ON THE DENSITY OF STABILIZABLE PLANTS IN THE CLASS OF UNSTABILIZABLE PLANTS

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Abstract. Let $A_{\mathbb{R}}$ denote the set of functions from the disk algebra having real Fourier coefficients. Generalizing a result of A. Quadrat we show that every unstabilizable multi-input multi-output plant is as close as we want to a stabilizable multi-input multi-output plant in the product topology.

1. Introduction

The fractional representation approach to analysis and synthesis problems was developed in the 1980s in order to express in a unique mathematical framework several questions on stabilization problems. In that framework, we can study internal stabilization (existence of an internally stabilizing controller), parametrization of all stabilizing controllers, strong stabilization (possibility of stabilizing a plant by means of a stable controller), simultaneous stabilization (possibility of a stabilizing a set of plants by means of a single controller), etc. See [11] for more details. In a recent paper, A. Quadrat proposed a generalisation of the well-known Youla-Kucera parametrization for stabilizable plants which do not necessarily admit a doubly coprime factorization. Consequently using the concept of topological stable rank, A. Quadrat showed that every unstable single input single output plant, defined by the transfer function $p = n/d$, with $0 \neq d, n \in H^\infty(\mathbb{D})$, is as close as we want to a stabilizable plant in the product topology, see 3.2.

The purpose of this research note is twofold: Unfortunately, Quadrat's transfer functions need not be real on the real axis, so no physically meaning can be given to his approximating plants. In the meantime this difficulty has been overcome, because the topological stable rank (Definition see below) of the algebra of real bounded analytic functions is 2, see [10]. Moreover, in [13], the authors proved the same for the real disk algebra.

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In this note, we generalize this approximation result to multi-input multi-output systems.

The notion of stable rank of a ring (which we call Bass stable rank) was introduced by H. Bass [2] to facilitate computations in algebraic K-theory. We recall the definition of the Bass stable rank of a ring below.

Definition 1.1 Let $\mathcal{A}$ be a commutative ring with identity 1. Let $n \in \mathbb{N}$. An element $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ is called *unimodular* if there exists an element $b = (b_1, \dots, b_n) \in \mathcal{A}^n$ such that

$$\sum_{k=1}^{n} b_k a_k = 1.$$

We denote by $U_n(\mathcal{A})$ the set of unimodular elements of $\mathcal{A}^n$.

The element $a = (a_1, \dots, a_n) \in U_n(\mathcal{A})$ is *reducible*, if there exist elements $h_1, \dots, h_{n-1} \in \mathcal{A}$ such that $(a_1 + h_1 a_n, \dots, a_{n-1} + h_{n-1} a_n) \in U_{n-1}(\mathcal{A})$. The *Bass stable rank of* $\mathcal{A}$, denoted by $\mathrm{bsr}(\mathcal{A})$, is the least integer n such that every $a \in U_{n+1}(\mathcal{A})$ is reducible, and is defined to be infinite if no such n exists.

The Bass stable rank is a purely algebraic notion, but when studying commutative Banach algebras of functions, analysis also plays a role. In [12], M. Rieffel introduced the notion of topological stable rank (abbreviated tsr), analogous to the K-theoretic concept of Bass stable rank. We recall this definition below.

Definition 1.2 Let $\mathcal{A}$ denote a (real or complex) commutative unital normed algebra. The *topological stable rank* of $\mathcal{A}$ is the minimum integer n such that $U_n(\mathcal{A})$ is dense in $\mathcal{A}^n$ (and it is infinite if no such integer exists).

The determination of the Bass and topological stable ranks of $A_{\mathbb{R}}$ plays an important role in control theory, in the problem of stabilization of linear systems. We refer the reader to Quadrat [11] for background on the connection between stable ranks and control theory; in particular, see Corollary 6.4 on page 2279 and Proposition 7.4 on page 2281. We briefly explain this in the last section of this note.

In applications in control theory, the linear systems and transfer functions have *real* coefficients, and so in this context it is important to consider *real* algebras, since otherwise the controllers obtained are physically meaningless. It was conjectured by Brett Wick in [17] that the Bass stable rank of the real disk algebra is equal to 2, and in [13] we prove this.

Jones, Marshall and Wolff showed that the Bass stable rank of the complex disk algebra A is equal to 1 (see [7]), and Rieffel showed that its topological stable rank is equal to 2 (see [12]). Recall that the disk algebra is the Banach algebra of all complex valued functions defined on the closed unit disk $\overline{\mathbb{D}}$ that are holomorphic in the open unit disk $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, equipped with the supremum norm: $\|f\|_\infty = \sup_{z\in\overline{\mathbb{D}}} |f(z)|$.

In this paper we show that the topological stable rank of the real disk algebra $A_{\mathbb{R}}$ (defined below) is equal to 2. From this result it follows, that the Bass stable rank of the real disk algebra is also 2, see [13].

DEFINITION 1.3 The *real disk algebra*, denoted by $A_{\mathbb{R}}$, is the set of all functions of A having real Fourier coefficients. Equivalently,

$$A_{\mathbb{R}} = \{f \in A \mid \forall z \in \overline{\mathbb{D}},\ f(z) = \overline{f(\overline{z})}\}.$$

The real algebra $A_{\mathbb{R}}$ is a Banach algebra with the supremum norm $\|\cdot\|_\infty$. The *real Wiener algebra*, denoted by $W^+_{\mathbb{R}}$, is the set of all functions of the Wiener algebra W^+ having real Fourier coefficients, that is

$$W^+_{\mathbb{R}} = \left\{ f(z) = \sum_{k=0}^{\infty} a_k z^k \;\middle|\; \text{all } a_k \text{ real},\ \sum_{k=0}^{\infty} |a_k| < \infty \right\}.$$

The real algebra $W^+_{\mathbb{R}}$ is a Banach algebra with the norm $\|f\| := \sum_{k=0}^{\infty} |a_k|$.

The real algebra of bounded analytic functions, denoted by $H^\infty_{\mathbb{R}}$, is given by all bounded analytic functions on the unit disk $\mathbb{D}$ having real Fourier coefficients

$$H^\infty_{\mathbb{R}} = \left\{ f(z) = \sum_{k=0}^{\infty} a_k z^k \;\middle|\; \text{all } a_k \text{ real},\ f \text{ bounded} \right\}.$$

The real algebra $H^\infty_{\mathbb{R}}$ is a Banach algebra when given the supremum norm $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|$.

All stable rational transfer fuctions belong to the real disk algebra, either in context of discrete systems or by using the well-known transformation of the extended right half plane to the closed unit disk. The transition to the uniform limits allows for a much wider class of transfer functions.

The functions in the real Wiener algebra represent the set of all l^∞-stable (bounded input bounded output stability), shift-invariant causual digital filters.

After the usual transformation to the right half plane the real bounded analytic functions are useful as transfer functions of certain PDEs, for example, the wave equation with inputs [5].

We make the observation that the polynomials with real coefficients are dense in $A_{\mathbb{R}}$. Indeed, given $f \in A_{\mathbb{R}}$, f has real Fourier coefficients, which are the same as coefficients in the Taylor expansion of the analytic function f about the point 0 in $\mathbb{D}$. Since f is continuous on the circle, and its negative Fourier coefficients vanish, the Cesàro means of the Fourier series for f comprise a sequence of trigonometric polynomials with real coefficients which converge uniformly to f. The corresponding polynomials in z give the desired sequence uniformly converging to f in $A_{\mathbb{R}}$.

To make this exposition self-contained, we include the proofs of the stable range results given in [13].

2. Topological stable rank

Theorem 2.1 *The topological stable rank of the Banach algebra $A_{\mathbb{R}}$ is equal to* 2.

Proof. First of all we note that $U_1(A_{\mathbb{R}})$ is not dense in $A_{\mathbb{R}}$. Indeed, $U_1(A_{\mathbb{R}})$ is the set of units in $A_{\mathbb{R}}$, and f is invertible as an element in $A_{\mathbb{R}}$ only if it has no zero in $\overline{\mathbb{D}}$. But the uniform limit of such a sequence is either identically zero or has no zeros in $\mathbb{D}$ (see Theorem 2, page 178 of [1]). So taking any function with finitely many zeros in $\mathbb{D}$, say z, we have a contradiction. So $\text{tsr}(A_{\mathbb{R}}) > 1$.

Next we show that $U_2(A_{\mathbb{R}})$ is dense in $A_{\mathbb{R}}^2$. Take $(f, g) \in A_{\mathbb{R}}^2$ and approximate f, g by polynomials p, q, respectively, having real coefficients. Since $p \in \mathbb{R}[z]$, we have the following product representation for p:

$$p(z) = C \prod (z - r_j) \prod (z^2 + s_j z + t_j),$$

where C, r_j, s_j, t_j are real numbers. If p and q have a common root in $\overline{\mathbb{D}}$, then we replace r_j, s_j, t_j by $r_j + \epsilon, s_j + \epsilon, t_j + \epsilon$ with a sufficiently small real ϵ so that the new polynomial $\widetilde{p}$ has no common root with q in $\overline{\mathbb{D}}$, and so $(\widetilde{p}, q) \in U_2(A_{\mathbb{R}})$. Consequently $\text{tsr}(A_{\mathbb{R}}) \leq 2$. $\square$

Remark 2.1 Theorem 2.1 (and Theorem 2.2) remain valid whenever we have a real (or complex) normed algebra R such that:

(1) The inclusion in $A_{\mathbb{R}}$ (or the complex disk algebra A, respectively) is continuous, that is, $\|f\|_\infty \leq C\|f\|$, $(f \in R)$. (Here $\|f\|$ denotes the norm of f in R.)
(2) The polynomials are dense in R.

The proofs are the same, mutatis mutandis. For instance, we have that the topological stable rank of the Wiener algebra W^+ and the real-symmetric Wiener algebra $W_{\mathbb{R}}^+$ are both equal to 2.

In [10], Mortini and Wick have proved that the topological stable rank of $H_{\mathbb{R}}^\infty$ is two.

We now give a matricial analogue of the above theorem, which is of independent interest in control theory, as explained in the next section following this result.

For a matrix $M \in A_{\mathbb{R}}^{p \times m}$, the transpose of M will be denoted by $M^\top$ in the sequel, and

$$\|M\| := \sup_{z \in \overline{\mathbb{D}}} \|M(z)\|_{\mathbb{C}^{p \times m}}.$$

THEOREM 2.2 *If $N \in A_{\mathbb{R}}^{p\times m}$ and $D \in A_{\mathbb{R}}^{m\times m}$, then for every $\epsilon > 0$, there exist $N_\epsilon \in A_{\mathbb{R}}^{p\times m}$ and $D_\epsilon \in A_{\mathbb{R}}^{m\times m}$ such that $\|N_\epsilon - N\| < \epsilon$, $\|D_\epsilon - D\| < \epsilon$ and $\begin{bmatrix} N_\epsilon & D_\epsilon \end{bmatrix}^\top$ has a left inverse with entries in $A_{\mathbb{R}}$.*

Proof. Since the polynomials with real coefficients are dense in $A_{\mathbb{R}}$, we can approximate N, D by polynomial matrices $P \in \mathbb{R}[z]^{p\times m}$ and $Q \in \mathbb{R}[z]^{m\times m}$. Let $G := \begin{bmatrix} P & Q \end{bmatrix}^\top$. By the Smith decomposition of G (see for instance [3, Theorem 4.3.2, p. 127]), there exist invertible matrices $U \in \mathbb{R}[z]^{(p+m)\times(p+m)}$, $V \in \mathbb{R}[z]^{m\times m}$, and polynomials $p_1, \dots, p_m \in \mathbb{R}[z]$ such that $G = U\Sigma V$, where

$$\Sigma = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_m \\ & 0_{p\times m} & \end{bmatrix}.$$

Now we perturb the roots of the polynomials $p_1, \dots, p_m$ (preserving symmetry with respect to the real line), so that with the new polynomials $\widetilde{p}_1, \dots, \widetilde{p}_m$, no two polynomials have a common root in $\overline{\mathbb{D}}$. Let $\widetilde{\Sigma}$ denote the matrix obtained by replacing the p_ks in Σ by the corresponding $\widetilde{p}_k$s, and by replacing one of the row of zeros in the $0_{p\times m}$ block of Σ by the row $\begin{bmatrix} \epsilon' & \dots & \epsilon' \end{bmatrix}$, where $\epsilon' > 0$. Then we note that

$$\forall z \in \overline{\mathbb{D}}, \quad \operatorname{rank}(\widetilde{\Sigma}(z)) = m. \tag{1}$$

We now show that (1) implies that $\widetilde{\Sigma}$ has a left inverse with entries in $A_{\mathbb{R}}$. Let $S(m, p+m)$ denote the set of all strictly increasing m-tuples taken from $\{1, \dots, m\}$. For each m-tuple $J \in S(m, p+m)$, let F_J denote the $m \times m$ submatrix of $\widetilde{\Sigma}$ corresponding to the rows with indices in J. Define f_J to be $\det F_J$. From (1), it follows that for all $z \in \overline{\mathbb{D}}$, there exists a $J \in S(m, p+m)$ such that $f_J(z) \neq 0$. Hence by the corona theorem for $A_{\mathbb{R}}$ [17], it follows that the minors f_J, $J \in S(m, p+m)$, together generate $A_{\mathbb{R}}$, that is, there exist $g_J \in A_{\mathbb{R}}$, $J \in S(m, p+m)$, such that

$$\sum_{J\in S(m,p+m)} g_J f_J = 1.$$

Next for each $J = (j_1, \dots, j_m) \in S(m, p+m)$, we construct a matrix $B_J \in A_{\mathbb{R}}^{m\times(p+m)}$ as follows:

(1) Let G_J be the adjoint of F_J.
(2) Let $g_J^1, \dots, g_J^m$ be the rows of G_J.
(3) Define B_J to be the $m \times (p+m)$ matrix whose rows with indices $j_1, \dots, j_m$ are equal to $g_J^1, \dots, g_J^m$, respectively, while all the other rows of B_J are zero.

Then by construction $B_J\widetilde{\Sigma} = G_J F_J = f_J I_m$. Now define

$$B = \sum_{J \in S(m,p+m)} g_J B_J.$$

Then $B\widetilde{\Sigma} = I_m$.

Consequently, defining the $p \times m$ and $m \times m$ matrices N_ϵ and D_ϵ, respectively, by $\begin{bmatrix} N_\epsilon & D_\epsilon \end{bmatrix}^\top = U\widetilde{\Sigma}V$, we have that $\begin{bmatrix} N_\epsilon & D_\epsilon \end{bmatrix}^\top$ has a left inverse with entries in $A_{\mathbb{R}}$ and moreover by choosing the ϵ' and the perturbation of the polynomial roots of $p_1, \dots, p_m$ small enough at the outset, we can also ensure that $\|N_\epsilon - N\| < \epsilon$ and $\|D_\epsilon - D\| < \epsilon$. This completes the proof. $\Box$

Of course this result also holds for the real Wiener algebra $W^+_{\mathbb{R}}$; see Remark 2.1.

Example 2.1 Let

$$\begin{aligned} N &= \begin{bmatrix} z^2 & 0 \end{bmatrix}, \\ D &= \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Then the matrix

$$\Sigma(z) := \begin{bmatrix} N(z) & D(z) \end{bmatrix} = \begin{bmatrix} z^2 & 0 \\ 0 & z \\ 0 & 0 \end{bmatrix}$$

is not left invertible. Indeed, if there were a matrix P with entries in $A_{\mathbb{R}}$ such that $P(z)\Sigma(z) = I_2$, then putting $z = 0$, we arrive at the contradiction that $0 = I$.

Now suppose that any $\epsilon > 0$ has been given. Following the procedure given in the proof above, we construct the matrices N_ϵ, D_ϵ as follows:

$$\begin{aligned} N_\epsilon &= \begin{bmatrix} z^2 + \epsilon & 0 \end{bmatrix}, \\ D_\epsilon &= \begin{bmatrix} 0 & z \\ \epsilon & \epsilon \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \|N - N_\epsilon\| &= \left\| \begin{bmatrix} \epsilon & 0 \end{bmatrix} \right\| \leq \epsilon, \\ \|D - D_\epsilon\| &= \left\| \begin{bmatrix} 0 & 0 \\ \epsilon & \epsilon \end{bmatrix} \right\| \leq \epsilon. \end{aligned}$$

Furthermore, the matrix

$$\begin{bmatrix} N_\epsilon(z) & D_\epsilon(z) \end{bmatrix} = \begin{bmatrix} z^2 + \epsilon & 0 \\ 0 & z \\ \epsilon & \epsilon \end{bmatrix}$$

is left invertible in $A_{\mathbb{R}}$, since

$$\begin{bmatrix} \frac{1}{\epsilon} & \frac{z}{\epsilon} & -\frac{z^2}{\epsilon^2} \\ -\frac{1}{\epsilon} & -\frac{z}{\epsilon} & \frac{z^2}{\epsilon^2}+\frac{1}{\epsilon} \end{bmatrix} \begin{bmatrix} z^2+\epsilon & 0 \\ 0 & z \\ \epsilon & \epsilon \end{bmatrix} = I_2.$$

$\Diamond$

3. Control theoretic consequences

Finally, we give consequences for systems theory of the results established in the previous section.

We consider unstable transfer functions which we write as a ratio of elements from $A_{\mathbb{R}}$.

Definition 3.1 Matrices with entries in $A_{\mathbb{R}}$ will be denoted by $\mathrm{Mat}(A_{\mathbb{R}})$. If $N, D \in \mathrm{Mat}(A_{\mathbb{R}})$, then the pair (N, D) is called *right coprime* (with respect to $A_{\mathbb{R}}$) if there exist $X, Y \in \mathrm{Mat}(A_{\mathbb{R}})$ such that the matrix Bézout identity holds: $XN+YD = I$. A *left coprime* pair of matrices is defined analogously.

We now consider unstable transfer functions that can be expressed as a quotient of two elements from $A_{\mathbb{R}}$. Since $A_{\mathbb{R}}$ is an integral domain, we can consider its field of fractions. We recall this notion below.

Definition 3.2 If R is an integral domain, then a *fraction* is a symbol $\frac{N}{D}$, where $N, D \in R$ and $D \neq 0$. Define the relation $\sim$ on the set of all fractions as follows: $\frac{N_1}{D_1} \sim \frac{N_2}{D_2}$ if $N_1D_2 = N_2D_1$. The relation $\sim$ is an equivalence relation on the set of all fractions. The equivalence class of $\frac{N}{D}$ is denoted by $[\frac{N}{D}]$. The *field of fractions*, denoted by $\mathbb{F}(R)$, is the set $\mathbb{F}(R) = \{[\frac{N}{D}] \mid N, D \in R \text{ and } D \neq 0\}$, of equivalence classes of the relation $\sim$, with addition and multiplication defined as follows: $[\frac{N_1}{D_1}] + [\frac{N_2}{D_2}] = [\frac{N_1D_2+N_2D_1}{D_1D_2}]$ and $[\frac{N_1}{D_1}][\frac{N_2}{D_2}] = [\frac{N_1N_2}{D_1D_2}]$. $\mathbb{F}(R)$ is then a field with these operations.

Matrices with entries in $\mathbb{F}(A_{\mathbb{R}})$ will be denoted by $\mathrm{Mat}(\mathbb{F}(A_{\mathbb{R}}))$. If $P \in \mathrm{Mat}(\mathbb{F}(A_{\mathbb{R}}))$, then P is said to have a *right coprime factorization* if there exists a pair (N, D) with $N, D \in \mathrm{Mat}(A_{\mathbb{R}})$ such that D is a square matrix, $\det(D) \neq 0$, $P = ND^{-1}$, and (N, D) is right coprime. A *left coprime factorization* is defined analogously. A transfer function having a right coprime factorization and a left coprime factorization is said to have a *doubly coprime factorization.*

It can be shown that in the case of $A_{\mathbb{R}}$, a plant P has a right coprime factorization iff it has a left coprime factorization. This is a consequence of the fact that the ring $A_{\mathbb{R}}$ is Hermite; see [9] and [15, Theorem 66, p. 347].

Coprime factorization plays an important role in stabilizing a plant using a factorization approach, where by ‘stabilization’, we mean the following.

Definition 3.3 Let $P, C \in \text{Mat}(\mathbb{F}(A_{\mathbb{R}}))$. The pair (P, C) is said to be *stable* if

$$\mathcal{H}(P,C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+PC)^{-1} \\ C(I+PC)^{-1} & (I+PC)^{-1} \end{bmatrix} \tag{2}$$

is well defined, and belongs to $\text{Mat}(A_{\mathbb{R}})$. We define

$$\mathcal{S}(P) = \{C \in \text{Mat}(\mathbb{F}(A_{\mathbb{R}})) \mid (P, C) \text{ is a stable pair}\}.$$

$P \in \mathbb{F}(A_{\mathbb{R}})^{p\times m}$ is said to be *stabilizable* if $\mathcal{S}(P) \neq \emptyset$.

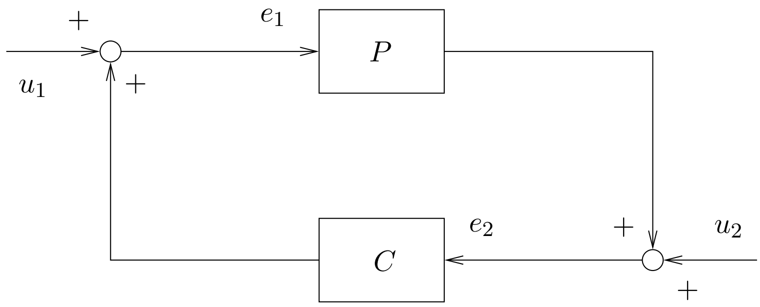

Figure 1. Closed loop interconnection of the plant P and the controller C.

As shown in Figure 1, $\mathcal{H}(P,C)$ in (2) is the transfer function of

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The stabilization problem for a plant is solved completely once a transfer function has a doubly coprime factorization. We recall the following well-known result from Vidyasagar [15, Theorem 12, p. 364]:

Theorem 3.1 *Let $P \in \text{Mat}(\mathbb{F}(A_{\mathbb{R}}))$ have a right coprime factorization (N_r, D_r) and a left coprime factorization (D_l, N_l). Let $X_r, Y_r, X_l, Y_l \in \text{Mat}(A_{\mathbb{R}})$ be such that $X_rN_r + Y_rD_r = I$ and $N_lX_l + D_lY_l = I$. Then*

$$\begin{aligned} \mathcal{S}(P) &= \{(Y_r - QN_l)^{-1}(X_r + QD_l) \mid Q \in \text{Mat}(A_{\mathbb{R}}),\ \det(Y_r - QN_l) \neq 0\} \\ &= \{(X_l + D_rQ)(Y_l - N_rQ)^{-1} \mid Q \in \text{Mat}(A_{\mathbb{R}}),\ \det(Y_l - N_rQ) \neq 0\}. \end{aligned}$$

We now recall the result of A. Quadrat, see [11, Proposition 7.4, p. 2281]:

Theorem 3.2 *If $(R, \|\cdot\|)$ is a Banach algebra such that $\text{tsr}(R) = 2$, then for every single input single output plant defined by the transfer function $p = n/d$, $n \in R$, $0 \neq d \in R$, and for given $\epsilon > 0$ there exist $n_\epsilon, d_\epsilon \in R$ such that*

$$\|n_\epsilon - n\| < \epsilon, \quad \|d_\epsilon - d\| < \epsilon,$$

and $p_\epsilon := \frac{n_\epsilon}{d_\epsilon}$ admits a coprime factorization.

Theorem 2.1 now implies that this approximation property holds for the real algebras $A_{\mathbb{R}}$ and $W^+_{\mathbb{R}}$, whereas the result by Mortini and Wick implies it for $H^\infty_{\mathbb{R}}$.

Not every $P \in \mathrm{Mat}(\mathbb{F}(A_{\mathbb{R}}))$ has a coprime factorization; see page 249 of [8], where an explicit construction of a finitely generated, non-principal ideal is given in terms of Blaschke products; this has the consequence that the ring $A_{\mathbb{R}}$ is not a Bézout domain, which implies that not every element in $\mathbb{F}(A_{\mathbb{R}})$ has a coprime factorization.

However, Theorem 2.2 rescues this undesirable situation in the following sense: even if the given “system” G does not have a coprime factorization, it can be replaced by a new system G_ϵ having a coprime factorization $G_\epsilon = N_\epsilon D_\epsilon^{-1}$, and the new system G_ϵ can be chosen to be arbitrarily “close” to G.

Corollary 3.3 *Let $P = ND^{-1} \in \mathrm{Mat}(\mathbb{F}(A_{\mathbb{R}}))$, with $N, D \in \mathrm{Mat}(A_{\mathbb{R}})$ and $\det D \neq 0$. Given any $\epsilon > 0$, there exist $N_\epsilon \in \mathrm{Mat}(A_{\mathbb{R}})$ and $D_\epsilon \in \mathrm{Mat}(A_{\mathbb{R}})$ such that $\det D_\epsilon \neq 0$, $\|N - N_\epsilon\| < \epsilon$ and $\|D - D_\epsilon\| < \epsilon$, and moreover (N_ϵ, D_ϵ) is right coprime.*

Again this holds also for $W^+_{\mathbb{R}}$, see Remark 2.1.

4. Open Problem

Does Corollary 3.3 hold in case of $H^\infty_{\mathbb{R}}$?

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