $\epsilon>0$, first choose a $r>0$ small enough so that with

$$B:=\{t\in\mathbb{H}^{[k]}\mid \|t\|_2\le r\},$$

we have $|\mu|(B)<\epsilon$. (This is possible, since $\mu^{[k-1]}=0$.) There exists a Borel measurable function w such that $d\mu_\theta(t)=e^{-iw(t)}d|\mu_\theta|(t)$. Thus

$$\begin{aligned}\|\mu_\theta\| &= |\mu_\theta|(\mathbb{H}^{[k]})=\int_{\mathbb{H}^{[k]}}e^{iw(t)}d\mu_\theta(t)=\int_{\mathbb{H}^{[k]}}e^{iw(t)}(1-\theta)^{t_1}d\mu(t)=\left|\int_{\mathbb{H}^{[k]}}e^{iw(t)}(1-\theta)^{t_1}d\mu(t)\right|\\ &\le \left|\int_B e^{iw(t)}(1-\theta)^{t_1}d\mu(t)\right|+\left|\int_{\mathbb{H}^{[k]}\setminus B}e^{iw(t)}(1-\theta)^{t_1}d\mu(t)\right|\\ &\le |\mu|(B)+(1-\theta)^r\cdot|\mu|(\mathbb{H}^{[k]}\setminus B)\le\epsilon+(1-\theta)^r\cdot|\mu|(\mathbb{H}^{[k]}).\end{aligned}$$

Consequently, $\limsup_{\theta\to 1}\|\mu_\theta\|\le\epsilon$. But $\epsilon>0$ was arbitrary, and so $\limsup_{\theta\to 1}\|\mu_\theta\|=0$. Since $\|\mu_\theta\|\ge 0$, it follows that $\lim_{\theta\to 1}\|\mu_\theta\|=0$.

If $\mu_{k-1}\neq 0$, then define $\nu:=\mu-\delta_0^1\otimes\mu^{[k-1]}\in\mathcal{M}(\mathbb{H}^{[k]})$. It is clear that $\nu^{[k-1]}=0$ and $\nu_\theta=\mu_\theta-\delta_0^1\otimes\mu^{[k-1]}$. From the above, $\lim_{\theta\to 1}\nu_\theta=0$, and so $\lim_{\theta\to 1}\mu_\theta=\delta_0^1\otimes\mu^{[k-1]}=\mu_1$ in $\mathcal{M}(\mathbb{H}^{[k]})$.

3° The case when $\theta_0=1$ and $k=1$ is analogous to 2° above. □

Finally we prove that every Banach subalgebra R of $\mathcal{M}(\mathbb{H}^{[n]})$ is semisimple.

Proof of Proposition 1.9. If $s\in\mathbb{C}$, $\mathrm{Re}(s)\ge 0$, and $k\in\{1,\dots,n\}$, then $\Phi_s^{[k]}$, given by

$$\Phi_s^{[k]}(\mu)=\int_{\{t\ \mid\ t=(0,\tau)\in\mathbb{R}^k\times\mathbb{H}^{[n-k]}\}}e^{-st_k}d\mu(t)\qquad(\mu\in R),$$

is an element of $X(R)$, and so the kernel of $\Phi_s^{[k]}$ is a maximal ideal in R. But if $\Phi_s^{[k]}(\mu)=0$ for all s and all k, then μ is zero on $\mathbb{H}^{[n]}$. So the radical of R is 0. □

3. Contractibility of $X(R)$

In this section we will prove our main result.

Proof of Theorem 1.5. Define $H : X(R) \times [0,1] \to X(R)$ as follows. If $\theta \in [0,1]$, $\Phi \in X(R)$ and $\mu \in R$, then

$$(H(\Phi,\theta))(\mu) = \begin{cases} \Phi(\mu_{n\theta}) & 0 \le \theta < \frac{1}{n}, \\ \Phi(\delta_0^k \otimes \mu_{n\theta-k}^{[n-k]}) & \frac{k}{n} \le \theta < \frac{k+1}{n}, \quad k = 1,\dots,n-1, \\ \Phi(\mu(\{0\})\delta_0^n) = \mu(\{0\}) & \theta = 1. \end{cases}$$

We show that H is well-defined. From the definition, $H(\Phi,1) \in X(R)$ for all $\Phi \in X(R)$. If $\theta \in [0,1)$, then the linearity of $H(\Phi,\theta) : R \to \mathbb{C}$ is obvious. Continuity of $H(\Phi,\theta)$ follows from the fact that Φ is continuous and $\|\mu_\theta\| \le \|\mu\|$ for $\theta \in [0,1]$. That $H(\Phi,\theta)$ is multiplicative is a consequence of Proposition 2.4, and the fact that Φ respects multiplication. Finally $(H(\Phi,\theta))(\delta_0^n) = \Phi((\delta_0^n)_\theta) = \Phi(\delta_0^n) = 1$.

It is obvious that $H(\cdot,0)$ is the identity map and $H(\cdot,1)$ is a constant map.

Finally, we show below that H is continuous. Since $X(\mathcal{M}(\mathbb{H}^{[n]}))$ is equipped with the Gelfand topology, we just have to prove that for every convergent net $(\Phi_i,\theta_i)_{i\in I}$ with limit (Φ,θ) in $X(\mathcal{M}(\mathbb{H}^{[n]})) \times [0,1]$, there holds that $(H(\Phi_i,\theta_i))(\mu) \to (H(\Phi,\theta))(\mu)$. We have

$$|(H(\Phi_i,\theta_i))(\mu) - (H(\Phi,\theta))(\mu)| \le |(H(\Phi_i,\theta_i))(\mu) - (H(\Phi_i,\theta))(\mu)| + |(H(\Phi_i,\theta) - H(\Phi,\theta))(\mu)|,$$

and from the definition of H, it is immediate that $|(H(\Phi_i,\theta) - H(\Phi,\theta))(\mu)| \to 0$. So it remains to show that $|(H(\Phi_i,\theta_i))(\mu) - (H(\Phi_i,\theta))(\mu)| \to 0$. There is no loss of generality in assuming that all the θ_i's belong to one of the intervals $\left[0,\frac{1}{n}\right)$, $\left[\frac{1}{n},\frac{2}{n}\right)$, $\dots$, $\left[\frac{n-1}{n},1\right)$. But then Proposition 2.5 yields the desired result: for example if $\theta_i \in [\frac{k}{n},\frac{k+1}{n})$ and $\theta = \frac{k+1}{n}$, then

$$\begin{aligned} |(H(\Phi_i,\theta_i))(\mu) - (H(\Phi_i,\theta))(\mu)| &= |\Phi_i(\delta_0^k \otimes \mu_{n\theta_i-k}^{[n-k]} - \delta_0^k \otimes (\delta_0^1 \otimes \mu^{[n-k-1]}))| \\ &\le \|\Phi_i\| \cdot \|\delta_0^k\| \cdot \|\mu_{n\theta_i-k}^{[n-k]} - \delta_0^1 \otimes \mu^{[n-k-1]}\| \\ &\le 1 \cdot 1 \cdot \|\mu_{n\theta_i-k}^{[n-k]} - \delta_0^1 \otimes \mu^{[n-k-1]}\| \to 0. \end{aligned}$$

This completes the proof. □

3.1. **Remarks about the conditions (P) and ($\widehat{\mathbf{P}}$) and the proof of Theorem 1.5.** Our definition of the map H is based on the following consideration, in the case of $n = 1$, when $\mathbb{H}^{[n]} = \mathbb{H}^{[1]} = [0,+\infty)$.

The result given below can be thought of as a generalization of the Riemann-Lebesgue Lemma for functions $f_a \in L^1(0,+\infty)$ (that the limit as $s \to +\infty$ of the Laplace transform of f_a is 0):

Proposition 3.1. *If $\mu \in \mathcal{M}(\mathbb{H}^{[1]})$, then* $\displaystyle\lim_{s\to+\infty} \int_0^{+\infty} e^{-st} d\mu(t) = \mu(\{0\})$.

The set $X(\mathcal{M}(\mathbb{H}^{[1]}))$ contains the half plane $\mathbb{C}_{\ge 0} := \{s \in \mathbb{C} \mid \operatorname{Re}(s) \ge 0\}$ in the sense that each $s \in \mathbb{C}_{\ge 0}$, gives rise to the corresponding complex homomorphism $\Phi_s : \mathcal{M}(\mathbb{H}^{[1]}) \to \mathbb{C}$, given simply by point evaluation of the Laplace transform of μ at s:

$$\mu \mapsto \Phi_s(\mu) = \int_0^{+\infty} e^{-st} d\mu(t), \quad \mu \in \mathcal{M}(\mathbb{H}^{[1]}).$$

If we imagine s tending to $+\infty$ along the real axis we see from Proposition 3.1, that Φ_s starts looking more and more like the complex homomorphism $\Phi_{+\infty}$ given by

$$\mu \mapsto \Phi_{+\infty}(\mu) := \mu(\{0\}), \quad \mu \in \mathcal{M}(\mathbb{H}^{[1]}).$$

So we may define $H(\Phi_s, \theta) = \Phi_{s-\log(1-\theta)}$, which would suggest that at least the part $\mathbb{C}_{\geq 0}$ of $X(\mathcal{M}(\mathbb{H}^{[1]}))$ is contractible to $\Phi_{+\infty}$. But we see that we can view the action of $H(\Phi_s, \theta)$ defined above as follows:

$$\begin{aligned}(H(\Phi_s, \theta))(\mu) &= \Phi_{s-\log(1-\theta)}(\mu) \\ &= \int_0^{+\infty} e^{-(s-\log(1-\theta))t} d\mu(t) = \int_0^{+\infty} e^{-st}(1-\theta)^t d\mu(t) \\ &= \Phi_s(\nu),\end{aligned}$$

where ν is the measure such that $d\nu(t) = (1-\theta)^t d\mu(t)$. This motivates the definition of μ_θ given in (1), and the definition of H in the proof of Theorem 1.5.

We note that the map $\mu \mapsto \mu_\theta$ is just a particular translation in the "frequency domain", that is, by taking Laplace transforms. Indeed, we have

$$\widehat{\mu_\theta}(s) = \int_0^{+\infty} e^{-st}(1-\theta)^t d\mu(t) = \int_0^{+\infty} e^{-(s-\log(1-\theta))t} d\mu(t) = \widehat{\mu}(s - \log(1-\theta)).$$

This explains the relation between the conditions (P) and $(\widehat{\mathrm{P}})$.

3.2. **The condition (P) or $(\widehat{\mathbf{P}})$ is not necessary for contractibility.** In this subsection, we will give an example of a unital Banach subalgebra of $\widehat{\mathcal{M}}(\mathbb{H}^{[1]})$ that has a contractible maximal ideal space, but fails to possess the property $(\widehat{\mathrm{P}})$. The example can be adapted also to get a counterexample for the necessity of $(\widehat{\mathrm{P}})$ for the contractibility of the maximal ideal space of $\widehat{\mathcal{M}}(\mathbb{H}^{[n]})$ for $n > 1$. By taking inverse Laplace transforms, we then also get the analogous result in the case of $\mathcal{M}(\mathbb{H}^{[n]})$.

The subalgebra R of $\widehat{\mathcal{M}}(\mathbb{H}^{[1]})$. Consider the element $\widehat{\mu} \in \widehat{\mathcal{M}}(\mathbb{H}^{[1]})$, given by

$$\widehat{\mu}(s) = \frac{s(s-1)}{(s+1)^2}, \quad s \in \mathbb{C}_{\geq 0} := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}.$$

Thus $\mu \in \mathcal{M}(\mathbb{H}^{[1]})$ is given by $d\mu(t) = (\delta_0^1 - 3e^{-t} + 2te^{-t})dt$. Consider the subalgebra $R = [\mathbf{1}, \widehat{\mu}]$ of $\widehat{\mathcal{M}}(\mathbb{H}^{[1]})$ generated by the identity element $\mathbf{1}$ (namely the map $s \mapsto 1$) and the element $\widehat{\mu}$, that is, R is the closure in $\widehat{\mathcal{M}}(\mathbb{H}^{[1]})$ of all polynomials in $\widehat{\mu}$. In other words, R is the closure in $\widehat{\mathcal{M}}(\mathbb{H}^{[1]})$ of elements of the form $p(\widehat{\mu}) := a_0\mathbf{1} + a_1\widehat{\mu} + a_2(\widehat{\mu})^2 + \cdots + a_n(\widehat{\mu})^n$, where $a_1, a_1, a_2, \ldots, a_n$ are complex scalars, and n denotes any nonnegative integer.

Contractibility of $X(R)$. The following result is known; see [5, Theorem 1.4, page 68]:

Proposition 3.2. *Let B be a finitely generated Banach algebra, generated by $x_1, \ldots, x_m$. Then the joint spectrum of $x_1, \ldots, x_m$ in B, namely the set*

$$\sigma_B(x_1, \ldots, x_m) = \{(\widehat{x_1}(\varphi), \ldots, \widehat{x_m}(\varphi)) \mid \varphi \in X(B)\} \ (\subset \mathbb{C}^m),$$

is homeomorphic to the maximal ideal space $X(B)$. (Here $\widehat{\cdot}$ denotes the Gelfand transform.)

So in our case, it suffices to show that the joint spectrum of $\mathbf{1}$ and $\widehat{\mu}$ in R is contractible. We observe that

$$\sigma_R(\mathbf{1},\widehat{\mu}) = \{(1,\widehat{\mu}(\varphi)) \mid \varphi \in X(R)\} = \{1\} \times \{\widehat{\mu}(\varphi) \mid \varphi \in X(R)\} = \{1\} \times \sigma_R(\widehat{\mu}). \qquad (2)$$

Hence it is enough to show that $\sigma_R(\widehat{\mu})$ is contractible. We recall the following result, which relates the spectrum of an element x of a subalgebra of a Banach algebra with the spectrum of x in the Banach algebra; see [11, Theorem 10.18, page 238].

Proposition 3.3. *Let B be a unital Banach algebra, and S be a Banach subalgebra of B that contains the unit of B. If $x \in S$, then $\sigma_S(x)$ is the union of $\sigma_B(x)$ and a (possibly empty) collection of bounded components of the complement of $\sigma_B(x)$.*

We apply the above with

$$B = \widehat{L^1(0,\infty) + \mathbb{C}\delta_0^1} = \left\{ s \mapsto \int_0^\infty e^{-st} f_a(t)dt + \alpha \;\middle|\; f_a \in L^1(0,\infty) \text{ and } \alpha \in \mathbb{C} \right\},$$

$S = R$, and $x = \widehat{\mu} \in R \subset \widehat{L^1(0,\infty) + \mathbb{C}\delta_0^1}$. The maximal ideal space of $\widehat{L^1(0,\infty) + \mathbb{C}\delta_0^1}$ can be identified with $\{s \in (\mathbb{C} \cup \{\infty\}) \mid \text{ and } \mathrm{Re}(s) \geq 0\}$; see [6, pages 112-113]. Consequently,

$$\sigma_{\widehat{L^1(0,\infty)+\mathbb{C}\delta_0^1}}(\widehat{\mu}) = \left\{ \frac{s(s-1)}{(s+1)^2} \;\middle|\; s \in (\mathbb{C} \cup \{\infty\}) \text{ and } \mathrm{Re}(s) \geq 0 \right\} \overset{\left(s=\frac{1+z}{1-z}\right)}{=} \left\{ \frac{z+z^2}{2} \;\middle|\; |z| \leq 1 \right\}.$$

It can be shown that this last set is the closure of the interior of a simple closed curve C; see Figure 1.

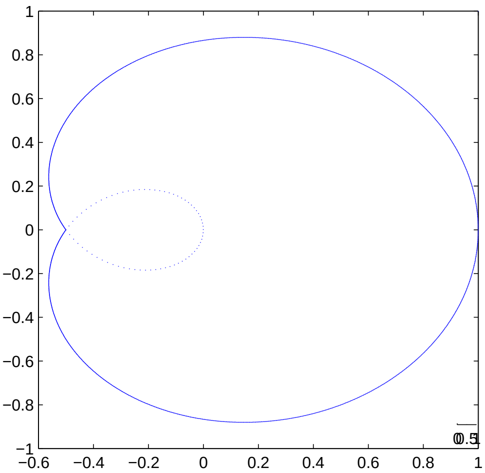

FIGURE 1. The simple closed curve C is depicted by the bold line. The bold line C together with the dotted line is the curve $\theta \mapsto \frac{e^{i\theta}+e^{2i\theta}}{2} : [0, 2\pi] \to \mathbb{C}$.

Thus the complement of $\sigma_{\widehat{L^1(0,\infty)+\mathbb{C}\delta_0^1}}(\widehat{\mu})$ has no bounded components, and so by Proposition 3.3 we conclude that $\sigma_R(\widehat{\mu}) = \sigma_{\widehat{L^1(0,\infty)+\mathbb{C}\delta_0^1}}(\widehat{\mu})$. Hence $\sigma_R(\widehat{\mu})$ is contractible and from (2) it follows that also $\sigma_R(\mathbf{1},\widehat{\mu}) = \{1\} \times \sigma_R(\widehat{\mu})$ is contractible. Finally, by Proposition 3.2, $X(R)$ is contractible.

$(\widehat{\mathrm{P}})$ *does not hold.* Now we show that the map

$$s \mapsto \widehat{\mu}(s - \log(1-\theta)) =: \widehat{\mu_\theta}(s) \quad (s \in \mathbb{C}_{\geq 0})$$

does not belong to R for a particular choice of $\theta \in [0,1)$, demonstrating that $(\widehat{\mathrm{P}})$ does not hold. In fact we take $\theta = 1 - \frac{1}{e}$, so that $-\log(1-\theta) = 1$. Suppose on the contrary that $\widehat{\mu_\theta} \in R$. Then by the density of polynomials in $\widehat{\mu}$ in R, it follows that given $\epsilon = \frac{1}{10}$, there exists a nonnegative integer n and scalars $a_1, a_1, a_2, \dots, a_n$ such that

$$\|\widehat{\mu_\theta} - (a_0\mathbf{1} + a_1\widehat{\mu} + a_2(\widehat{\mu})^2 + \cdots + a_n(\widehat{\mu})^n)\| < \epsilon. \tag{3}$$

But for every $\nu \in \mathcal{M}(\mathbb{H}^{[1]})$, we have that

$$|\widehat{\nu}(s)| = \left|\int_0^\infty e^{-st} d\nu(t)\right| = \left|\int_0^\infty e^{-iw(t)}e^{-st} d|\nu|(t)\right| \leq \int_0^\infty 1 \cdot d|\nu|(t) = \|\nu\| \quad (s \in \mathbb{C}_{\geq 0}),$$

where in the above, w denotes a Borel measurable function such that $d|\nu|(t) = e^{iw(t)}d\nu(t)$. In light of this, we have from (3) that

$$|\widehat{\mu}(s+1) - (a_0\mathbf{1}(s) + a_1\widehat{\mu}(s) + a_2(\widehat{\mu}(s))^2 + \cdots + a_n(\widehat{\mu}(s))^n)| < \epsilon \quad (s \in \mathbb{C}_{\geq 0}).$$

Now putting $s = 0$ and $s = 1$, respectively, we obtain the inequalities

$$\left|\frac{(0+1)(0+1-1)}{(0+1+1)^2} - a_0\right| = |a_0| < \epsilon \quad \text{and} \quad \left|\frac{(1+1)(1+1-1)}{(1+1+1)^2} - a_0\right| = \left|\frac{2}{9} - a_0\right| < \epsilon.$$

Adding these, we obtain $\frac{2}{9} \leq \left|\frac{2}{9} - a_0\right| + |a_0| < 2\epsilon = \frac{2}{10}$, a contradiction.

4. Examples

As specific examples of R in Theorem 1.5 and Theorem 1.6, we have the following:

4.1. **The algebra $L^1(\mathbb{H}^{[n]})+\mathbb{C}\delta_0^n$.** Consider the Banach subalgebra $L^1(\mathbb{H}^{[n]})+\mathbb{C}\delta_0^n$ of $\mathcal{M}(\mathbb{H}^{[n]})$, consisting of all complex Borel measures of the type $\mu_a + \alpha\delta_0^n$, where μ_a is absolutely continuous (with respect to the Lebesgue measure) and $\alpha \in \mathbb{C}$. It can be checked that this Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ has the property (P) in the statement of Theorem 1.5.

4.2. **The algebra $\mathcal{A}(\mathbb{H}^{[n]})$.** The Banach subalgebra $\mathcal{A}(\mathbb{H}^{[n]})$ of $\mathcal{M}(\mathbb{H}^{[n]})$ consists of all complex Borel measures that do not have a singular non-atomic part. Then it can be verified that $\mathcal{A}(\mathbb{H}^{[n]})$ also possesses the property (P). (So in the case when $n = 1$, we recover the main result in [13], but this time without recourse to the explicit description of the maximal ideal space.)

4.3. **Algebras of almost periodic functions.** The algebra AP^n of complex valued (uniformly) almost periodic functions is, by definition, the smallest closed subalgebra of $L^\infty(\mathbb{R}^n)$ (with all operations defined pointwise), that contains all the functions $e_\lambda(x) := e^{i\langle\lambda,x\rangle}$. Here the variable $x = (x_1, \dots, x_n)$, the parameter $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, and $\langle\lambda, x\rangle := \sum_{k=1}^n \lambda_k x_k$. For any $f \in AP^n$, its *Bohr-Fourier series* is defined by the formal sum $\sum_\lambda f_\lambda e^{i\langle\lambda,x\rangle}$ $(x \in \mathbb{R}^n)$, where

$$f_\lambda := \lim_{N\to\infty} \frac{1}{(2N)^n} \int_{[-N,N]^n} e^{-i\langle\lambda,x\rangle} a(x)dx, \quad \lambda \in \mathbb{R}^n,$$

and the sum $\sum_\lambda f_\lambda e^{i\langle\lambda,x\rangle}$ is taken over the set $\sigma(f) := \{\lambda \in \mathbb{R}^n \mid f_\lambda \neq 0\}$, called the *Bohr-Fourier spectrum* of f. The Bohr-Fourier spectrum of every $f \in AP^n$ is at most a countable set.

The *almost periodic Wiener algebra* APW^n is defined as the set of all AP^n such that the Bohr-Fourier series $\sum_\lambda f_\lambda e^{i\langle \lambda, x\rangle}$ of f converges absolutely. The almost periodic Wiener algebra is a Banach algebra with pointwise operations and the norm $\|f\| := \sum_{\lambda\in\mathbb{R}^n} |f_\lambda|$. Let Δ be a nonempty subset of $\mathbb{R}^n$. Denote

$$\begin{aligned} AP^n_\Delta &= \{f \in AP^n \mid \sigma(f) \subset \Delta\} \\ APW^n_\Delta &= \{f \in APW^n \mid \sigma(f) \subset \Delta\}. \end{aligned}$$

If Δ is an additive subset of $\mathbb{R}^n$, then AP^n_Δ (respectively APW^n_Δ) is a Banach subalgebra of AP^n (respectively APW^n). Moreover, if $0 \in \Delta$, then AP^n_Δ and APW^n_Δ are also unital.

Let $\Sigma \subset \mathbb{H}^{[n]}$ be an *additive semigroup* (if $\lambda, \mu \in \Sigma$, then $\lambda + \mu \in \Sigma$) and suppose $0 \in \Sigma$. The Banach algebra APW^n_Σ is isometrically isomorphic to the following Banach subalgebra R of $\mathcal{M}(\mathbb{H}^{[n]})$:

$$R = \left\{ \sum_\lambda f_\lambda \delta^n_0(\lambda) \;\middle|\; \sum_\lambda f_\lambda e^{i\langle\lambda,x\rangle} \in APW^n_\Sigma \right\}.$$

Then $APW^n_\Sigma = \widehat{R}$. In the above, $\delta^n_0(\lambda) \in \mathcal{M}(\mathbb{H}^{[n]})$ denotes the Dirac measure supported at $\lambda \in \mathbb{H}^{[n]}$, that is,

$$(\delta^n_0(\lambda))(E) = \begin{cases} 1 & \text{if } \lambda \in E, \\ 0 & \text{if } \lambda \notin E, \end{cases}$$

where E is any Borel subset of $\mathbb{H}^{[n]}$. It can be seen that the subalgebra R has the property (P), and $\widehat{R}$ has the property $(\widehat{\text{P}})$. Thus the maximal ideal space of APW^n_Σ is contractible. The maximal ideal spaces of AP^n_Σ and APW^n_Σ are homeomorphic as topological spaces; see for example [1, Theorem 3.1]. So the maximal ideal space of AP^n_Σ is contractible as well. Thus we recover the main result from [10]. (In [10], instead of the canonical half space $\mathbb{H}^{[n]}$, more general half spaces S were considered. There a subset S of $\mathbb{R}^n$ was called a *half space* in $\mathbb{R}^n$ if it satisfied the properties $S \cup (-S) = \mathbb{R}^n$, $S \cap (-S) = \{0\}$, $x + y \in S$ for all $x, y \in S$, $\alpha x \in S$ for all $x \in S$ and $\alpha \geq 0$. However, it was shown in [10, Proposition 1.2] that any such half space S is of the form $Z\mathbb{H}^{[n]}$ for an invertible matrix $Z \in \mathbb{R}^{n\times n}$.)

Summarizing the results of this section, we have shown Corollary 1.11 as a particular consequence of our main results in Theorems 1.5 and 1.6.

5. Open question

We have seen that the condition (P) is sufficient but not necessary for the contractibility of the maximal ideal space of the unital Banach subalgebra R of $\mathcal{M}(\mathbb{H}^{[n]})$. The following natural question then arises:

Can the condition (P) be replaced by a weaker condition (P′) so that the new condition (P′) is necessary *and* sufficient for the contractibility of the maximal ideal space of a unital Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ in Theorem 1.5?

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