AN ABSTRACT NYQUIST CRITERION CONTAINING OLD AND NEW RESULTS

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Abstract. We prove an abstract Nyquist criterion in a general set up. As applications, we recover various versions of the Nyquist criterion, some of which are new.

1. Introduction

Harry Nyquist, in his fundamental paper [17], gave a criterion for the stability of a feedback system, which is one of the basic tools in the frequency domain approach to feedback control. This test, which is expressed in terms of the winding number around zero of a certain curve in the complex plane, is well known for finite dimensional systems; see for example [26] or Theorem 5.2 in this article. There are several extensions of this test for other classes of systems as well; see for example [3], [5], [6]. Thus the problem of obtaining a Nyquist criterion encompassing the different transfer function classes of systems is a natural one; see [14], [19, p.65].

In this article, we will prove an "abstract Nyquist theorem", where we only start with a commutative ring R (thought of as the class of stable transfer functions of a linear control system) possessing certain properties, and then give a criterion for the stability of a closed loop feedback system formed by a plant and a controller (which have transfer functions that are matrices with entries from the field of fractions of R). We then specialize R to several classes of stable transfer functions and obtain various versions of the Nyquist criterion. In the section on applications, we have given references to the known results; all other results seem to be new.

The article is organized as follows:

(1) In Section 2, we describe the basic objects in our abstract set up in which we will prove our abstract Nyquist criterion. The starting point will be a commutative ring R. We will also give a systematic procedure to build the other basic objects starting from R in teh case when R is a Banach algebra.
(2) In section 3, we will recall the standard definitions from the factorization approach to feedback control theory.
(3) In Section 4, we prove our main result, the abstract Nyquist criterion, in Theorem 4.1.

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(4) Finally in various subsections of Section 5, we recover some old versions of the Nyquist criterion as well as obtain new ones, as special instances of our abstract Nyquist criterion.

2. General setup and assumptions

Our set up is a triple (R, S, ι), satisfying the following:

(A1) R be a unital commutative ring.

(A2) S is a unital commutative Banach algebra such that $R \subset S$. The invertible elements of S will be denoted by inv S.

(A3) There exists a map $\iota : \text{inv } S \to G$, where $(G, \star)$ is an Abelian group with identity denoted by $\circ$, and ι satisfies

$$\iota(ab) = \iota(a) \star \iota(b) \quad (a, b \in \text{inv } S).$$

The function ι will be called an *abstract index*.

(A4) $x \in R \cap (\text{inv } S)$ is invertible as an element of R iff $\iota(x) = \circ$.

Typically, one has R available. So the natural question which arises is: How does one find S and ι that satisfy (A1)-(A4)? We outline a systematic procedure for doing this below when R is a commutative unital complex Banach algebra (or more generally a full subring of such a Banach algebra; the definition of a full subring is recalled below).

Definition 2.1. Let R_1, R_2 be commutative unital rings, and let R_1 be a subring of R_2. Then R_1 is said to be a *full* subring of R_2 if for every $x \in R_1$ such that x is invertible in R_2, there holds that x is invertible in R_1.

2.1. **A choice of ι.** If $\exp S$ denotes the connected component in inv S which contains the identity element of S, then we can take G as the (discrete) group $(\text{inv } S)/(\exp S)$, and ι can be taken to be the natural homomorphism ι_S from inv S to $(\text{inv } S)/(\exp S)$. Then (A3) holds; see [7, Proposition 2.9].

2.2. **A choice of S.** On the other hand, one possible construction of an S is as follows. First we recall a definition from [15].

Definition 2.2. Let X_R denote the maximal ideal space of a unital commutative Banach algebra R. A closed subset $Y \subset X_R$ is said to satisfy the *generalized argument principle* for R if whenever $a \in R$ and $\log \widehat{a}$ is defined continuously on Y, then a is invertible in R. (Here $\widehat{a}$ denotes the Gelfand transform of a, Y is equipped with the topology it inherits from X_R and X_R has the usual Gelfand topology).

It was shown in [15, Theorem 2.2] that any Y satisfying the generalized argument principle is a boundary for R and so it contains the Šilov boundary of R. Moreover, given any R, there always exists a minimal closed set Y_R of X_R which satisfies the generalized argument principle for R [15, Theorem 2.7].

So if we know a set $Y \subset X_R$ that satisfies the generalized argument principle for R, then one can take S to be equal to $S_Y := C(Y)$. The topology on $C(Y)$ is the one given by the supremum norm.

Lemma 2.3. *Let R be a commutative unital complex Banach algebra, and let $Y \subset X_R$ satisfy the generalized argument principle for R. Let $S := S_Y$ and $\iota := \iota_{S_Y}$ be as described in the previous two subsections. Let $f \in \text{inv } S$. Then f has a continuous logarithm iff $\iota(f) = \circ$. In particular the triple (R, S, ι) satisfies* (A1)-(A3) *and the 'if' part of* (A4).

Proof. Suppose that f has a continuous logarithm. Then $f = e^g$ for some $g \in C(S)$. But then by the definition of ι, $\iota(f) = \circ$.

Conversely, suppose that $\iota(f) = \circ$. This means that $f = e^g$ for some $g \in C(S)$. Hence f has a continuous logarithm.

(A1) is trivial. Given $f \in R$, we see that $\widehat{f}|_Y \in C(Y)$. Moreover the map $f \mapsto \widehat{f}|_Y$ is one-to-one since Y contains the Šilov boundary of R. Indeed if $\widehat{f}|_Y = 0$, then we have

$$\max_{\varphi \in X_R} |\widehat{f}(\varphi)| = \max_{\varphi \in Y} |\widehat{f}(\varphi)| = 0,$$

and so $\widehat{f} \equiv 0$, that is $f = 0$. Hence (A2) holds as well. (A3) follows from the definition of ι. Finally we show (A4) below.

Suppose that $f \in R \cap \text{inv } S$. If $\iota(f) = \circ$, then we know that f has a continuous logarithm on Y. But Y satisfies the generalized argument principle for R. Thus f is invertible as an element of R. $\square$

For the 'only if' part, we will need a stronger property on Y than the generalized argument principle.

Definition 2.4. A closed subset $Y \subset X_R$ is said to satisfy the *strong generalized argument principle* for R if $a \in R$ is invertible as an element in R iff $\log \widehat{a}$ is defined continuously on Y.

Lemma 2.5. *Let R be a commutative unital complex Banach algebra, and let $Y \subset X_R$ satisfy the strong generalized argument principle for R. Let $S := S_Y$ and $\iota := \iota_{S_Y}$ be as described in the previous subsection. Then the triple (R, S, ι) satisfies* (A1)-(A4).

Proof. (A1)-(A3) and the 'if' part of (A4) have been verified already in Lemma 2.3. We just verify the 'only if' part of (A4). So suppose that $f \in R \cap \text{inv } C(Y)$ and that f is invertible as an element of R. Then f has a continuous logarithm on Y, and so $\iota(f) = \circ$, again by Lemma 2.3. $\square$

In Subsection 5.1 and 5.2, in the case of the disk algebra $A(\mathbb{D})$ and the analytic almost periodic algebra AP^+, we will see that our choices of S and ι are precisely of the type described above.

3. Feedback stabilization

We recall the following definitions from the factorization approach to control theory.

Definition 3.1. The field of fractions of R will be denoted by $\mathbb{F}(R)$. Let $P \in (\mathbb{F}(R))^{p\times m}$ and let $P = ND^{-1}$, where N, D are matrices with entries from R. Here D^{-1} denotes a matrix with entries from $\mathbb{F}(R)$ such that $DD^{-1} = D^{-1}D = I$. The factorization $P = ND^{-1}$ is called a *right coprime factorization of* P if there exist matrices X, Y with entries from R such that $XN + YD = I_m$. Similarly, a factorization $P = \widetilde{D}^{-1}\widetilde{N}$, where $\widetilde{N}, \widetilde{D}$ are matrices with entries from R, is called a *left coprime factorization of* P if there exist matrices $\widetilde{X}, \widetilde{Y}$ with entries from R such that $\widetilde{N}\widetilde{X} + \widetilde{D}\widetilde{Y} = I_p$. Given $P \in (\mathbb{F}(R))^{p\times m}$ with right and left factorizations

$$P = ND^{-1} \quad \text{and} \quad P = \widetilde{D}^{-1}\widetilde{N},$$

respectively, we introduce the following matrices with entries from R:

$$G_P = \begin{bmatrix} N \\ D \end{bmatrix} \quad \text{and} \quad \widetilde{G}_P = \begin{bmatrix} -\widetilde{N} & \widetilde{D} \end{bmatrix}.$$

We denote by $\mathbb{S}(R, p, m)$ the set of all $P \in (\mathbb{F}(R))^{p\times m}$ that possess a right coprime factorization and a left coprime factorization.

Given $P \in (\mathbb{F}(R))^{p\times m}$ and $C \in (\mathbb{F}(R))^{m\times p}$, define the *closed loop transfer function*

$$H(P, C) := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \in (\mathbb{F}(R))^{(p+m)\times(p+m)}.$$

C is said to *stabilize* P if $H(P, C) \in R^{(p+m)\times(p+m)}$, and P is called *stabilizable* if $\{C \in (\mathbb{F}(R))^{m\times p} : H(P, C) \in R^{(p+m)\times(p+m)}\} \neq \emptyset$. If $P \in \mathbb{S}(R, p, m)$, then P is a *stabilizable*; see for example [26, Chapter 8]. Thus

$$\mathbb{S}(R, p, m) = \left\{ P \in (\mathbb{F}(R))^{p\times m} \;\middle|\; \begin{array}{l} \exists C \in (\mathbb{F}(R))^{m\times p} \text{ such that} \\ H(P, C) \in R^{(p+m)\times(p+m)} \end{array} \right\}.$$

It was shown in [18, Theorem 6.3] that if the ring R is projective free, then every stabilizable P admits a right coprime factorization and a left coprime factorization.

We will use the following in order to prove our main result in the next section.

Lemma 3.2. *Suppose that* $F \in R^{m\times m}$. *Then* F *is invertible as an element of* $R^{m\times m}$ *iff* $\det F \in \operatorname{inv} S$ *and* $\iota(\det F) = \circ$.

Proof. Using Cramer's rule, we see that F is invertible as an element of $R^{m\times m}$ iff $\det F$ is invertible as an element of R. The result now follows from (A4). $\square$

4. Abstract Nyquist criterion

Theorem 4.1. *Let* (A1)-(A4) *hold. Suppose that* $P \in \mathbb{S}(R, p, m)$ *and that* $C \in \mathbb{S}(R, m, p)$. *Moreover, let* $P = N_P D_P^{-1}$ *be a right coprime factorization of* P, *and let* $C = \widetilde{D}_C^{-1}\widetilde{N}_C$ *be a left coprime factorization of* C. *Then the following are equivalent:*

(1) *C stabilizes P.*
(2) (a) $\det(I-CP), \det D_P, \det \widetilde{D}_C \in \text{inv } S$ *and*
(b) $\iota(\det(I-CP)) \star \iota(\det D_P) \star \iota(\det \widetilde{D}_C) = \circ$.

Proof. We note that

$$\begin{aligned} H(P,C) &= \begin{bmatrix} P \\ I \end{bmatrix} (I-CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \\ &= \begin{bmatrix} N_P D_P^{-1} \\ I \end{bmatrix} (I - \widetilde{D}_C^{-1}\widetilde{N}_C N_P D_P^{-1})^{-1} \begin{bmatrix} -\widetilde{D}_C^{-1}\widetilde{N}_C & I \end{bmatrix} \\ &= \begin{bmatrix} N_P \\ D_P \end{bmatrix} (\widetilde{D}_C D_P - \widetilde{N}_C N_P)^{-1} \begin{bmatrix} -\widetilde{N}_C & \widetilde{D}_C \end{bmatrix} \\ &= G_P(\widetilde{G}_C G_P)^{-1}\widetilde{G}_C. \end{aligned}$$

So if $(\widetilde{G}_C G_P)^{-1} \in R^{p\times p}$, then $H(P,C) \in R^{(p+m)\times(p+m)}$. Conversely, using the fact that there exist matrices Θ and $\widetilde{\Theta}$ with R entries such that $\Theta G_P = I$ and $\widetilde{G}_C\widetilde{\Theta} = I$, it follows from the above that if $H(P,C) \in R^{(p+m)\times(p+m)}$, then $(\widetilde{G}_C G_P)^{-1} \in R^{p\times p}$. So C stabilizes P iff $(\widetilde{G}_C G_P)^{-1} \in R^{p\times p}$. We will use this fact below.

(1)⇒(2): Suppose that C stabilizes P. Then $(\widetilde{G}_C G_P)^{-1} \in R^{p\times p}$. So $\det(\widetilde{G}_C G_P)$ is invertible as an element of R. By (A4), it follows that $\det(\widetilde{G}_C G_P)$ is invertible as an element of S and $\iota(\det(\widetilde{G}_C G_P)) = \circ$. But

$$\widetilde{G}_C G_P = \widetilde{D}_C D_P - \widetilde{N}_C N_P = \widetilde{D}_C(I-CP)D_P.$$

Thus $\det(\widetilde{G}_C G_P) = (\det \widetilde{D}_C)\cdot(\det(I-CP))\cdot(\det D_P)$ and so $(\det \widetilde{D}_C)\cdot(\det(I-CP))\cdot(\det D_P) \in \text{inv } S$. Hence $\det \widetilde{D}_C$, $\det(I-CP)$, $\det D_P$ are each invertible elements of S. From (A3) we obtain

$$\circ = \iota(\det(\widetilde{G}_C G_P)) = \iota(\det \widetilde{D}_C) \star \iota(\det(I-CP)) \star \iota(\det D_P).$$

(2)⇒(1): Suppose that $\det(I-CP), \det D_P, \det \widetilde{D}_C \in \text{inv } S$ and that

$$\iota(\det(I-CP)) \star \iota(\det D_P) \star \iota(\det \widetilde{D}_C) = \circ.$$

Then retracing the above steps in the reverse order, we see that $\det(\widetilde{G}_C G_P)$ is invertible in S, and moreover,

$$\iota(\det(\widetilde{G}_C G_P)) = \iota(\det \widetilde{D}_C) \star \iota(\det(I-CP)) \star \iota(\det D_P) = \circ.$$

From (A4) it follows that $\det(\widetilde{G}_C G_P)$ is invertible as an element of R. Thus $\widetilde{G}_C G_P$ is invertible as an element of $R^{p\times p}$. Consequently C stabilizes P. □

5. Applications

Now we specialize R to several classes of stable transfer functions and obtain various versions of the Nyquist criterion. In particular, we begin with Subsection 5.1, where we recover the classical Nyquist criterion.

5.1. **The disk algebra.** Let

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}, \quad \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

The *disk algebra* $A(\mathbb{D})$ is the set of all functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ such that f is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Let $C(\mathbb{T})$ denote the set of complex-valued continuous functions on the unit circle $\mathbb{T}$. For each $f \in \text{inv } C(\mathbb{T})$, we can define the *winding number* $\mathtt{w}(f) \in \mathbb{Z}$ of f as follows:

$$\mathtt{w}(f) = \frac{1}{2\pi}(\Theta(2\pi) - \Theta(0)),$$

where $\Theta : [0, 2\pi] \to \mathbb{R}$ is a continuous function such that

$$f(e^{it}) = |f(e^{it})|e^{i\Theta(t)}, \quad t \in [0, 2\pi].$$

The existence of such a Θ can be proved; see [24, Lemma 4.6]. Also, it can be checked that $\mathtt{w}$ is well-defined and integer-valued. Geometrically, $\mathtt{w}(f)$ is the number of times the curve $t \mapsto f(e^{it}) : [0, 2\pi] \to \mathbb{C}$ winds around the origin in a counterclockwise direction. Also, [24, Lemma 4.6.(ii)] shows that the map $\mathtt{w} : \text{inv } C(\mathbb{T}) \to \mathbb{R}$ is locally constant. Here the local constancy of $\mathtt{w}$ means continuity relative to the discrete topology on $\mathbb{R}$, while $C(\mathbb{T})$ is equipped with the usual sup-norm.

Lemma 5.1. *Let*

$$\begin{aligned} R &= \text{a unital full subring of } A(\mathbb{D}), \\ S &:= C(\mathbb{T}), \\ G &:= \mathbb{Z}, \\ \iota &:= \mathtt{w}. \end{aligned}$$

Then (A1)-(A4) *are satisfied.*

Proof. (A1) and (A2) are clear. (A3) is evident from the definition of $\mathtt{w}$. Finally, we will show below that (A4) holds.

Suppose that $f \in R \cap (\text{inv } C(\mathbb{T}))$ is invertible as an element of R. Then obviously f is also invertible as an element of $A(\mathbb{D})$. Hence it has no zeros or poles in $\overline{\mathbb{D}}$. For $r \in (0, 1)$, define $f_r \in A(\mathbb{D})$ by $f_r(z) = f(rz)$ $(z \in \overline{\mathbb{D}})$. Then f_r also has no zeros or poles in $\overline{\mathbb{D}}$, and has a holomorphic extension across $\mathbb{T}$. From the Argument Principle (applied to f_r), it follows that $\mathtt{w}(f_r) = 0$. But $\|f_r - f\|_\infty \to 0$ as $r \nearrow 1$. Hence $\mathtt{w}(f) = \lim_{r\to1} \mathtt{w}(f_r) = \lim_{r\to1} 0 = 0$.

Suppose, conversely, that $f \in R \cap (\text{inv } C(\mathbb{T}))$ is such that $\mathtt{w}(f) = 0$. For all $r \in (0, 1)$ sufficiently close to 1, we have that $f_r \in \text{inv } C(\mathbb{T})$. Also, by the local constancy of $\mathtt{w}$, for r sufficiently close to 1, $\mathtt{w}(f_r) = \mathtt{w}(f) = 0$. By the Argument principle, it then follows that f_r has no zeros in $\overline{\mathbb{D}}$. Equivalently, f has no zeros in $r\overline{\mathbb{D}}$. But letting $r \nearrow 1$, we see that f has no zeros in $\mathbb{D}$. Moreover, f has no zeros on $\mathbb{T}$ either, since $f \in \text{inv } C(\mathbb{T})$. Thus f has no zeros in $\overline{\mathbb{D}}$. Consequently, we conclude that f is invertible as an element of $A(\mathbb{D})$. (Indeed, f is invertible as an element of $C(\overline{\mathbb{D}}$, and it is also then clear that this inverse is holomorphic in $\mathbb{D}$.) Finally, since R is a full subring of $A(\mathbb{D})$, we can conclude that f is invertible also as an element of R. □

Besides $A(\mathbb{D})$ itself, some other examples of such R are:

(1) $\mathcal{P}$, the set of all polynomial functions in the variable $z \in \mathbb{C}$.
(2) $RH^\infty(\mathbb{D})$, the set of all rational functions without poles in $\overline{\mathbb{D}}$.
(3) The Wiener algebra $W^+(\mathbb{D})$ of all functions $f \in A(\mathbb{D})$ that have an absolutely convergent Taylor series about the origin:
$$\sum_{n=0}^{\infty} |f_n| < +\infty, \text{ where } f(z) = \sum_{n=0}^{\infty} f_n z^n \ (z \in \mathbb{D}).$$
(4) $\partial^{-n} H^\infty(\mathbb{D})$, the set of $f : \mathbb{D} \to \mathbb{C}$ such that $f, f^{(1)}, f^{(2)}, \dots, f^{(n)}$ belong to $H^\infty(\mathbb{D})$. Here $H^\infty(\mathbb{D})$ denotes the Hardy algebra of all bounded and holomorphic functions on $\mathbb{D}$.

An application of our main result (Theorem 4.1) yields the following Nyquist criterion. We note that invertibility of f in $C(\mathbb{T})$ just means that f belongs to $C(\mathbb{T})$ and it has no zeros on $\mathbb{T}$.

Corollary 5.2. *Let R be a unital full subring of $A(\mathbb{D})$. Let $P \in \mathbb{S}(R, p, m)$ and $C \in \mathbb{S}(R, m, p)$. Moreover, let $P = N_P D_P^{-1}$ be a right coprime factorization of P, and $C = \widetilde{D}_C^{-1} \widetilde{N}_C$ be a left coprime factorization of C. Then the following are equivalent:*

(1) *C stabilizes P.*
(2) (a) $\det(I - CP)$ *belongs to* $C(\mathbb{T})$*,*
 (b) $\det(I - CP)$*,* $\det D_P$*,* $\det \widetilde{D}_C$ *have no zeros on* $\mathbb{T}$*, and*
 (c) $\mathtt{w}(\det(I - CP)) + \mathtt{w}(\det D_P) + \mathtt{w}(\det \widetilde{D}_C) = 0$*.*

It can be shown that $Y = \mathbb{T}$ satisfies the generalized argument principle for $A(\mathbb{D})$; see [15, Corollary 1.25]. Moreover, we know that if a function in $A(\mathbb{D})$ is invertible, then by considering the map $r \mapsto f_r|_{\mathbb{T}} : [0, 1] \to \text{inv } C(\mathbb{T})$, we see that f belongs to the connected component of inv $C(\mathbb{T})$ that contains 1. So it is of the form $f|_{\mathbb{T}} = e^g$ for some $g \in C(\mathbb{T})$. Hence $f|_{\mathbb{T}}$ has a continuous logarithm on $\mathbb{T}$. So we can take $S = C(\mathbb{T})$. Moreover, if $\exp C(\mathbb{T})$ denotes the connected component in inv $C(\mathbb{T})$ which contains the constant function 1 on $\mathbb{T}$, then $G = (\text{inv } C(\mathbb{T})/(\exp C(\mathbb{T}))$ is isomorphic to $\mathbb{Z}$ (see for example [7, Corollary 2.20]), and ι can be taken as the the natural homomorphism from inv $C(\mathbb{T})$ to $\mathbb{Z}$ given by the winding number.

Remark 5.3. $\mathcal{P}$, $RH^\infty(\mathbb{D})$ are projective free rings since they are both Bezout domains. Also $A(\mathbb{D})$, $W^+(\mathbb{D})$, or $\partial^{-n} H^\infty(\mathbb{D})$ are projective free rings, since their maximal ideal space is $\overline{\mathbb{D}}$, which is contractible; see [1]. Thus if R is one of $\mathcal{P}$, $RH^\infty(\mathbb{D})$, $A(\mathbb{D})$, $W^+(\mathbb{D})$ or $\partial^{-n} H^\infty(\mathbb{D})$, then the set $\mathbb{S}(R, p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

The result in Corollary 5.2 was known in the special cases when R is $\mathcal{P}$, $RH^\infty(\mathbb{D})$ or $A(\mathbb{D})$; see [26].

5.2. **Almost periodic functions.** The algebra AP of complex valued (uniformly) *almost periodic functions* is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions $e_\lambda := e^{i\lambda y}$. Here the parameter λ belongs to $\mathbb{R}$. For any $f \in AP$, its *Bohr-Fourier series* is defined by the formal sum

$$\sum_\lambda f_\lambda e^{i\lambda y}, \quad y \in \mathbb{R}, \tag{1}$$

where

$$f_\lambda := \lim_{N\to\infty} \frac{1}{2N} \int_{[-N,N]} e^{-i\lambda y} f(y) dy, \quad \lambda \in \mathbb{R},$$

and the sum in (1) is taken over the set $\sigma(f) := \{\lambda \in \mathbb{R} \mid f_\lambda \neq 0\}$, called the *Bohr-Fourier spectrum* of f. The Bohr-Fourier spectrum of every $f \in AP$ is at most a countable set.

The *almost periodic Wiener algebra* APW is defined as the set of all AP such that the Bohr-Fourier series (1) of f converges absolutely. The almost periodic Wiener algebra is a Banach algebra with pointwise operations and the norm $\|f\| := \sum_{\lambda\in\mathbb{R}} |f_\lambda|$. Set

$$\begin{aligned} AP^+ &= \{f \in AP \mid \sigma(f) \subset [0,\infty)\} \\ APW^+ &= \{f \in APW \mid \sigma(f) \subset [0,\infty)\}. \end{aligned}$$

Then AP^+ (respectively APW^+) is a Banach subalgebra of AP (respectively APW). For each $f \in \text{inv } AP$, we can define the *average winding number* $w(f) \in \mathbb{R}$ of f as follows:

$$w(f) = \lim_{T\to\infty} \frac{1}{2T} \bigg(\arg(f(T)) - \arg(f(-T)) \bigg).$$

See [13, Theorem 1, p. 167].

Lemma 5.4. *Let*

$$\begin{aligned} R &:= \text{a unital full subring of } AP^+ \\ S &:= AP, \\ G &:= \mathbb{R}, \\ \iota &:= w. \end{aligned}$$

Then (A1)-(A4) *are satisfied.*

Proof. (A1) and (A2) are clear. (A3) follows from the definition of w. Finally, (A4) follows from [3, Theorem 1, p.776] which says that $f \in AP^+$ satisfies

$$\inf_{\text{Im}(s)\geq 0} |f(s)| > 0 \tag{2}$$

iff $\inf_{y\in\mathbb{R}} |f(y)| > 0$ and $w(f) = 0$. But

$$\inf_{y\in\mathbb{R}} |f(y)| > 0$$

is equivalent to f being an invertible element of AP by the corona theorem for AP (see for example [9, Exercise 18, p.24]). Also the equivalence of (2) with that of the invertibility of f as an element of AP^+ follows from the Arens-Singer corona theorem for AP^+ (see for example [2, Theorems 3.1, 4.3]). Finally, the invertibility of $f \in R$ in R is equivalent to the invertibility of f as an element of AP^+ since R is a full subring of AP^+. $\square$

Remark 5.5. Specific examples of such R are AP^+ and APW^+. More generally, let $\Sigma \subset [0, +\infty)$ be an *additive semigroup* (if $\lambda, \mu \in \Sigma$, then $\lambda + \mu \in \Sigma$) and suppose $0 \in \Sigma$. Denote

$$\begin{aligned} AP_\Sigma &= \{f \in AP \mid \sigma(f) \subset \Sigma\} \\ APW_\Sigma &= \{f \in APW \mid \sigma(f) \subset \Sigma\}. \end{aligned}$$

Then AP_Σ (respectively APW_Σ) is a unital Banach subalgebra of AP^+ (respectively APW^+). Let $\overline{Y_\Sigma}$ denote the set of all maps $\theta : \Sigma \to [0, +\infty]$ such that $\theta(0) = 0$ and $\theta(\lambda + \mu) = \theta(\lambda) + \theta(\mu)$ for all $\lambda, \mu \in \Sigma$. Examples of such maps θ are the following. If $y \in [0, +\infty)$, then θ_y, defined by $\theta_y(\lambda) = \lambda y$, $\lambda \in \Sigma$, belongs to $\overline{Y_\Sigma}$. Another example is θ_∞, defined as follows:

$$\theta_\infty(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0, \\ +\infty & \text{if } \lambda \neq 0. \end{cases}$$

So in this way we can consider $[0, +\infty]$ as a subset of $\overline{Y_\Sigma}$.

The results [2, Proposition 4.2, Theorem 4.3] say that if $\overline{Y_\Sigma} \subset [0, +\infty]$, and $f \in AP_\Sigma$ (respectively APW_Σ), then $f \in \operatorname{inv} AP_\Sigma$ (respectively $\in \operatorname{inv} APW_\Sigma$) iff (2) holds. So in this case AP_Σ and APW_Σ are unital full subalgebras of AP^+.

An application of our main result (Theorem 4.1) yields the following Nyquist criterion. We note that invertibility of f in AP just means that f belongs to AP and is bounded away from zero on $\mathbb{R}$ again by the corona theorem for AP.

Corollary 5.6. *Let R be a unital full subring of AP^+. Let $P \in \mathbb{S}(R, p, m)$ and $C \in \mathbb{S}(R, m, p)$. Moreover, let $P = N_P D_P^{-1}$ be a right coprime factorization of P, and $C = \widetilde{D}_C^{-1} \widetilde{N}_C$ be a left coprime factorization of C. Then the following are equivalent:*

(1) *C stabilizes P.*

(2) (a) $\det(I - CP)$ *belongs to AP,*

 (b) $\det(I - CP)$, $\det D_P$, $\det \widetilde{D}_C$ *are bounded away from 0 on $\mathbb{R}$,*

 (c) $w(\det(I - CP)) + w(\det D_P) + w(\det \widetilde{D}_C) = 0$.

Finally, in the case of the analytic almost periodic algebra AP^+, we show below that the choices of S and ι are precisely of the type described in Subsections 2.1 and 2.2. Let $\mathbb{R}_B$ denote the Bohr compactification of $\mathbb{R}$. Then X_{AP^+} contains a copy of $\mathbb{R}_B$ (since $X_{AP} = \mathbb{R}_B$, and $AP^+ \subset AP$), and we show below that $Y := \mathbb{R}_B$ satisfies the strong generalized argument principle for AP^+. Thus we can take $S = C(\mathbb{R}_B) = AP$, and we will also show that the ι_{AP} coincides with the average winding number defined above.

Lemma 5.7. $\mathbb{R}_B$ *satisfies the strong generalized argument principle for* AP^+.

Proof. First of all, suppose that $f \in AP^+$ has a continuous logarithm on $\mathbb{R}_B$. Then $f = e^g$ for some $g \in C(\mathbb{R}_B) = AP$. But then since $g \in AP$, we have that $\operatorname{Im}(g)$ is bounded on $\mathbb{R}$.

$$\begin{aligned} w(f) &= \lim_{T\to\infty} \frac{1}{2R}\Big(\arg(f(T)) - \arg(f(-T))\Big) \\ &= \lim_{T\to\infty} \frac{1}{2T}\Big(\operatorname{Im}(g(T)) - \operatorname{Im}(g(-T))\Big) = 0. \end{aligned}$$

But by (A4) (shown in Lemma 5.4), it follows that f is invertible as an element of AP^+.

Conversely, suppose that

$$f = \sum_{n=1}^{\infty} f_n e^{i\lambda_n \cdot}$$

is invertible as an element of AP^+. Consider the map $\Phi : [0, 1] \to \operatorname{inv} AP$ given by $\Phi(t) = f(\cdot - i\log(1 - t))$ if $t \in [0, 1)$ and $\Phi(1) = f_0$. Thus $\widehat{f}|_{\mathbb{R}_B}$ belongs to the connected component of $\operatorname{inv} AP$ that contains the constant function 1. Hence $\widehat{f}|_{\mathbb{R}_B} = e^g$ for some $g \in C(\mathbb{R}_B)$. This shows that $\widehat{f}$ has a continuous logarithm on $\mathbb{R}_B$. $\square$

Moreover, ι_{AP} coincides with the average winding number. Indeed, the result [13, Theorem 1, p. 167] says that if $f \in \operatorname{inv} AP$, then there exists a $g \in AP$ such that $\arg f(t) = w(f)t + g(t)$ $(t \in \mathbb{R})$. Hence

$$f = |f|e^{i(w(f)t+g)} = e^{\log|f|+i(w(f)t+g)} = e^{\log|f|+ig}e^{iw(f)t}.$$

Since $\log|f| + ig \in AP$, it follows that $\iota_{AP}(f) = \iota_{AP}(e^{iw(f)t})$. But now with the association $\iota_{AP}(e^{iw(f)t}) \leftrightarrow w(f)$, we see that the maps ι_{AP} and w are the same.

So AP and w are precisely S_Y and $\iota_{C(Y)}$, respectively, described in Subsections 2.1 and 2.2 when $Y = \mathbb{R}_B$.

Remark 5.8. It was shown in [1] that AP^+ and APW^+ are projective free rings. Thus if $R = AP^+$ or APW^+, then the set $\mathbb{S}(R, p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

Corollary 5.6 was known in the special case when $R = APW^+$; see [3].

5.3. **Algebras of Laplace transforms of measures without a singular nonatomic part.** Let $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 0\}$ and let $\mathcal{A}^+$ denote the Banach algebra

$$\mathcal{A}^+ = \left\{ s(\in \mathbb{C}_+) \mapsto \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \;\middle|\; \begin{array}{l} f_a \in L^1(0,\infty),\ (f_k)_{k\geq 0} \in \ell^1, \\ 0 = t_0 < t_1, t_2, t_3, \dots \end{array} \right\}$$

equipped with pointwise operations and the norm:

$$\|F\| = \|f_a\|_{L^1} + \|(f_k)_{k\geq 0}\|_{\ell^1}, \quad F(s) = \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in \mathbb{C}_+).$$

Here $\widehat{f_a}$ denotes the *Laplace transform of* f_a, given by

$$\widehat{f_a}(s) = \int_0^\infty e^{-st} f_a(t)dt, \quad s \in \mathbb{C}_+.$$

Similarly, define the Banach algebra $\mathcal{A}$ as follows ([11]):

$$\mathcal{A} = \left\{ iy(\in i\mathbb{R}) \mapsto \widehat{f_a}(iy) + \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \;\middle|\; \begin{array}{l} f_a \in L^1(\mathbb{R}),\ (f_k)_{k\in\mathbb{Z}} \in \ell^1, \\ \dots, t_{-2}, t_{-1} < 0 = t_0 < t_1, t_2, \dots \end{array} \right\}$$

equipped with pointwise operations and the norm:

$$\|F\| = \|f_a\|_{L^1} + \|(f_k)_{k\in\mathbb{Z}}\|_{\ell^1}, \quad F(iy) := \widehat{f_a}(iy) + \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \quad (y \in \mathbb{R}).$$

Here $\widehat{f_a}$ is the *Fourier transform of* f_a, $\widehat{f_a}(iy) = \displaystyle\int_{-\infty}^{\infty} e^{-iyt} f_a(t)dt$, $(y \in \mathbb{R})$.

It can be shown that $\widehat{L^1(\mathbb{R})}$ is an ideal of $\mathcal{A}$.

For $F = \widehat{f_a} + \displaystyle\sum_{k=-\infty}^{\infty} f_k e^{-i\cdot t_k} \in \mathcal{A}$, we set $F_{AP}(iy) = \displaystyle\sum_{k=-\infty}^{\infty} f_k e^{-iyt_k}$ $(y \in \mathbb{R})$.

If $F = \widehat{f_a} + F_{AP} \in \text{inv } \mathcal{A}$, then it can be shown that $F_{AP}(i\cdot) \in \text{inv } AP$ as follows. First of all, the maximal ideal space of $\mathcal{A}$ contains a copy of the maximal ideal space of APW in the following manner: if $\varphi \in M(APW)$, then the map $\Phi : \mathcal{A} \to \mathbb{C}$ defined by $\Phi(F) = \Phi(\widehat{f_a} + F_{AP}) = \varphi(F_{AP}(i\cdot))$, $(F \in \mathcal{A})$, belongs to $M(\mathcal{A})$. So if F is invertible in $\mathcal{A}$, in particular for every Φ of the type describe above, $0 \neq \Phi(F) = \varphi(F_{AP}(i\cdot))$. Thus by the elementary theory of Banach algebras, $F_{AP}(i\cdot)$ is an invertible element of AP.

Moreover, since $\widehat{L^1(\mathbb{R})}$ is an ideal in $\mathcal{A}$, $F_{AP}^{-1}\widehat{f_a}$ is the Fourier transform of a function in $L^1(\mathbb{R})$, and so the map $y \mapsto 1 + (F_{AP}(iy))^{-1}\widehat{f_a}(iy) = \frac{F(iy)}{F_{AP}(iy)}$ has a well-defined winding number $\mathtt{w}$ around 0. Define $W : \text{inv } \mathcal{A} \to \mathbb{R} \times \mathbb{Z}$ by $W(F) = (w(F_{AP}), \mathtt{w}(1 + F_{AP}^{-1}\widehat{f_a}))$, where $F = \widehat{f_a} + F_{AP} \in \text{inv } \mathcal{A}$, and

$$\begin{aligned} w(F_{AP}) &:= \lim_{R\to\infty} \frac{1}{2R}\Big(\arg\big(F_{AP}(iR)\big) - \arg\big(F_{AP}(-iR)\big)\Big), \\ \mathtt{w}(1 + F_{AP}^{-1}\widehat{f_a}) &:= \frac{1}{2\pi}\Big(\arg\big(1 + (F_{AP}(iy))^{-1}\widehat{f_a}(iy)\big)\Big|_{y=-\infty}^{y=+\infty}\Big). \end{aligned}$$

Lemma 5.9. *$F = \widehat{f_a} + F_{AP} \in \mathcal{A}$ is invertible iff for all $y \in \mathbb{R}$, $F(iy) \neq 0$ and $\inf_{y\in\mathbb{R}} |F_{AP}(iy)| > 0$.*

Proof. The 'only if' part is clear. We simply show the 'if' part below.

Let $F = \widehat{f_a} + F_{AP} \in \mathcal{A}$ be such that

$$\inf_{y\in\mathbb{R}} |F_{AP}(iy)| > 0.$$

Thus $F(i\cdot)$ is invertible as an element of AP. Hence $F = F_{AP}(1 + \widehat{f_a}F_{AP}^{-1})$ and so it follows that $(1 + \widehat{f_a}F_{AP}^{-1})(iy) \neq 0$ for all $y \in \mathbb{R}$. But by the corona theorem for

$$\mathcal{W} := \widehat{L^1(\mathbb{R})} + \mathbb{C}$$

(see [10, Corollary 1, p.109]), it follows that $1 + \widehat{f_a}F_{AP}^{-1}$ is invertible as an element of $\mathcal{W}$ an in particular, also as an element of $\mathcal{A}$. This completes the proof. $\square$

Lemma 5.10. *Let*

$$\begin{array}{rcl} R & := & \text{a unital full subring of } \mathcal{A}^+, \\ S & := & \mathcal{A}, \\ G & := & \mathbb{R}\times\mathbb{Z}, \\ \iota & := & W. \end{array}$$

Then (A1)-(A4) *are satisfied.*

Proof. (A1) and (A2) are clear. (A3) follows from the definition of i as follows. Let $F = \widehat{f_a} + F_{AP}$ and $G = \widehat{g_a} + G_{AP}$. Then we have

$$w(F_{AP}G_{AP}) = w(F_{AP}) + w(G_{AP})$$

from the definition of w. Thus

$$\begin{array}{rcl} W(FG) & = & W((\widehat{f_a} + F_{AP})(\widehat{g_a} + G_{AP}) \\ & = & W(\widehat{f_a}\widehat{g_a} + \widehat{f_a}G_{AP} + \widehat{g_a}F_{AP} + F_{AP}G_{AP}) \\ & = & (\mathtt{w}(1 + (F_{AP}G_{AP})^{-1}(\widehat{f_a}\widehat{g_a} + \widehat{f_a}G_{AP} + \widehat{g_a}F_{AP}), w(F_{AP}G_{AP})) \\ & = & (\mathtt{w}((1 + F_{AP}^{-1}\widehat{f_a})(1 + G_{AP}^{-1}\widehat{g_a})), w(F_{AP}) + w(G_{AP})) \\ & = & (\mathtt{w}(1 + F_{AP}^{-1}\widehat{f_a}) + \mathtt{w}(1 + G_{AP}^{-1}\widehat{g_a}), w(F_{AP}) + w(G_{AP})) \\ & = & W(\widehat{f_a} + F_{AP}) + W(\widehat{g_a} + G_{AP}). \end{array}$$

So (A3) holds.

Finally we check that (A4) holds. Suppose that $F = \widehat{f_a} + F_{AP}$ belonging to $(\mathcal{A}^+) \cap (\text{inv } \mathcal{A})$, is such that $W(F) = 0$. Since F is invertible in $\mathcal{A}$, it follows that $F_{AP}(i\cdot)$ is invertible as an element of AP. But $w(F_{AP}) = 0$, and so $F_{AP}(i\cdot) \in AP^+$ is invertible as an element of AP^+. But this implies that $1 + F_{AP}^{-1}\widehat{f_a}$ belongs to the Banach algebra

$$\mathcal{W}^+ := \widehat{L^1(0,\infty)} + \mathbb{C}.$$

Moreover, it is bounded away from 0 on $i\mathbb{R}$ since

$$1 + F_{AP}^{-1}\widehat{f_a} = \frac{F}{F_{AP}},$$

and F is bounded away from zero on $i\mathbb{R}$. Moreover $\mathtt{w}(1 + F_{AP}^{-1}\widehat{f}_a) = 0$, and so it follows that $1 + F_{AP}^{-1}\widehat{f}_a$ is invertible as an element of $\mathcal{W}^+$, and in particular in $\mathcal{A}^+$. Since $F = (1 + F_{AP}^{-1}\widehat{f}_a)F_{AP}$ and we have shown that both $(1 + F_{AP}^{-1}\widehat{f}_a)$ as well as F_{AP} are invertible as elements of $\mathcal{A}^+$, it follows that F is invertible in $\mathcal{A}^+$. □

An example of such a R (besides $\mathcal{A}^+$) is the algebra

$$\widehat{L^1(0,+\infty)} + APW_\Sigma(i\cdot) := \{\widehat{f}_a + F_{AP} : f_a \in L^1(0,+\infty),\ F_{AP}(i\cdot) \in APW_\Sigma\},$$

where Σ is as described in Remark 5.5.

An application of our main result (Theorem 4.1) yields the following Nyquist criterion. We note that invertibility of f in $\mathcal{A}$ just means that $f \in \mathcal{A}$, it is nonzero on $i\mathbb{R}$ and the almost periodic part of f is bounded away from zero on $i\mathbb{R}$ by Lemma 5.9.

Corollary 5.11. *Let R be a unital full subring of $\mathcal{A}^+$. Let $P \in \mathbb{S}(R, p, m)$ and $C \in \mathbb{S}(R, m, p)$. Moreover, let $P = N_P D_P^{-1}$ be a right coprime factorization of P, and $C = \widetilde{D}_C^{-1}\widetilde{N}_C$ be a left coprime factorization of C. Then the following are equivalent:*

(1) *C stabilizes P.*
(2) (a) $\det(I - CP) \in \mathcal{A}$,
  (b) $\det(I - CP)$, $\det D_P$, $\det \widetilde{D}_C$ *are all nonzero on $i\mathbb{R}$ and their almost periodic parts are bounded away from zero on $i\mathbb{R}$, and*
  (c) $W(\det(I - CP)) + W(\det D_P) + W(\det \widetilde{D}_C) = (0, 0)$.

Remark 5.12. It was shown in [1] that $\mathcal{A}^+$ is a projective free ring. Thus the set $\mathbb{S}(\mathcal{A}^+, p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

Corollary 5.11 was known in the special case when $R = \mathcal{A}^+$; see [3].

5.4. **The complex Borel measure algebra.** Let $\mathcal{M}$ denote the set of all complex Borel measures on $\mathbb{R}$. Then $\mathcal{M}_+$ is a complex vector space with addition and scalar multiplication defined as usual, and it becomes a complex algebra if we take convolution of measures as the operation of multiplication. With the norm of μ taken as the total variation of μ, $\mathcal{M}$ is a Banach algebra. Recall that the *total variation* $\|\mu\|$ *of* μ is defined by

$$\|\mu\| = \sup \sum_{n=1}^{\infty} |\mu(E_n)|,$$

the supremum being taken over all *partitions of* $\mathbb{R}$, that is over all countable collections $(E_n)_{n\in\mathbb{N}}$ of Borel subsets of $\mathbb{R}$ such that $E_n \bigcap E_m = \emptyset$ whenever $m \neq n$ and $\mathbb{R} = \bigcup_{n\in\mathbb{N}} E_n$. Let $\mathcal{M}^+$ denote the Banach subalgebra of $\mathcal{M}$ consisting of all measures $\mu \in \mathcal{M}$ whose support is contained in the half-line $[0, +\infty)$. The following result was obtained in [23]:

Proposition 5.13. *If μ is an invertible measure in $\mathcal{M}$, then there exist an integer $n \in \mathbb{Z}$, a real number $c \in \mathbb{R}$ and a measure $\nu \in \mathcal{M}$ such that*

$$\mu = \rho^n * e^\nu * \delta_c.$$

Here δ_c denotes the Dirac measure supported at c. The measure ρ is given by $d\rho(t) = d\delta_0(t) + 2\mathbf{1}_{[0,\infty)}(t)e^{-t}dt$, where $\mathbf{1}_{[0,+\infty)}$ is the indicator function of the interval $[0, +\infty)$.

We now define $I : \text{inv } \mathcal{M} \to \mathbb{R} \times \mathbb{Z}$ as follows:

$$I(\mu) = (c, n),$$

where $\mu = \rho^n * e^\nu * \delta_c \in \text{inv } \mathcal{M}$. It can be shown that I is well-defined, since in any such decomposition, the n, ν and c are unique.

Lemma 5.14. *Let*

$$\begin{array}{rcl} R & := & \text{be a unital full subring of } \mathcal{M}^+, \\ S & := & \mathcal{M}, \\ G & := & \mathbb{R} \times \mathbb{Z}, \\ \iota & := & I. \end{array}$$

Then (A1)-(A4) *are satisfied.*

Proof. (A1) and (A2) are clear. (A3) follows from the definition of I, since $\rho^n * \rho^{\widetilde{n}} = \rho^{n+\widetilde{n}}$ for all integers n, m and $\delta_c * \delta_{\widetilde{c}} = \delta_{c+\widetilde{c}}$.

Finally we check that (A4) holds. Suppose that $\mu \in R \cap (\text{inv } \mathcal{M})$ is such that $I(\mu) = 0$. Then from Proposition 5.13 above, $\mu = \rho^0 * e^\nu * \delta_0 = e^\nu$ for some $\nu \in \mathcal{M}$. But this implies that ν also has support in $[0, +\infty)$, which can be seen as follows. Write $\nu = \nu_1 + \nu_2$, where ν_1 has support in $[0, +\infty)$ and ν_2 has support in $(-\infty, 0]$. It follows from $\mu = e^\nu$ that $\mu * e^{-\nu_1} = e^{\nu_2}$. But $\mu * e^{-\nu_1}$ has support in $[0, +\infty)$, while e^{ν_2} has support in $(-\infty, 0]$. Hence the support of ν_2 must be contained in $\{0\}$, and so ν has support in $[0, +\infty)$. But then clearly $e^{-\nu} \in \mathcal{M}^+$ is an inverse of μ. As R is a full subring of $\mathcal{M}^+$, we conclude that μ is invertible in R as well.

Conversely, suppose that $\mu \in R \cap (\text{inv } \mathcal{M})$ is invertible as an element of R. Then μ is also invertible as an element of $\mathcal{M}_+$. Consider the Toeplitz operator $W_\mu : L^2(0, +\infty) \to L^2(0, +\infty)$ given by $W_\mu f = P(\mu * f)$, where P is the canonical projection from $L^2(\mathbb{R})$ onto $L^2(0, +\infty)$. Since μ is in invertible element of $\mathcal{M}^+$, it is immediate that W_μ is invertible. In particular, W_μ is Fredholm with Fredholm index 0. But [8, Theorem 2, p.139] says that for $\nu \in \text{inv } \mathcal{M}$, W_ν is Fredholm iff $I(\nu) = (0, n)$ for some integer n, and moreover the Fredholm index of W_ν is then $-n$. Applying this result in our case, we obtain that $I(\mu) = (0, 0)$. This completes the proof. ☐

An application of our main result (Theorem 4.1) yields the following Nyquist criterion.

Corollary 5.15. *Let R be a unital full subring of $\mathcal{M}^+$. Let $P \in \mathbb{S}(R,p,m)$ and $C \in \mathbb{S}(R,m,p)$. Moreover, let $P = N_P D_P^{-1}$ be a right coprime factorization of P, and $C = \widetilde{D}_C^{-1}\widetilde{N}_C$ be a left coprime factorization of C. Then the following are equivalent:*

(1) *C stabilizes P.*
(2) (a) $\det(I - CP)$, $\det D_P$, $\det \widetilde{D}_C$ *belong to* $\text{inv}\,\mathcal{M}$, *and*
 (b) $I(\det(I - CP)) + I(\det D_P) + I(\det \widetilde{D}_C) = (0,0)$.

Remark 5.16. It was shown in [1] that $\mathcal{M}^+$ is a projective free ring. Thus the set $\mathbb{S}(\mathcal{M}^+,p,m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

5.5. **The Hardy algebra.** Let $H^\infty(\mathbb{D})$ denote the Hardy algebra of all bounded and holomorphic functions $f : \mathbb{D} \to \mathbb{C}$. Let $H^2(\mathbb{D})$ denote the Hardy Hilbert space. For $f \in L^\infty(\mathbb{T})$, we denote by T_f the Toeplitz operator corresponding to f, that is, $T_f\varphi = P_+(M_f\varphi)$, $\varphi \in H^2(\mathbb{D})$. Here M_f denotes the pointwise multiplication map by f, taking $\varphi \in L^2(\mathbb{T})$ to $f\varphi \in L^2(\mathbb{T})$, while $P_+ : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ is the canonical orthogonal projection.

If $f \in \text{inv}\,(H^\infty(\mathbb{D}) + C(\mathbb{T}))$, then T_f is a Fredholm operator; see [7, Corollary 7.34]. In this case, let $\text{ind}\,T_f$ denote the index of the Fredholm operator T_f.

Recall the definition of the *harmonic extension* of an $L^\infty(\mathbb{T})$-function.

Definition 5.17. If $z = re^{it}$ is in $\mathbb{D}$ and $f \in L^\infty(\mathbb{T})$, then we define

$$F(z) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})k_r(t-\theta)d\theta,$$

where $k_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$ and $a_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\theta})e^{-2\pi i n\theta}d\theta.$

We will also use the result given below; see [7, Theorem 7.36].

Proposition 5.18. *If $f \in H^\infty(\mathbb{D}) + C(\mathbb{T})$, then T_f is Fredholm iff there exist $\delta, \epsilon > 0$ such that*

$$|F(re^{it})| \geq \epsilon \textit{ for } 1-\delta < r < 1,$$

where F is the harmonic extension of f to $\mathbb{D}$. Moreover, in this case the index of T_f is the negative of the winding number with respect to the origin of the curve $F(re^{it})$ for $1-\delta < r < 1$.

Lemma 5.19. *Let*

$$\begin{aligned} R &:= H^\infty(\mathbb{D}), \\ S &:= H^\infty(\mathbb{D}) + C(\mathbb{T}), \\ G &:= \mathbb{Z}, \\ \iota &:= -\text{ind}\,T_\bullet. \end{aligned}$$

Then (A1)-(A4) *are satisfied.*

Proof. (A1) and (A2) are clear. (A3) follows from the fact that the index of the product of two Fredholm operators is the sum of their respective indices; see for example [16, Exercise 2.5.1.(f)]. The 'only if' part of (A4) is immediate, since if f is invertible as an element of $H^\infty(\mathbb{D})$, then T_f is invertible, and so ind $T_f = 0$. The 'if' part of (A4) follows from Proposition 5.18. Suppose that $f \in H^\infty(\mathbb{D})$, that f is invertible as an element of $H^\infty(\mathbb{D}) + C(\mathbb{T})$ and that ind $T_f = 0$. By Proposition 5.18, it follows that there exist $\delta, \epsilon > 0$ such that $|F(re^{it})| \geq \epsilon$ for $1 - \delta < r < 1$, where F is the harmonic extension of f to $\mathbb{D}$. But since $f \in H^\infty(\mathbb{D})$, its harmonic extension F is equal to f. So $|f(re^{it})| \geq \epsilon$ for $1 - \delta < r < 1$. Also since $\iota(f) = 0$, the winding number with respect to the origin of the curve $f(re^{it})$ for $1-\delta < r < 1$ is equal to 0. By the Argument principle, it follows that f cannot have any zeros inside $r\mathbb{T}$ for $1 - \delta < r < 1$. In light of the above, we can now conclude that there is an $\epsilon' > 0$ such that $|f(z)| > \epsilon'$ for all $z \in \mathbb{D}$. It follows from the corona theorem for $H^\infty(\mathbb{D})$ that f is invertible as an element of $H^\infty(\mathbb{D})$. ☐

An application of Theorem 4.1 yields the following Nyquist criterion.

Corollary 5.20. *Let $P \in \mathbb{S}(H^\infty(\mathbb{D}), p, m)$ and $C \in \mathbb{S}(H^\infty(\mathbb{D}), m, p)$. Moreover, let $P = N_P D_P^{-1}$ be a right coprime factorization of P, and $C = \widetilde{D}_C^{-1}\widetilde{N}_C$ be a left coprime factorization of C. Then the following are equivalent:*

(1) *C stabilizes P.*

(2) (a) $\det(I - CP) \in H^\infty(\mathbb{D}) + C(\mathbb{T})$.

(b) *Let F_1, F_2, F_3 be the harmonic extensions to $\mathbb{D}$, of*

$$f_1 := \det(I - CP), \quad f_2 := \det D_P, \quad f_3 := \det \widetilde{D}_C,$$

respectively. There exist $\delta, \epsilon > 0$ such that

$$|F_i(re^{it})| \geq \epsilon, \quad 1 - \delta < r < 1, \quad i = 1, 2, 3.$$

(c) $\iota(\det(I - CP)) + \iota(\det D_P) + \iota(\det \widetilde{D}_C) = 0.$

Remark 5.21. It was proved by Inouye [12] that the set $\mathbb{S}(H^\infty(\mathbb{D}), p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable.

5.6. **The polydisk algebra.** Let

$$\begin{aligned} \mathbb{D}^n &:= \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1 \text{ for } i = 1, \dots, n\}, \\ \overline{\mathbb{D}^n} &:= \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| \leq 1 \text{ for } i = 1, \dots, n\}, \\ \mathbb{T}^n &:= \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| = 1 \text{ for } i = 1, \dots, n\}. \end{aligned}$$

The *polydisk algebra* $A(\mathbb{D}^n)$ is the set of all functions $f : \overline{\mathbb{D}^n} \to \mathbb{C}$ such that f is holomorphic in $\mathbb{D}^n$ and continuous on $\overline{\mathbb{D}^n}$.

If $f \in A(\mathbb{D}^n)$, then the function f_d defined by $z \mapsto f(z, \dots, z) : \overline{\mathbb{D}} \to \mathbb{C}$ belongs to the disk algebra $A(\mathbb{D})$, and in particular also to $C(\mathbb{T})$. The map

$$f \mapsto (f|_{\mathbb{T}^n}, f_d) : A(\mathbb{D}^n) \to C(\mathbb{T}^n) \times C(\mathbb{T})$$

is a ring homomorphism. This map is also injective, and this is an immediate consequence of Cauchy's formula; see [20, p.4-5]. We recall the following result; see [20, Theorem 4.7.2, p.87].

Proposition 5.22. *Suppose that $\Psi = (\psi_1, \dots, \psi_n)$ is a continuous map from $\overline{\mathbb{D}}$ into $\overline{\mathbb{D}^n}$, which carries $\mathbb{T}$ into $\mathbb{T}^n$ and the winding number of each ψ_i is positive. Then for every $f \in A(\mathbb{D}^n)$, $f(\Psi(\overline{\mathbb{D}}) \cup \mathbb{T}^n) = f(\overline{\mathbb{D}^n})$.*

Lemma 5.23. *Let*

$$\begin{array}{rcl} R & = & \text{a unital full subring of } A(\mathbb{D}^n), \\ S & := & C(\mathbb{T}^n) \times C(\mathbb{T}), \\ G & := & \mathbb{Z}, \\ \iota & := & ((g,h) \mapsto \mathtt{w}(h)). \end{array}$$

Then (A1)-(A4) *are satisfied.*

Proof. (A1) and (A2) are clear. (A3) was proved earlier in Subsection 5.1. Finally, we will show below that (A4) holds, following [6].

Suppose that $f \in A(\mathbb{D}^n)$ is such that $f|_{\mathbb{T}^n} \in \text{inv } C(\mathbb{T}^n)$, $f_d \in \text{inv } C(\mathbb{T})$ and that $\mathtt{w}(f_d) = 0$. We use Proposition 5.22, with $\Psi(z) := (z, \dots, z)$ $(z \in \overline{\mathbb{D}})$. Then we know that f will have no zeros in $\overline{\mathbb{D}^n}$ if $f(\Psi(\overline{\mathbb{D}}))$ does not contain 0. But since $f_d \in \text{inv } C(\mathbb{T})$ and $\mathtt{w}(f_d) = 0$, it follows that f_d is invertible as an element of $A(\mathbb{D})$ by the result in Subsection 5.1. But this implies that $f(\Psi(\overline{\mathbb{D}}))$ does not contain 0.

Now suppose that $f \in A(\mathbb{D}^n)$ with $f|_{\mathbb{T}^n} \in \text{inv } C(\mathbb{T}^n)$, $f_d \in \text{inv } C(\mathbb{T})$, and that it is invertible as an element of $A(\mathbb{D}^n)$. But then in particular, f_d is an invertible element of $A(\mathbb{D})$, and so again by the result in Subsection 5.1, it follows that $\mathtt{w}(f_d) = 0$. ☐

Besides $A(\mathbb{D}^n)$ itself, some other examples of such R are:

(1) $\mathcal{P}$, the set of all polynomials $p : \mathbb{C}^n \to \mathbb{C}$,
(2) $RH^\infty(\mathbb{D}^n)$, the set of all rational functions without poles in $\overline{\mathbb{D}^n}$.

An application of our main result (Theorem 4.1) yields the following Nyquist criterion.

Corollary 5.24. *Let R be a unital full subring of $A(\mathbb{D}^n)$. Let $P \in \mathbb{S}(R, p, m)$ and $C \in \mathbb{S}(R, m, p)$. Moreover, let $P = N_P D_P^{-1}$ be a right coprime factorization of P, and $C = \widetilde{D}_C^{-1} \widetilde{N}_C$ be a left coprime factorization of C. Then the following are equivalent:*

(1) *C stabilizes P.*
(2) (a) $\det(I - CP)$, $\det D_P$, $\det \widetilde{D}_C$ *belong to* inv $(C(\mathbb{T}^n) \times C(\mathbb{T}))$, *and*
(b) $\iota(\det(I - CP)) + \iota(\det D_P) + \iota(\det \widetilde{D}_C) = 0$.

Remark 5.25. By [1], it follows that $A(\mathbb{D}^n)$ is a projective free ring, since its maximal ideal space the polydisk $\overline{\mathbb{D}^n}$ is contractible. Thus the set $\mathbb{S}(A(\mathbb{D}^n), p, m)$ of plants possessing a left and a right coprime factorization coincides with the class of plants that are stabilizable by [18, Theorem 6.3].

Corollary 5.24 was known in the special case when $R = \mathcal{P}$; see [6].

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