SIAM J. CONTROL OPTIM.
Vol. 44, No. 2, pp. 531–563

STATE-SPACE FORMULAS FOR THE NEHARI–TAKAGI PROBLEM FOR NONEXPONENTIALLY STABLE INFINITE-DIMENSIONAL SYSTEMS*

JOSEPH A. BALL†, KALLE M. MIKKOLA‡, AND AMOL J. SASANE§

Abstract. We obtain state-space formulas for the solution of the Nehari–Takagi/suboptimal Hankel norm approximation problem for infinite-dimensional systems with a nonexponentially stable generator, via the method of J-spectral factorization. We make key use of a purely frequency-domain solution of the problem.

Key words. Nehari–Takagi problem, infinite-dimensional systems, state-space formulas

AMS subject classifications. 41A30, 47B35, 47B50, 47N70, 93B28

DOI. 10.1137/S0363012903433024

1. Introduction.

The Hankel norm approximation problem has received a lot of attention, both in the mathematical and engineering literature (see Adamjan, Arov, and Kreĭn [1], Ball and Helton [4], Ball and Ran [7], Glover [19], and Doyle, Glover, and Zhou [17]). Its importance in control theory is due to its connections with the model reduction problem (see [19]).

In order to state the suboptimal Hankel norm approximation problem, we will need a few preliminaries. First we recall the definition of the (frequency-domain) Hankel operator corresponding to a symbol $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$ and the definition of its singular values. Let $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ and $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$.

Let $H_2(\mathbb{C}_+, \mathbb{C}^k)$ denote the set of all analytic functions $f : \mathbb{C}_+ \to \mathbb{C}^k$ such that

$$\|f\|_2 := \sup_{\zeta>0} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(\zeta + i\omega)\|^2 d\omega \right)^{\frac{1}{2}} < \infty.$$

Analogously one defines $H_2(\mathbb{C}_-, \mathbb{C}^k)$. For $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$ we define the *Hankel operator with symbol* G, denoted by H_G, acting from $H_2(\mathbb{C}_-, \mathbb{C}^m)$ to $H_2(\mathbb{C}_+, \mathbb{C}^p)$, as follows:

$$H_G f = P_{H_2(\mathbb{C}_+,\mathbb{C}^p)}(M_G f) \quad \text{for } f \in H_2(\mathbb{C}_-, \mathbb{C}^m),$$

where M_G is the multiplication map on $L_2(i\mathbb{R}, \mathbb{C}^m)$ induced by G, and $P_{H_2(\mathbb{C}_+,\mathbb{C}^p)}$ is the orthogonal projection operator from $L_2(i\mathbb{R}, \mathbb{C}^p)$ onto $H_2(\mathbb{C}_+, \mathbb{C}^p)$. The Hankel operator is bounded, that is, $H_G \in \mathcal{L}(H_2(\mathbb{C}_-, \mathbb{C}^m), H_2(\mathbb{C}_+, \mathbb{C}^p))$.

Now we recall the notion of singular values of a bounded linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$. For $k \in \{1, 2, \dots\}$ the kth *singular value*

*Received by the editors August 12, 2003; accepted for publication (in revised form) August 27, 2004; published electronically August 31, 2005.
http://www.siam.org/journals/sicon/44-2/43302.html
†Department of Mathematics, 460 McBryde, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0123 (ball@ math.vt.edu).
‡Institute of Mathematics, Box 1100, FIN-02015 Helsinki University of Technology, Helsinki, Finland (kalle.mikkola@iki.fi). The work of this author was supported by the Academy of Finland under grant 203946.
§Department of Mathematics, London School of Economics, Houghton St., London WC2A 2AE, UK (a.j.sasane@lse.ac.uk).

(denoted by $\sigma_k(\mathtt{H})$) of an operator $\mathtt{H} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is defined to be the distance with respect to the norm in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ of $\mathtt{H}$ from the set of operators in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ of rank at most $k-1$. Thus $\sigma_1(\mathtt{H}) = \|\mathtt{H}\|$, and $\sigma_1(\mathtt{H}) \geq \sigma_2(\mathtt{H}) \geq \sigma_3(\mathtt{H}) \geq \cdots \geq 0$. For $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$, we refer to the singular values of H_G simply as the *Hankel singular values of* G.

Let $H_{\infty,k}(\mathbb{C}_-, \mathbb{C}^{p\times m})$ denote the set of all $p \times m$ matrix-valued functions K of a complex variable defined in the open left half-plane such that $K = G_{\mathtt{f}} + F$, where F is an element in $H_\infty(\mathbb{C}_-, \mathbb{C}^{p\times \mathtt{m}})$ and $G_{\mathtt{f}}$ is the transfer function of a finite-dimensional system with order at most k, with all its poles in the open left half-plane. The set $H_{\infty,k}(\mathbb{C}_-, \mathbb{C}^{p\times m})$ is a subset of $L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$.

We recall the following well-known result of Adamjan, Arov, and Kreĭn [1], adapted here to the right half-plane setting: If $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$, then

$$\inf_{K \in H_{\infty,k}(\mathbb{C}_-, \mathbb{C}^{p\times m})} \|G(i\cdot) + K(i\cdot)\|_\infty = \sigma_{k+1}(G).$$

We are now ready to give the statement of the suboptimal Hankel norm approximation problem, which is also known as the *Nehari–Takagi problem.* The *suboptimal Hankel norm approximation problem* is the following: Let $G(i\cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{p\times m})$. If $\sigma_{k+1} < \sigma < \sigma_k$, then find $K \in H_{\infty,k}(\mathbb{C}_-, \mathbb{C}^{p\times m})$ such that $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$. In fact, the authors of [1], working with Schmidt pairs of the Hankel operator, also gave a linear-fractional description for the set of all solutions of the suboptimal Hankel norm approximation problem; later work of Ball and Helton [4] obtained such a linear-fractional description, but via an indefinite-metric Beurling–Lax theorem combined with some Kreĭn-space projective geometry.

Now suppose that G is in fact the transfer function of some well-posed linear system; that is, G is not simply an L_∞ function, but it has the special form $G(s) = C(sI - A)^{-1}B$, where (A, B, C) are the generators of the system. Then by a *state-space solution* to the suboptimal Hankel norm approximation problem we mean a K given explicitly in terms of the A, B, C operators. For the case of rational $G(s)$ with system-generators (A, B, C) equal to finite matrices, a state-space solution of the Hankel norm approximation problem has been obtained by Kung and Lin [29], Glover [19], Ball and Ran [7], and Ball, Gohberg, and Rodman [3, Chapter 20].

In Curtain and Sasane [14, 13], state-space solutions to the suboptimal Hankel norm approximation problem were given for two classes of infinite-dimensional state-linear systems, but under the assumption that A generates an *exponentially* stable, strongly continuous semigroup. Recall that a semigroup $\{T(t)\}_{t\geq 0}$ on a Hilbert space X is said to be exponentially stable if there exist positive constants M and ϵ such that

$$\|T(t)\| \leq Me^{-\epsilon t} \quad \text{for all } \ t \geq 0.$$

However, there exists an important class of systems with a transfer function $G \in H_\infty(\mathbb{C}_-, \mathbb{C}^{p\times m})$ for which A does not generate an exponentially stable semigroup (see, for example, Oostveen [33]), for example, if A is the generator of a *strongly stable semigroup*, that is, a semigroup satisfying

$$T(t)x \to 0 \ \text{ as } \ t \to \infty \quad \text{for all } \ x \in X.$$

Roughly speaking, the rate of convergence to zero is not uniform but depends on the choice of the element in the Hilbert space. An elementary example of a semigroup which is strongly stable but not exponentially stable is given by e^{tA} on ℓ_2, where

$$A = \begin{bmatrix} -1 & & & \\ & -\frac{1}{2} & & \\ & & -\frac{1}{3} & \\ & & & \ddots \end{bmatrix} \in \mathcal{L}(\ell_2).$$

In this article, we consider an even weaker notion of stability, the so-called *nonexponentially stable semigroup*, namely, a semigroup whose generator has a nonnegative growth bound. Clearly this class encompasses both strongly stable semigroups and (hence surely) exponentially stable semigroups; thus we emphasize that the prefix "non" is really short for "not necessarily."

Earlier work on the problem for infinite-dimensional systems includes the work of Curtain and Ran [12], which handled the case of Pritchard–Salamon systems, and of Glover, Curtain, and Partington [20], where approximating solutions to the optimal Hankel norm approximation problem were obtained without assuming exponential stability, but only for the case that the Hankel operator is nuclear, a rather strong assumption. In this paper, we give solutions to the suboptimal Hankel norm approximation problem for infinite-dimensional systems having a nonexponentially stable semigroup. Our solution depends on a preliminary result which obtains the linear-fractional parameterization of the set of all solutions in purely frequency-domain terms via the solution Θ of a certain J-spectral factorization problem. The fact that Θ may be unbounded in our general setting makes the analysis much more delicate. We give three proofs of this key frequency-domain result in order to point out the close connections with results already existing in the literature. The first proof shows how the result can be reduced to the result of Adamjan, Arov, and Kreĭn in [1]. The second proof revisits the proof of Ball and Helton [4] with special care given to the details required to handle the general case where Θ may be unbounded. The third proof revisits the homotopy argument appearing in [3, 40]. The standard homotopy argument works well in case the coefficients of the linear-fractional parameterization and the free parameter are continuous up to the boundary. We show how an approximation argument can be used to reduce the general case here to the classical situation, at least for the proof that every admissible free parameter leads to a solution of the Nehari–Takagi problem. The proof that any solution of the Nehari–Takagi problem necessarily is of the linear-fractional form follows the ideas appearing in the second proof.

The outline of the paper is as follows. In section 2, we give the key frequency-domain result (the reduction of the parameterization of the set of all solutions of the suboptimal Hankel norm approximation problem to solving a certain J-spectral factorization problem), along with our three proofs of this result. In section 3 we use this frequency-domain result to parameterize all solutions to the suboptimal Hankel norm approximation problem for infinite-dimensional state-space systems for which the generator is not necessarily exponentially stable. Finally in the last section, we give state-space solutions for well-posed linear systems by applying the result in section 3 to the associated reciprocal system.

2. The key frequency-domain result.

THEOREM 2.1. *Let $G \in H_\infty(\mathbb{C}_+, \mathbb{C}^{p\times m})$ and let $H_G\colon H_2(\mathbb{C}_-, \mathbb{C}^m) \mapsto H_2(\mathbb{C}_+, \mathbb{C}^p)$ denote the corresponding Hankel operator, with the singular values $\sigma_1 \geq \sigma_2 \geq \cdots (\geq 0)$. Suppose that $\sigma_k > \sigma > \sigma_{k+1}$. Then there exists a matrix function $\Lambda\colon \mathbb{C}_- \mapsto$*

$\mathbb{C}^{(p+m)\times(p+m)}$, *uniquely determined up to a* $(p+m)\times(p+m)$*-matrix right constant factor* U *satisfying* $U^*\left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]U = \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]$, *such that*

S1. $\Lambda(i\omega)^*\left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]\Lambda(i\omega) = \left[\begin{smallmatrix} I_p & G(i\omega) \\ 0 & I_m \end{smallmatrix}\right]^*\left[\begin{smallmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{smallmatrix}\right]\left[\begin{smallmatrix} I_p & G(i\omega) \\ 0 & I_m \end{smallmatrix}\right]$ *for* $\omega \in \mathbb{R}$;

S2. $\frac{1}{\cdot - 1}\Lambda \in H_2(\mathbb{C}_-, \mathbb{C}^{(p+m)\times(p+m)})$;

S3. Λ *is invertible* (*i.e., there exists a* $V\colon \mathbb{C}_- \mapsto \mathbb{C}^{(p+m)\times(p+m)}$ *such that* $\Lambda(s)V(s) = I_{p+m}$ *for* $s \in \mathbb{C}_-$) *and* $\frac{1}{\cdot - 1}V \in H_2(\mathbb{C}_-, \mathbb{C}^{(p+m)\times(p+m)})$.

Define

$$\Theta(i\omega)\left(= \begin{bmatrix} \Theta_{11}(i\omega) & \Theta_{12}(i\omega) \\ \Theta_{21}(i\omega) & \Theta_{22}(i\omega) \end{bmatrix}\right) = \begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix} V(i\omega) \quad \text{for } \omega \in \mathbb{R}.$$

Then we have the following: $K\colon \mathbb{C}_- \mapsto \mathbb{C}^{p\times m}$ *such that* $K \in H_{\infty,k}(\mathbb{C}_-, \mathbb{C}^{p\times m})$ *and* $\|G(i\cdot) + K(i\cdot)\|_\infty \le \sigma$ *if and only if*

(2.1)
$$G(i\omega) + K(i\omega) = (\Theta_{11}(i\omega)Q(i\omega) + \Theta_{12}(i\omega))(\Theta_{21}(i\omega)Q(i\omega) + \Theta_{22}(i\omega))^{-1} \quad \text{for } \omega \in \mathbb{R}$$

for some $Q\colon \mathbb{C}_- \mapsto \mathbb{C}^{p\times m}$ *such that* $Q \in H_\infty(\mathbb{C}_-, \mathbb{C}^{p\times m})$ *and* $\|Q(i\cdot)\|_\infty \le 1$.

For the application of Theorem 2.1 in section 3, we note that a sufficient condition for the validity of S2 is the existence of a constant $\Lambda(\infty) \in \mathbb{C}^{(p+m)\times(p+m)}$ such that $\Lambda - \Lambda(\infty) \in H_2(\mathbb{C}_-, \mathbb{C}^{(p+m)\times(p+m)})$.

By using the transformation

$$f(s) \mapsto \widetilde{f}(z) := f\left(\frac{1-z}{1+z}\right)$$

and observing (via the Jacobi change-of-variable formula) that

$$\int_{\mathbb{T}} |\widetilde{f}(z)|^2\ |dz| = \int_{i\mathbb{R}} |f(s)|^2\ \frac{|ds|}{1+|s|^2},$$

we see that Theorem 2.1 is exactly equivalent to the following discrete-time version. Here $\mathbb{D}$ denotes the unit disk, $\mathbb{D}_e$ denotes the exterior of the unit disk (including the point at infinity), and $\mathbb{T}$ denotes the unit torus (equal to the boundary of $\mathbb{D}$).

THEOREM 2.2. *Let* $G \in H_\infty(\mathbb{D}_e)$ *and let* $H_G\colon H_2(\mathbb{D}, \mathbb{C}^m) \mapsto H_2(\mathbb{D}, \mathbb{C}^p)^\perp$ *be the associated Hankel operator, with singular values* $\sigma_1 \ge \sigma_2 \ge \cdots (\ge 0)$. *Suppose that* $\sigma_k > \sigma > \sigma_{k+1}$. *Then there exists a unique matrix function* $\Lambda\colon \mathbb{D} \mapsto \mathbb{C}^{(p+m)\times(p+m)}$, *uniquely determined up to a* $(p+m)\times(p+m)$*-matrix right constant factor* U *satisfying* $U^*\left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]U = \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]$, *such that*

S′1. $\Lambda(\zeta)^*\left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]\Lambda(\zeta) = \left[\begin{smallmatrix} I_p & G(\zeta) \\ 0 & I_m \end{smallmatrix}\right]^*\left[\begin{smallmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{smallmatrix}\right]\left[\begin{smallmatrix} I_p & G(\zeta) \\ 0 & I_m \end{smallmatrix}\right]$ *for* $\zeta \in \mathbb{T}$;

S′2. $\Lambda \in H_2(\mathbb{D}, \mathbb{C}^{(p+m)\times(p+m)})$;

S′3. Λ *is invertible* (*i.e., there exists a* $V\colon \mathbb{D} \mapsto \mathbb{C}^{(p+m)\times(p+m)}$ *such that* $\Lambda(z)V(z) = I_{p+m}$ *for* $z \in \mathbb{D}$) *and* $V \in H_2(\mathbb{D}, \mathbb{C}^{(p+m)\times(p+m)})$.

Define

$$\Theta(\zeta)\left(= \begin{bmatrix} \Theta_{11}(\zeta) & \Theta_{12}(\zeta) \\ \Theta_{21}(\zeta) & \Theta_{22}(\zeta) \end{bmatrix}\right) = \begin{bmatrix} I_p & G(\zeta) \\ 0 & I_m \end{bmatrix} V(\zeta) \quad \text{for } \zeta \in \mathbb{T}. \tag{2.2}$$

Then we have the following: $K\colon \mathbb{D} \mapsto \mathbb{C}^{p\times m}$ *is such that* $K \in H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p\times m})$ *and* $\|(G+K)|_{\mathbb{T}}\|_\infty \le \sigma$ *if and only if*

$$G(\zeta) + K(\zeta) = (\Theta_{11}(\zeta)Q(\zeta) + \Theta_{12}(\zeta))(\Theta_{21}(\zeta)Q(\zeta) + \Theta_{22}(\zeta))^{-1} \quad \text{for } \zeta \in \mathbb{T} \tag{2.3}$$

for some $Q\colon \mathbb{D} \mapsto \mathbb{C}^{p\times m}$ *such that* $Q \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ *and* $\|Q\|_\infty \le 1$.

We next indicate several proofs of Theorem 2.2 based on various different points of view. We first need to lay out a few preliminaries.

2.1. Preliminaries. For p and m positive integers we let $\mathbb{C}^{p\times m}$ be the space of complex $p \times m$ matrices M with norm $\|M\|$ equal to the induced operator norm:

$$\|M\| = \sup_{x\in\mathbb{C}^m:\ \|x\|_2\leq 1} \|Mx\|_2,$$

where $\|x\|_2$ is the standard Euclidean 2-norm on $\mathbb{C}^m$. The trace norm $\mathrm{Tr}(M)$ of a $p \times m$ matrix M is defined by

$$\mathrm{Tr}(M) = \mathrm{tr}(M^*M)^{1/2},$$

where the *trace* $\mathrm{tr}(A)$ of an $m \times m$ matrix A is defined by

$$\mathrm{tr}(A) = \sum_{k=1}^{p} \langle Ae_k, e_k\rangle,$$

where $\{e_1, \dots, e_n\}$ is any orthonormal basis for $\mathbb{C}^p$—a good reference for the natural infinite-dimensional setting for this material is [21, Chapter VII]. We let $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ denote the space of measurable $p \times m$ matrix-valued functions on the unit circle $\mathbb{T}$ with finite essential supremum (supremum up to sets of measure zero) norm uniformly bounded:

$$\|F\|_\infty = \operatorname*{ess-sup}_{\zeta\in\mathbb{T}} \|F(\zeta)\| < \infty.$$

We let $L_1(\mathbb{T}, \mathbb{C}^{m\times p})$ be the space of measurable $m \times p$ matrix-valued functions f on $\mathbb{T}$ with integrable trace norm:

$$\|f\|_1 = \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{Tr}(f(\zeta))|d\zeta|.$$

It is well known (see, e.g., [38, page 197]) that the Banach space $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ can be identified as the dual of the Banach space $L_1(\mathbb{T}, \mathbb{C}^{m\times p})$ under the duality pairing

$$[F, f] = \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{tr}(F(\zeta)f(\zeta))\ d|\zeta| \quad \text{for } F \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m}) \text{ and } f \in L_1(\mathbb{T}, \mathbb{C}^{m\times p}).$$

Therefore, in addition to its norm topology, $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ carries a *weak-$*$ topology* induced by its duality with respect to $L_1(\mathbb{T}, \mathbb{C}^{p\times m})$. We shall have use of the following facts concerning this weak-$*$ topology.

PROPOSITION 2.3.

(1) *A subspace $\mathcal{S}$ of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ is closed in the weak-$*$ topology of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ if and only if whenever $\{F_n\}_{n=1,2,\dots}$ is a sequence of elements of $\mathcal{S}$ converging weak-$*$ to $F \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$, then in fact $F \in \mathcal{S}$.*

(2) *Suppose that $\{F_n\}_{n=1,2,\dots}$ is a sequence of elements of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ converging* pointwise boundedly *to the element $F \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$, i.e.,*

$$\lim_{n\to\infty} F_n(\zeta) = F(\zeta) \quad \textit{for almost all } \zeta \in \mathbb{T}, \textit{ and}$$
$$\|F_n(\zeta)\| \leq M \quad \textit{for some } M < \infty \textit{ for all } n = 1, 2, \dots.$$

Then F_n converges to F in the weak-$*$ topology of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$.

Proof. By the Kreĭn–Šmulian theorem (see [42, Theorem 10.1, page 173]), a subspace $\mathcal{S}$ of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ (or more generally, a convex subset) is weak-$*$ closed if and only if $\mathcal{S} \cap \{F \colon \|F\|_\infty \le r\}$ is weak-$*$ closed for each $r > 0$. Since $\mathcal{S}$ is a subspace, by homogeneity it suffices to consider only the case $r = 1$. As $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ is the dual of the *separable* space $L_1(\mathbb{T}, \mathbb{C}^{m\times p})$, it follows that the weak-$*$ topology on the unit ball of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ is metrizable (see [18, Theorem 102, page 174]). Hence, to show that $\mathcal{S}$ is closed in the weak-$*$ topology, it suffices to show that $\mathcal{S}$ is closed under sequential weak-$*$ limits as asserted. This proves part (1) of Proposition 2.3.

Suppose now that $\{F_n\}_{n=1,2,\dots}$ is a sequence of elements of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ converging pointwise boundedly to F. To show that F_n converges to F in the weak-$*$ topology, we must show that

$$\lim_{n\to\infty}[F_n, f] = \lim_{n\to\infty} \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{tr}(F_n(\zeta)f(\zeta))|d\zeta| = \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{tr}(F(\zeta)f(\zeta))|d\zeta| \tag{2.4}$$

for each choice of $f \in L_1(\mathbb{T}, \mathbb{C}^{m\times p})$. Note that the assumptions imply that

$$\lim_{n\to\infty} \operatorname{tr}(F_n(\zeta)f(\zeta)) = \operatorname{tr}(F_n(\zeta)f(\zeta)) \quad \text{for almost all } \zeta \in \mathbb{T}.$$

By the standard trace estimate

$$|\operatorname{tr}(AB)| \le \operatorname{Tr}(AB) \le \|A\| \operatorname{Tr}(B),$$

we have

$$|\operatorname{tr}(F_n(\zeta)f(\zeta))| \le \|F_n(\zeta)\| \operatorname{Tr}(f(\zeta)) \le M \operatorname{Tr}(f(\zeta)),$$

where $M \operatorname{Tr}(f(\cdot))$ is integrable by the definition of $f \in L_1(\mathbb{T}, \mathbb{C}^{p\times m})$. It now follows from the Lebesgue dominated convergence theorem (see, e.g., [37, Theorem 16, page 91]) that (2.4) follows as required. This completes the proof of part (2) of Proposition 2.3. $\square$

The subspace $H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ can be viewed as the subspace of $L_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ consisting of functions $F \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ such that the Fourier coefficients of negative index vanish:

$$\frac{1}{2\pi}\int_{\mathbb{T}} F(\zeta)\zeta^n |d\zeta| = 0 \quad \text{for } n = -1, -2, \dots.$$

The subspace $H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ can also be viewed (via identification through nontangential-limit boundary values) as the space of analytic $p \times m$ matrix-valued functions on the unit disk which are uniformly bounded there:

$$\|F\|_\infty = \sup_{z\in\mathbb{D}} \|F(z)\| < \infty.$$

We define $H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p\times m})$ as consisting of all elements G of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ for which the associated Hankel operator $H_G \colon H_2(\mathbb{D}, \mathbb{C}^m) \mapsto H_2(\mathbb{D}, \mathbb{C}^p)^\perp$ given by

$$H_G \colon f \mapsto P_{H_2(\mathbb{D},\mathbb{C}^p)^\perp} M_G|_{H_2(\mathbb{D},\mathbb{C}^m)}$$

has rank equal to k. Here M_G denotes the multiplication operator associated with G. Equivalently, the *Hankel matrix*

$$[H_G] = \begin{bmatrix} g_{-1} & g_{-2} & g_{-3} & \cdots \\ g_{-2} & g_{-3} & g_{-4} & \cdots \\ g_{-3} & g_{-4} & g_{-5} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} : \ell^2(\mathbb{Z}_+, \mathbb{C}^m) \mapsto \ell^2(\mathbb{Z}_+, \mathbb{C}^p)$$

based on the Fourier coefficients for G,

$$G(z) \sim \sum_{n=-\infty}^{\infty} g_n z^n \quad \text{for } z \in \mathbb{T},$$

has rank equal to k. In what follows we shall use the following result.

PROPOSITION 2.4. *For a given $k \in \{0, 1, 2, \dots\}$, the set*

$$\bigcup_{k' \colon k' \le k} H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$$

is closed in the weak-$*$ topology of $L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$.

Proof. Let us suppose that $G_n \in H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$ for some $k' \le k$ for all $n = 1, 2, \dots$, and that G_n converges to $G \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ in the weak-$*$ topology. By part (1) of Proposition 2.3, Proposition 2.4 follows if we are able to show that necessarily the limit G is again in $\bigcup_{k' \colon k' \le k} H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$. For $f \in H_2(\mathbb{D}, \mathbb{C}^m)$ and $g \in H_2(\mathbb{D}, \mathbb{C}^p)^\perp$ we then have

$$\begin{aligned}
\langle H_{G_n} f, g\rangle_{H_2^\perp} &= \frac{1}{2\pi} \int_{\mathbb{T}} g(\zeta)^* G_n(\zeta) f(\zeta) |d\zeta| \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} \operatorname{tr}\left(G_n(\zeta) f(\zeta) g(\zeta)^*\right) |d\zeta|. \qquad (2.5)
\end{aligned}$$

As $f(\zeta)g(\zeta)^* \in L_1(\mathbb{T}, \mathbb{C}^{m\times p})$ and G_n converges weak-$*$ to G by assumption, we conclude from (2.5) that

$$\begin{aligned}
\lim_{n\to\infty} \langle H_{G_n} f, g\rangle_{H_2^\perp} &= \frac{1}{2\pi} \int_{\mathbb{T}} \operatorname{tr}\left(G(\zeta) f(\zeta) g(\zeta)^*\right) |d\zeta| \\
&= \langle H_G f, g\rangle_{H_2^\perp}, \qquad (2.6)
\end{aligned}$$

i.e., H_{G_n} *converges to* H_G *in the weak operator topology of* $\mathcal{L}(H_2(\mathbb{D}, \mathbb{C}^m), H_2(\mathbb{D}, \mathbb{C}^p)^\perp)$. The fact that $G_n \in H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$ with $k' \le k$ means that

$$\det[\langle H_{G_n} e_j, f_\ell\rangle_{H_2^\perp}]_{j,l=1,\dots,k+1} = 0 \qquad (2.7)$$

for all $n = 1, 2, 3, \dots$ for any choice of $k+1$ linearly independent vectors $\{e_1, \dots, e_{k+1}\}$ in $H_2(\mathbb{D}, \mathbb{C}^m)$ and $k+1$ linearly independent vectors $\{f_1, \dots, f_{k+1}\}$ in $H_2(\mathbb{D}, \mathbb{C}^p)^\perp$. Using (2.5) and taking limits in (2.7) then implies that

$$\det[\langle H_G e_j, f_l\rangle_{H_2^\perp}]_{j,l=1,\dots,k+1} = 0 \qquad (2.8)$$

for all such $\{e_1, \dots, e_{k+1}\}$ and $\{f_1, \dots, f_{+1}\}$. This then implies that H_G has rank at most k, or, by definition, $G \in H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$ for some $k' \le k$. ∎

Sometimes it is of interest to focus on the "unit ball" of $H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p\times m})$, namely, the set of functions $G \in H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p\times m})$ with $\|G\|_\infty \le 1$. This class is often given a special name, namely, the *generalized Schur class of index* k, denoted as $\mathcal{S}_k(\mathbb{D}, \mathbb{C}^{p\times m})$. The following result concerning the class $\mathcal{S}_k(\mathbb{D}, \mathbb{C}^{p\times m})$ originates in the work of Kreĭn and Langer (see [27, 28]).

PROPOSITION 2.5. *Let $G \in L^\infty(\mathbb{T}, \mathbb{C}^{p\times m})$. Then the following are equivalent:*

(1) $G \in \mathcal{S}_k(\mathbb{D}, \mathbb{C}^{p\times m})$.

(2) *G has a factorization $G = F \cdot B^{-1}$, where F is in $H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ with $\|F\|_\infty \leq 1$ and B is an $m \times m$ Blaschke–Potapov product of degree k, and no such representation $G = f' \cdot B'^{-1}$ is possible with B' an $m \times m$ matrix Blaschke–Potapov product of degree $k' < k$.*

(3) *G has meromorphic continuation to $\mathbb{D}$ and, for any choice of vectors $x_1, \dots, x_N \in \mathbb{C}^p$, points $z_1, \dots, z_N \in \Omega_G$ (where $\Omega_G \subset \mathbb{D}$ is the domain of analyticity for G), and $N = 1, 2, 3, \dots$, the Hermitian matrix*

$$\left[\frac{x_i^* x_j - x_i^* G(z_i) G(z_j)^* x_j}{1 - z_i \overline{z_j}}\right] \tag{2.9}$$

has at most k negative eigenvalues, and there is at least one choice of $x_1, \dots, x_N$, $z_1, \dots, z_N$, and N for which (2.9) *has exactly k negative eigenvalues.*

We shall also need an asymptotic version of the maximum modulus theorem for the generalized Schur class $\mathcal{S}_k(\mathbb{D}, \mathbb{C}^{p\times m})$ (with $k < \infty$).

PROPOSITION 2.6. *Suppose G is in the generalized Schur class $\mathcal{S}_k(\mathbb{D}, \mathbb{C}^{p\times m})$, where $k < \infty$, and let $s > 1$. Then there exists an $r < 1$ so that*

$$z \in \mathbb{D}, \quad r < |z| < 1 \Rightarrow z \in \Omega_G, \quad \textit{and} \quad \|G(z)\| \leq s.$$

Proof. Let $G = F \cdot B^{-1}$ be the Kreĭn–Langer factorization G and suppose that we are given a number $s > 1$. As $F \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ with $\|F\|_\infty \leq 1$, we have $\|F(z)\| \leq 1$ for all $z \in \mathbb{D}$ by the maximum modulus theorem for H_∞. As B is a finite matrix Blaschke–Potapov product, B is uniformly continuous on the closed disk $\overline{\mathbb{D}}$, and B^{-1} is uniformly continuous on any annulus $\mathbb{A}_r = \{z \colon r \leq |z| \leq 1\}$ which misses the zeros of B. As B^{-1} has norm 1 on the unit circle, we can therefore guarantee that $\|B^{-1}(z)\| \leq s$ (for any preassigned $s > 1$) as long as we restrict to an annulus $\mathbb{A}_r$ with r sufficiently close to 1. The result now follows. ∎

We also need the following elementary result.

PROPOSITION 2.7. *Suppose that $G \in H_\infty(\mathbb{D}_e, \mathbb{C}^{p\times m})$ with Hankel singular values $\sigma_1 \geq \sigma_2 \geq \cdots (\geq 0)$. If $Q \in H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p\times m})$, then $\|G + Q\|_\infty \geq \sigma_{k+1}$.*

Proof. The Hankel singular values are characterized by

$$\sigma_{k+1}(H_G) = \inf_{X \colon \operatorname{rank} X \leq k} \|H_G - X\|,$$

where X here is an operator from $H_2(\mathbb{D}, \mathbb{C}^m)$ to $H_2(\mathbb{D}, \mathbb{C}^p)$ (see, e.g., [21, Chapter VI, Theorem 1.5, page 98]). In particular, if $K \in H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p\times m})$, then $X = H_K$ has rank equal to k. Hence

$$\|G + K\|_\infty \geq \|H_{G+K}\|_{op} = \|H_G + H_K\|_{op} \geq \inf_{X \colon \operatorname{rank} X \leq k} \|H_G + X\|_{op} = \sigma_{k+1},$$

and the assertion follows. ∎

2.2. Existence of Λ and Θ in Theorem 2.2. In this section we point out some general considerations which guarantee the existence of a function $\Lambda \colon \mathbb{D} \mapsto \mathbb{C}^{(p+m)\times(p+m)}$ satisfying conditions S$'$1, S$'$2, and S$'$3. It then remains to prove that such a Λ leads to a parameterization of the set of all solutions of the Nehari–Takagi problem as in Theorem 2.2. In practice, it then remains to compute Λ (and Θ) in some explicit form in terms of known parameters in the application; this is what we do in section 3 (for the setting of the left half-plane rather than of the unit disk), where $G(s) = C(sI - A)^{-1}B$ is assumed to be the transfer function of a continuous-time linear system having certain (nonexponential) stability properties.

First we need to make a few general observations. The invertibility of $H_G^* H_G - \sigma^2 I$ on $L_2(\mathbb{T}, \mathbb{C}^m)$ is equivalent to σ being in the resolvent set of $[H_G^* H_G]^{1/2}$, i.e., of σ being in a gap of the spectrum of $[H_G^* H_G]^{1/2}$. It is well known that the singular values $\sigma_1 > \sigma_2 > \cdots$ of H_G consist of the points of the spectrum of $[H_G^* H_G]^{1/2}$ which are isolated eigenvalues of finite multiplicity positioned to the right of the continuous spectrum. The condition that σ is in a gap between Hankel singular values implies in particular that σ is in a gap of the spectrum of $[H_G^* H_G]^{1/2}$, and hence implies the invertibility of $H_G^* H_G - \sigma^2 I$ on $L_2(\mathbb{T}, \mathbb{C}^m)$. Further details on singular values in general are given in Lemma 6.2 in Appendix B. For a given matrix function $G \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$, in addition to the notation $H_G \colon H_2(\mathbb{D}, \mathbb{C}^m) \mapsto H_2(\mathbb{D}, \mathbb{C}^p)^\perp$ for the Hankel operator $H_G \colon f \mapsto P_{H_2(\mathbb{D},\mathbb{C}^p)^\perp}(G \cdot f)$ associated with G, we let $T_G \colon H_2(\mathbb{D}, \mathbb{C}^m) \mapsto H_2(\mathbb{D}, \mathbb{C}^p)$ denote the *Toeplitz operator* associated with G,

$$T_G \colon f \mapsto P_{H_2(\mathbb{D},\mathbb{C}^p)}(G \cdot f) \quad \text{for } f \in H_2(\mathbb{D}, \mathbb{C}^m),$$

and we let $M_G \colon L_2(\mathbb{T}, \mathbb{C}^m) \mapsto L_2(\mathbb{T}, \mathbb{C}^p)$ denote the *multiplication* (sometimes also called the *Laurent*) *operator* associated with G,

$$M_G \colon f \mapsto G \cdot f.$$

The next proposition gives a number of conditions equivalent to the invertibility of $H_G^* H_G - \sigma^2 I$ on $H_2(\mathbb{D}, \mathbb{C}^m)$.

Proposition 2.8. *Let $G \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ and set $A = \left[\begin{smallmatrix} I & G \\ 0 & I \end{smallmatrix}\right] \in L_\infty(\mathbb{T}, \mathbb{C}^{(p+m)\times(p+m)})$. Then the following conditions are equivalent:*

(1) *$H_G^* H_G - \sigma^2 I$ is invertible.*
(2) *The Toeplitz operator $T_{A^* J_\sigma A}$ is invertible on $H_2(\mathbb{D}, \mathbb{C}^{p+m})$.*
(3) *The singular integral operator $S := M_{A^* J_\sigma A} P_{H_2(\mathbb{D},\mathbb{C}^{p+m})} + P_{H_2(\mathbb{D},\mathbb{C}^{p+m})^\perp}$ is invertible on $L_2(\mathbb{T}, \mathbb{C}^m)$.*

Proof. To see that (1) $\Rightarrow$ (2), note that

$$A^* J_\sigma A = \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix}^* \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_p & G \\ G^* & G^* G - \sigma^2 I_m \end{bmatrix}.$$

Taking Schur complements, we see that invertibility of $T_{A^* J_\sigma A}$ is equivalent to invertibility of

$$\begin{aligned} T_{G^* G - \sigma^2 I_m} - T_{G^*} T_G &= P_{H_2} \left(M_{G^*} M_G - M_{G^*} P_{H_2} M_G \right)|_{H_2} - \sigma^2 I_{H_2} \\ &= P_{H_2} M_{G^*} P_{H_2^\perp} M_G|_{H_2} - \sigma^2 I_{H_2} \\ &= H_G^* H_G - \sigma^2 I_{H_2}, \end{aligned}$$

and (1) $\iff$ (2) follows.

If we decompose $L_2(\mathbb{C}^{p+m})$ in the form $L_2(\mathbb{C}^{p+m}) = \left[\begin{smallmatrix} H_2(\mathbb{D},\mathbb{C}^{p+m}) \\ H_2(\mathbb{D},\mathbb{C}^{p+m})^\perp \end{smallmatrix}\right]$, then the singular integral operator $S := M_{A^* J_\sigma A} P_{H_2} + P_{H_2^\perp}$ has the operator-block representation

$$S = \begin{bmatrix} T_{A^* J_\sigma A} & 0 \\ H_{A^* J_\sigma A} & I_{H_2^\perp} \end{bmatrix}.$$

From the triangular form of this block operator matrix, we see (2) $\iff$ (3). ∎

Theorem VII.2.1 combined with Theorem VIII.4.1 from [10], adapted to our setting, gives the following.

Theorem 2.9 (see [10, Theorem VII.2.1 and Theorem VIII.4.1] or [31]). *Let $G \in L_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$. Then the following are equivalent:*

(1) *Any of the equivalent conditions* (1), (2), *or* (3) *in Proposition* 2.8 *holds.*
(2) *There exists a function* $\Lambda \in H_2(\mathbb{D}, \mathbb{C}^{(p+m)\times(p+m)})$ *meeting the conditions* $\mathrm{S}'1$, $\mathrm{S}'2$, *and* $\mathrm{S}'3$ *of Theorem* 2.2 *and satisfying the additional condition:* The operator $M_V P_{H_2} M_\Lambda$ $(= M_{\Lambda^{-1}} P_{H_2} M_\Lambda)$ defines a bounded projection operator on $L_2(\mathbb{T}, \mathbb{C}^{p+m})$. *Moreover,* Λ *is uniquely determined up to a* $(p+m)\times(p+m)$*-matrix right constant factor* U *satisfying* $U^* \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right] U = \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]$.

We point out that in fact conditions $\mathrm{S}'1$, $\mathrm{S}'2$, and $\mathrm{S}'3$ already determine Λ uniquely up to a constant (without the additional condition on the boundedness of $M_V P_{H_2} M_\Lambda$). Indeed if Λ and Λ' satisfy $\mathrm{S}'1$, $\mathrm{S}'2$, and $\mathrm{S}'3$, then $\Lambda\Lambda'^{-1}(z)$ is analytic on $\mathbb{D}$ and satisfies

$$(\Lambda\Lambda'^{-1})^*(\zeta) \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} (\Lambda\Lambda'^{-1})(\zeta) = \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} \quad \text{for } \zeta \in \mathbb{T}. \tag{2.10}$$

We then use the formula $\left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right] (\Lambda\Lambda'^{-1})^{*-1}(1/\overline{z}) \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]$ to analytically continue $\Lambda\Lambda'^{-1}$ to the exterior of the unit disk. From (2.10) we see that the nontangential boundary values from outside the disk agree with the nontangential boundary values from inside the disk. By using Lemma 6.6 from [32, page 223], we see that the analytic continuation passes through the unit circle as well. Then by Liouville's theorem we see that $\Lambda\Lambda'^{-1}$ must be an invertible constant matrix U. Since Λ and Λ' both satisfy $\mathrm{S}'1$, we see next that the constant matrix U must also satisfy $U^* \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right] U = \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]$.

We remark that the version of Theorem 2.9 as formulated in [10] uses the invertibility of the singular integral operator (condition (3) in Proposition 2.8) as the operator theory condition equivalent to the existence of the so-called canonical generalized factorization with respect to L_2.

2.3. Proof of Theorem 2.2 via the Adamjan–Arov–Kreĭn (AAK) theorem. The following result of Adamjan, Arov, and Kreĭn (see [1]) also gives a parameterization of the set of all solutions of the (discrete-time) Nehari–Takagi problem under the assumption that $\sigma_k > \sigma > \sigma_{k+1}$. A thorough recent treatment of the AAK approach can be found in Peller [34].

Theorem 2.10. *Let* $G \in H_\infty(\mathbb{D}_e)$ *and let* $H_G \colon H_2(\mathbb{D}, \mathbb{C}^m) \mapsto H_2(\mathbb{D}, \mathbb{C}^p)^\perp$ *be the associated Hankel operator, with singular values* $\sigma_1 \geq \sigma_2 \geq \cdots (\geq 0)$. *Suppose that* $\sigma_k > \sigma > \sigma_{k+1}$. *Define*

$$\Theta(\zeta) = \begin{bmatrix} \Theta_{11}(\zeta) & \Theta_{12}(\zeta) \\ \Theta_{21}(\zeta) & \Theta_{22}(\zeta) \end{bmatrix} \in L_2(\mathbb{T}, \mathbb{C}^{(p+m)\times(p+m)})$$

by (*viewed as an operator from* $\mathbb{C}^{p+m}$ *into* $L_2(\mathbb{T}, \mathbb{C}^{p+m})$)

$$\Theta = \begin{bmatrix} \zeta \cdot Z_* e_*^* \gamma_* & H_G Z e^* \gamma \\ \zeta \cdot H_G^* Z_* e_*^* \gamma_* & Z e^* \gamma \end{bmatrix}, \tag{2.11}$$

where $e_* \colon L^2(\mathbb{T}, \mathbb{C}^p) \mapsto \mathbb{C}^p$, $e \colon L^2(\mathbb{T}, \mathbb{C}^m) \mapsto \mathbb{C}^m$, $Z \colon H_2(\mathbb{D}, \mathbb{C}^m) \mapsto H_2(\mathbb{D}, \mathbb{C}^m)$, $Z_* \colon H_2(\mathbb{D}, \mathbb{C}^p) \mapsto H_2(\mathbb{D}, \mathbb{C}^p)$, $\gamma \colon \mathbb{C}^m \mapsto \mathbb{C}^m$, *and* $\gamma_* \colon \mathbb{C}^p \mapsto \mathbb{C}^p$ *are given by*

$$\begin{aligned} &e_* \colon \textstyle\sum_{j=-\infty}^{\infty} \zeta^j f_j \mapsto f_{-1}, && e \colon \textstyle\sum_{j=-\infty}^{\infty} \zeta^j g_j \mapsto g_0, \\ &Z = (I - \sigma^{-2} H_G^* H_G)^{-1}, && Z_* = (I - \sigma^{-2} H_G H_G^*)^{-1}, \\ &\gamma = (eZe^*)^{-1/2}, && \gamma_* = (e_* Z_* e_*^*)^{-1/2}. \end{aligned}$$

Then the conclusion of Theorem 2.2 *holds with* Θ *given by* (2.11) *rather than by* (2.2).

In [4] it is argued that one way to compute a function $\Theta \in L_2(\mathbb{D}, \mathbb{C}^{(p+m)\times(p+m)})$ meeting the requirement in Theorem 2.2 is as follows: Θ should satisfy the conditions

C′1. $\Theta(\zeta)^* \left[\begin{smallmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{smallmatrix}\right] \Theta(\zeta) = \left[\begin{smallmatrix} I_p & 0 \\ 0 & I_q \end{smallmatrix}\right]$ and

C′2. the columns of Θ should form a basis for the "wandering subspace" $\mathcal{L}$ associated with the problem

$$\Theta \cdot \mathbb{C}^{p+m} = \mathcal{L} := \mathcal{M} \ominus_{J_\sigma} \zeta \cdot \mathcal{M},$$

where we have set

$$\mathcal{M} := \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} \cdot H_2(\mathbb{D}, \mathbb{C}^{p+m})$$

and where the notation $\ominus_{J_\sigma}$ refers to the orthogonal difference in the indefinite inner product $\langle \cdot, \cdot \rangle_{J_\sigma}$ on $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ induced by J_σ:

$$\langle f, g \rangle_{J_\sigma} = \frac{1}{2\pi} \int_{\mathbb{T}} \langle J_\sigma f(\zeta), g(\zeta) \rangle_{\mathbb{C}^{p+m}} \, |d\zeta|.$$

In fact, this construction is very close to that in [10] for the construction of Wiener–Hopf factors under the assumption that the associated singular integral operator is invertible as discussed in section 2.2. Furthermore, in [2] it is verified that Θ as defined in (2.11) meets the criteria C′1 and C′2. In this way we have a proof of Theorem 2.2 which ultimately rests on the main result from [1].

2.4. Proof of Theorem 2.2 via Kreĭn-space projective geometry: The Ball–Helton approach. This approach, originating in [4] (see also [39] and [2]), relies on a projective geometry of Kreĭn spaces. The method is reasonably straightforward in case the spectral factor Λ and its inverse V are bounded (and hence also Θ is bounded), but there are some extra complications for the general case. Since these extra complications remained a little obscure in the original exposition [4], we now revisit the ideas there in an attempt to make them more accessible for the system-theory community. For basic background concerning Kreĭn spaces, we refer to [9].

We first observe that the space $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ is a Kreĭn space in the J_σ inner product given by

$$\langle f, g \rangle_{J_\sigma} = \frac{1}{2\pi} \int_{\mathbb{T}} \langle J_\sigma f(\zeta), g(\zeta) \rangle_{\mathbb{C}^{p+m}} \, |d\zeta|.$$

A key role is played by the subspace $\mathcal{M}$ given by

$$\mathcal{M} = \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \cdot H_2(\mathbb{D}, \mathbb{C}^{p+m}) \subset L_2(\mathbb{T}, \mathbb{C}^{p+m}). \tag{2.12}$$

In general a subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{K}$ is said to be *regular* if it has a good orthogonal complement in the Kreĭn space inner product (i.e., if $\mathcal{K} = \mathcal{M} \dot{+} \mathcal{M}^{[\perp]}$, where $\dot{+}$ indicates direct-sum decomposition), where $\mathcal{M}^{[\perp]}$ indicates the orthogonal complement in the indefinite Kreĭn space inner product. We have the following characterization of when the subspace $\mathcal{M}$ given by (2.12) is a regular subspace of the Kreĭn spaces $(L_2(\mathbb{T}, \mathbb{C}^{p+m}), \langle \cdot, \cdot \rangle_{J_\sigma})$.

PROPOSITION 2.11. *The subspace $\mathcal{M}$ given by* (2.12) *is a regular subspace of the Kreĭn space $(L_2(\mathbb{T}, \mathbb{C}^{p+m}), \langle \cdot, \cdot \rangle_{J_\sigma})$ if and only if any one of the equivalent conditions in Proposition* 2.8 *holds.*

Proof. Note that, for $f, g \in H_2(\mathbb{D}, \mathbb{C}^{p+m})$, we have

$$\left\langle \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} f, g \right\rangle_{J_\sigma} = \left\langle J_\sigma \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} f, g \right\rangle_{L_2(\mathbb{T}, \mathbb{C}^{p+m})} = \langle T_{A^* J_\sigma A} f, g \rangle_{L_2(\mathbb{T}, \mathbb{C}^{p+m})},$$

where $A = \left[\begin{smallmatrix} I & G \\ 0 & I \end{smallmatrix}\right]$ is as in Proposition 2.8. Thus the map $U\colon f \mapsto \left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right] \cdot f$ is unitary from $H_2(\mathbb{D}, \mathbb{C}^{p+m})$ with the inner product induced by the Toeplitz operator $T_{A^* J_\sigma A}$ to $\mathcal{M}$ with the inner product induced by J_σ. A standard fact concerning Kreĭn spaces (see, e.g., [9]) is that a subspace of a Kreĭn space $\mathcal{K}$ is regular if and only if it is itself a Kreĭn space in the inner product inherited from $\mathcal{K}$. In the case at hand, by the indefinite-metric unitary property of U, this happens if and only if $H_2(\mathbb{T}, \mathbb{C}^{p+m})$ is a Kreĭn space in the inner product induced by $T_{A^* J_\sigma A}$; this in turn is equivalent to the invertibility of the Toeplitz operator $T_{A^* J_\sigma A}$, i.e., condition (2) in Proposition 2.8. Proposition 2.11 now follows. □

When $\mathcal{M}$ is a regular subspace of $(L_2(\mathbb{T}, \mathbb{C}^{p+m}), \langle \cdot, \cdot \rangle_{J_\sigma})$, we denote by $P_\mathcal{M}$ the projection of $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ onto $\mathcal{M}$ along $\mathcal{M}^{[\perp]}$. Then $P_\mathcal{M}$ is bounded as an operator on $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ and is self-adjoint in the J_σ-inner product:

$$\langle P_\mathcal{M} f, g \rangle_{J_\sigma} = \langle f, P_\mathcal{M} g \rangle_{J_\sigma}.$$

A key result from [4] is that when $\mathcal{M}$ is regular, then $\mathcal{M}$ has the following Beurling–Lax-type representation.

Theorem 2.12 (see [4, 5]). *Assume that the subspace $\mathcal{M}$ as in (2.12) is a regular subspace of $(L_2(\mathbb{T}, \mathbb{C}^{p+m}), \langle \cdot, \cdot \rangle_{J_\sigma})$. Then there is a matrix function $\Theta \in L_2(\mathbb{T}, \mathbb{C}^{p+m})$ such that*

(1) *$\mathcal{M} = L_2(\mathbb{T}, \mathbb{C}^{p+m})$-closure of $\Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$;*
(2) *$\Theta(\zeta)^* J_\sigma \Theta(\zeta) = J_1 := \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]$ for almost all $\zeta \in \mathbb{T}$;*
(3) *the operator $M_\Theta P_{H_2(\mathbb{D}, \mathbb{C}^{p+m})} M_\Theta^{-1}$ defines a bounded operator, namely, the J_σ-orthogonal projection $P_\mathcal{M}$ of $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ onto $\mathcal{M}$ along $\mathcal{M}^{[\perp]}$.*

Moreover, Θ is uniquely determined up to a constant J-unitary factor on the right, and in principle can be computed from the (J_1, J_σ)-unitary property (2) above, along with the condition that

$$\Theta \cdot \mathbb{C}^{p+m} = \mathcal{M} \ominus_{J_\sigma} \zeta \cdot \mathcal{M}.$$

Alternatively, Θ arises as $\Theta = \left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right] V$, where $V = \Lambda^{-1}$ and Λ is the spectral factor for $A^* J_\sigma A$ as in Theorem 2.9.

Remark 2.13. Note that if we set $A = \left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right]$ (with then $A^{-1} = \left[\begin{smallmatrix} I_p & -G \\ 0 & I_m \end{smallmatrix}\right]$), we have

$$M_\Theta P_{H_2(\mathbb{D}, \mathbb{C}^{p+m})} M_{\Theta^{-1}} = M_A M_V P_{H_2(\mathbb{D}, \mathbb{C}^{p+m})} M_\Lambda M_A^{-1},$$

where M_A and its inverse M_A^{-1} are bounded on $L_2(\mathbb{T}, \mathbb{C}^{p+m})$. In this way we see that the last part of condition (2) in Theorem 2.9 fits with condition (3) in Theorem 2.12.

The next step is to reformulate the Nehari–Takagi problem itself in terms of a certain graph subspace of $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ instead of in terms of the matrix function $K \in H_{\infty,k}(\mathbb{T}, \mathbb{C}^{p \times m})$. We shall work with the Kreĭn–Langer representation for an element K of $H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p \times m})$. Specifically, a matrix function $K \in L_\infty(\mathbb{T}, \mathbb{C}^{p \times m})$ is in the class $H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p \times m})$ if and only if K has a representation as $K = F \cdot B^{-1}$, where $F \in H_\infty(\mathbb{D}, \mathbb{C}^{p \times m})$ and $B \in H_\infty(\mathbb{D}, \mathbb{C}^{m \times m})$ is a Blaschke–Potapov product of degree k, and k is the smallest nonnegative integer for which such a representation is possible. Then we have the following reformulation of the Nehari–Takagi problem.

Proposition 2.14 (see [4, 2]). *The angle-operator–graph correspondence induces a one-to-one correspondence between solutions $K \in H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p \times m})$ of the Nehari–Takagi problem with datum $G \in L_\infty(\mathbb{T}, \mathbb{C}^{p \times m})$ and with index $k' \leq k$, on the one* *hand, and subspaces $\mathcal{G}$ of the Kreĭn space*

$$\mathcal{K} = \left(\begin{bmatrix} L_2(\mathbb{T}, \mathbb{C}^p) \\ H_2(\mathbb{D}, \mathbb{C}^m) \end{bmatrix}, \langle \cdot, \cdot \rangle_{J_\sigma} \right) \tag{2.13}$$

such that

(1) $\mathcal{G} \subset \mathcal{M} := \left[\begin{smallmatrix} I_p & 0 \\ 0 & I_m \end{smallmatrix}\right] \cdot H_2(\mathbb{D}, \mathbb{C}^{p+m})$,

(2) *$\mathcal{G}$ has codimension k in a maximal negative subspace of $\mathcal{K}$, and*

(3) *$\mathcal{G}$ is shift invariant, i.e., $\zeta \cdot \mathcal{G} \subset \mathcal{G}$,*

on the other hand, as follows. If $K \in H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$ has a representation as $K = FB^{-1}$, with $F \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ and with $B \in H_\infty(\mathbb{D}, \mathbb{C}^{m\times m})$ a Blaschke–Potapov product of degree k, and is such that $\|G + K\|_\infty \leq \sigma$, and if we set

$$\mathcal{G}_K = \begin{bmatrix} G + K \\ I \end{bmatrix} B \cdot H_2(\mathbb{T}, \mathbb{C}^{p\times m}) = \begin{bmatrix} GB + F \\ B \end{bmatrix} \cdot H_2(\mathbb{D}, \mathbb{C}^{p\times m}), \tag{2.14}$$

then $\mathcal{G}_K$ satisfies conditions (1), (2), *and* (3) *listed above. Conversely, if $\mathcal{G}$ satisfies conditions* (1), (2), *and* (3) *listed above, then necessarily there is a $K \in H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$ with $k' \leq k$ and with $K = FB^{-1}$ for a Blaschke–Potapov product of degree k' such that $\|G + K\|_\infty \leq \sigma$ and $\mathcal{G}$ has the form $\mathcal{G}_K$ as in* (2.14).

Proof. We defer the proof to Appendix A (see section 5.1). ◻

The next step is to note the geometric significance of the fact that $\sigma_k > \sigma > \sigma_{k+1}$.

PROPOSITION 2.15. *Assume that $G \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ has Hankel singular values $\sigma_1 \geq \sigma_2 \geq \cdots$ with $\sigma_k > \sigma > \sigma_{k+1}$, and define subspaces $\mathcal{M}$ and $\mathcal{K}$ of $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ as in* (2.12) *and* (2.13). *Then $\mathcal{M}$ is a regular subspace of $\mathcal{K}$, and the J_σ-orthogonal complement $\mathcal{K} \ominus_{J_\sigma} \mathcal{M}$ of $\mathcal{M}$ inside $\mathcal{K}$ has k negative squares.*

Proof. One can compute that the relative J_σ-orthogonal complement $\mathcal{K} \ominus_{J_\sigma} \mathcal{M}$ is given by

$$\mathcal{K} \ominus_{J_\sigma} \mathcal{M} = \begin{bmatrix} I \\ \sigma^{-2} H_G^* \end{bmatrix} H_2(\mathbb{D}, \mathbb{C}^p)^\perp.$$

Hence, the negative signature of $\mathcal{K} \ominus_{J_\sigma} \mathcal{M}$ is equal to the number of negative eigenvalues of the self-adjoint operator

$$\begin{bmatrix} I & \sigma^{-2} H_G \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I \\ \sigma^{-2} H_G^* \end{bmatrix} = I - \sigma^{-2} H_G H_G^*.$$

From the definition of singular values, we see that this quantity in turn is equal to k if $\sigma_k > \sigma > \sigma_{k+1}$, and the assertion follows. ◻

Proposition 2.15 enables us to adjust Proposition 2.14 to a more useful form as follows.

PROPOSITION 2.16. *Assume that $G \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ has Hankel singular values $\sigma_1 \geq \sigma_2 \geq \cdots$ satisfying $\sigma_k > \sigma > \sigma_{k+1}$ as in Proposition* 2.15. *Then the angle-operator–graph correspondence as sketched in Proposition* 2.14 *induces a one-to-one correspondence between solutions $K \in H_{\infty,k}(\mathbb{D}, \mathbb{C}^{p\times m})$ of the Nehari–Takagi problem with datum $G \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ and with index k, on the one hand, and subspaces $\mathcal{G}$ of the Kreĭn space $\mathcal{K}$ as in* (2.13) *such that*

(1) $\mathcal{G} \subset \mathcal{M} := \left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right] \cdot H_2(\mathbb{D}, \mathbb{C}^{p+m})$,

(2) *$\mathcal{G}$ is a maximal negative subspace as a subspace of $\mathcal{M}$, and*

(3) *$\mathcal{G}$ is shift invariant, i.e., $\zeta \cdot \mathcal{G} \subset \mathcal{G}$,*

on the other hand.

Proof. We defer the proof to Appendix A (see section 5.2). ◻

Proposition 2.16 reduces the description of all solutions K of the Nehari–Takagi problem to a description of all shift-invariant subspaces $\mathcal{G}$ of $\mathcal{M}$ which are maximal negative as a subspace of $\mathcal{M}$. The next proposition gives a characterization of these subspaces; it is at this point that we use the Beurling–Lax representation of $\mathcal{M}$ given in Theorem 2.12.

PROPOSITION 2.17. *Assume that $K \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ with Hankel singular values $\sigma_1 > \sigma_2 > \cdots$ satisfying $\sigma_k > \sigma > \sigma_{k+1}$ as in Proposition* 2.16. *As in* (2.12), *let $\mathcal{M}$ be considered as a Kreĭn space in the J_σ-inner product, and let $\Theta \in L_2(\mathbb{T}, \mathbb{C}^{p\times m})$ be the J_σ-Beurling–Lax representer for $\mathcal{M}$ as in Theorem* 2.12. *Then a subspace $\mathcal{G}$ of $\mathcal{M}$ satisfies conditions* (1), (2), *and* (3) *in Proposition* 2.16 *if and only if there is a matrix function $Q \in H_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ such that*

$$\mathcal{G} = L_2(\mathbb{T}, \mathbb{C}^{p+m})\text{-}closure\ of\ \Theta \begin{bmatrix} Q \\ I \end{bmatrix} \cdot H_\infty(\mathbb{D}, \mathbb{C}^m) \tag{2.15}$$

for a uniquely determined matrix function $Q \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ with $\|Q\|_\infty \le 1$.

Proof. The proof is deferred to Appendix A (see section 5.3). ◻

We are now ready to put all the pieces together to complete the proof of Theorem 2.2

Proof of Theorem 2.2. By combining Propositions 2.14 and 2.16 with Proposition 2.17, we see that K solves the Nehari–Takagi problem if and only if K has a Kreĭn–Langer factorization $Q = FB^{-1}$ (where $F \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ and $B \in H_\infty(\mathbb{D}, \mathbb{C}^{m\times m})$ is a Blaschke–Potapov product of degree k) such that

$$\begin{bmatrix} G+K \\ I_m \end{bmatrix} B \cdot H_2(\mathbb{D}, \mathbb{C}^m) \cap \Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m}) = \Theta \begin{bmatrix} Q \\ I_m \end{bmatrix} \cdot H_\infty(\mathbb{D}, \mathbb{C}^m)$$

for a uniquely determined $Q \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ with $\|Q\|_\infty \le 1$. In particular, we see that for each of the standard basis vectors $e_1, \dots, e_m$ in $\mathbb{C}^m$ there must be corresponding vector functions $f_1, \dots, f_m \in BH_2(\mathbb{D}, \mathbb{C}^m)$ so that

$$\begin{bmatrix} G+K \\ I_m \end{bmatrix} f_j = \Theta \begin{bmatrix} Q \\ I_m \end{bmatrix} e_j,$$

or, in operator form,

$$\begin{bmatrix} G+K \\ I_m \end{bmatrix} F = \begin{bmatrix} \Theta_{11}Q + \Theta_{12} \\ \Theta_{21}Q + \Theta_{22} \end{bmatrix}.$$

From the bottom component we read off that $F = \Theta_{21}Q + \Theta_{22}$; then the top component gives

$$(G+K)(\Theta_{21}Q + \Theta_{22}) = \Theta_{11}Q + \Theta_{12}.$$

Once we confirm that $F(\zeta)^{-1} = (\Theta_{21}(\zeta)Q(\zeta) + \Theta_{22}(\zeta))^{-1}$ makes sense for almost all $\zeta \in \mathbb{T}$, we can solve for $G + K$ and arrive at the formula (2.3) for $G + K$. As all the analysis is necessary and sufficient, this will then complete the proof of Theorem 2.2.

We can see that $\Theta_{21}(\zeta)Q(\zeta)+\Theta_{22}(\zeta)$ is invertible for almost all $\zeta \in \mathbb{T}$ by the following geometric argument; for those readers who would prefer an analytic argument, we also give an analytic proof of the same point in the next section. By our construction we have that the linear manifold

$$\begin{bmatrix} G+K \\ I_m \end{bmatrix} F \cdot H_\infty(\mathbb{D}, \mathbb{C}^m)$$

is dense in a shift-invariant subspace $\mathcal{G}_{G+K}$ of $\mathcal{K} = \left[\begin{smallmatrix} L_2(\mathbb{T},\mathbb{C}^p) \\ H_2(\mathbb{D},\mathbb{C}^m) \end{smallmatrix}\right]$ which has codimension k in a maximal J_σ-negative subspace of $\mathcal{K}$. By the angle-operator–graph correspondence for J_σ-negative subspaces, equivalently $\|G+K\|_\infty \le \sigma$ and the L_2-closure of $FH_\infty(\mathbb{D},\mathbb{C}^m)$ has the form $BH_2(\mathbb{D},\mathbb{C}^m)$ for a Blaschke–Potapov product in $H_\infty(\mathbb{D},\mathbb{C}^{m\times m})$ of degree k. For this to occur, it is necessarily the case that $\det F(\zeta) \neq 0$ for almost all $\zeta \in \mathbb{T}$. The proof of Theorem 2.2 (via Kreĭn-space projective geometry) is now complete. $\square$

2.5. Proof of Theorem 2.2 via a winding number argument. It is also possible to bypass the Kreĭn-space geometry ideas and give a more analytic, less geometric proof for most of the content of Theorem 2.2, as we now show. The main idea for this approach comes from [6]; it can also be considered as a purely frequency-domain version of the state-space solution given in [3] for the rational case. One key point of Theorem 2.2 is that every solution of the Nehar–Takagi problem arises from a contractive H_∞-free parameter via the linear-fractional map; for the proof of this part we translate the ideas from the Grassmannian approach to the more analytic setting here.

The starting point for this alternative derivation is still the Beurling–Lax representation for the subspace $\mathcal{M}$ given in Theorem 2.12. Under the assumption that $\mathcal{M}$ is a *regular* subspace of $(L_2(\mathbb{T},\mathbb{C}^{p+m}), \langle\cdot,\cdot\rangle_{J_\sigma})$, we know that $L_2(\mathbb{T},\mathbb{C}^{p+m})$ has a direct sum decomposition

$$L_2(\mathbb{T},\mathbb{C}^{p+m}) = \mathcal{M}^{[\perp]} \dot{+} \mathcal{M},$$

and hence there is a bounded projection operator $P_\mathcal{M}$ from $L_2(\mathbb{T},\mathbb{C}^{p+m})$ onto $\mathcal{M}$ along the J_σ-orthogonal complement $\mathcal{M}^{[\perp]}$ of $\mathcal{M}$. In addition, in this setup the projection operator $P_\mathcal{M}$ is J_σ-self-adjoint in the sense that

$$\langle P_\mathcal{M} f, g\rangle_{J_\sigma} = \langle f, P_\mathcal{M} g\rangle_{J_\sigma} \quad \text{for all } f, g \in L_2(\mathbb{T},\mathbb{C}^{p+m}).$$

In addition, we shall have use for the operator $P_-^* P_\mathcal{M} P_-$ on $H_2(\mathbb{D},\mathbb{C}^m)$, where we have set

$$P_- = \begin{bmatrix} 0_{p\times m} \\ I_m \end{bmatrix} : H_2(\mathbb{D},\mathbb{C}^m) \mapsto L_2(\mathbb{T},\mathbb{C}^{p+m}).$$

Note that then

$$P_-^* = \begin{bmatrix} 0_{m\times p} & P_{H_2(\mathbb{D},\mathbb{C}^m)} \end{bmatrix} : L_2(\mathbb{T},\mathbb{C}^{p+m}) \mapsto H_2(\mathbb{D},\mathbb{C}^m).$$

Proposition 2.18. *Assume that $\mathcal{M}$ as in* (2.12) *is regular and that $\mathcal{M}$ has the J_σ-Beurling–Lax representation $\mathcal{M} = L_2$-closure of $\Theta \cdot H_\infty(\mathbb{D},\mathbb{C}^{p+m})$ as in Theorem* 2.12. *Then the following hold:*

(1) *J_σ-orthogonal projection of $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ onto $\mathcal{M}$ can be computed either in terms of G as*

$$P_{\mathcal{M}} = \begin{bmatrix} P_{H_2(\mathbb{D},\mathbb{C}^p)} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H_G \\ I \end{bmatrix} \left(H_G^* H_G - \sigma^2 I_m\right)^{-1} \begin{bmatrix} H_G^* & -\sigma^2 I_m \end{bmatrix} \tag{2.16}$$

or in terms of Θ as

$$P_{\mathcal{M}} = M_\Theta P_{H_2(\mathbb{D},\mathbb{C}^{p+m})} M_\Theta^{-1}. \tag{2.17}$$

(2) *The operator $P_-^* P_{\mathcal{M}} P_-$ can be expressed in two ways:*

$$P_-^* P_{\mathcal{M}} P_- = -\sigma^2 (H_G^* H_G - \sigma^2 I_m)^{-1} \tag{2.18}$$

$$= -\sigma^2 M_{[\Theta_{21} \ \ \Theta_{22}]} P_{H_2(\mathbb{D},\mathbb{C}^{p+m})} M_{[\Theta_{21} \ \ -\Theta_{22}]^*}. \tag{2.19}$$

(3) *The number k of negative eigenvalues of the self-adjoint operator $P_-^* P_{\mathcal{M}} P_-$ on $H_2(\mathbb{D}, \mathbb{C}^m)$ can be expressed either as*

$$k = \text{ the number of Hankel singular values } > \sigma \tag{2.20}$$

or as the number of negative squares of the kernel

$$\frac{\Theta_{22}(z)\Theta_{22}(w)^* - \Theta_{21}(z)\Theta_{21}(w)^*}{1 - z\overline{w}}. \tag{2.21}$$

Consequently, the matrix function $\Theta_{22}^{-1}\Theta_{21} \in H_{\infty,k}(\mathbb{D}, \mathbb{C}^{m\times p})$ with $\|\Theta_{22}^{-1} \cdot \Theta_{21}\|_\infty \le 1$, and Θ_{22} has outer-inner factorization $\Theta_{22} = F \cdot B$, where $F \in H_2(\mathbb{D}, \mathbb{C}^{m\times m})$ is outer and $B \in H_\infty(\mathbb{D}, \mathbb{C}^{m\times m})$ is a Blaschke–Potapov product of degree k.

Proof. To prove (2.16) note that $\mathcal{M}$ has J_σ-orthogonal decomposition

$$\mathcal{M} = \begin{bmatrix} H_2(\mathbb{D}, \mathbb{C}^p) \\ 0 \end{bmatrix} \oplus_{J_\sigma} \begin{bmatrix} H_G \\ I_m \end{bmatrix} H_2(\mathbb{D}, \mathbb{C}^m). \tag{2.22}$$

The J_σ-orthogonal projection onto im $\left[\begin{smallmatrix} H_G \\ I_m \end{smallmatrix}\right]$ can be computed as

$$P_{\mathrm{im}\left[\begin{smallmatrix} H_G \\ I \end{smallmatrix}\right]} = \begin{bmatrix} H_G \\ I_m \end{bmatrix} \left(\begin{bmatrix} H_G \\ I \end{bmatrix}^{[*]} \begin{bmatrix} H_G \\ I_m \end{bmatrix} \right)^{-1} \begin{bmatrix} H_G \\ I \end{bmatrix}^{[*]}. \tag{2.23}$$

Here we view $\left[\begin{smallmatrix} H_G \\ I_m \end{smallmatrix}\right]$ as an operator acting from $H_2(\mathbb{D}, \mathbb{C}^m)$ with the standard Hilbert space inner product into $L_2(\mathbb{T}, \mathbb{C}^{p+m})$ with the J_σ-inner product. Hence

$$\begin{bmatrix} H_G \\ I_m \end{bmatrix}^{[*]} = \begin{bmatrix} H_G \\ I_m \end{bmatrix}^* J_\sigma = \begin{bmatrix} H_G^* & -\sigma^2 I \end{bmatrix}. \tag{2.24}$$

Substituting (2.24) into (2.23) and using (2.22) then gives the formula (2.16) for $P_{\mathcal{M}}$.

Formula (2.17) for $P_{\mathcal{M}}$ was already noted as condition (3) of Theorem 2.12.

Formula (2.18) now follows upon multiplying (2.16) on the left by $[0 \ \ P_{H_2(\mathbb{D},\mathbb{C}^m)}]$ and on the right by $\left[\begin{smallmatrix} 0 \\ I_m \end{smallmatrix}\right]$ (considered as acting from $H_2(\mathbb{D}, \mathbb{C}^m)$ into $L_2(\mathbb{T}, \mathbb{C}^{p+m})$).

Formula (2.20) for the number of negative eigenvalues of $P_-^* P_{\mathcal{M}} P_-$ can now be read off immediately from formula (2.18) for $P_- P_{\mathcal{M}} P_-$. To get formula (2.21) for the number of negative eigenvalues of $P_-^* P_{\mathcal{M}} P_-$, we use (2.19) to compute, where we set

$k_w(\zeta) = \frac{1}{1-\zeta\overline{w}}$ equal to the kernel function for $H_2(\mathbb{D}, \mathbb{C})$, for any $w_1, \dots, w_N \in \mathbb{D}$ and $x_1, \dots, x_N \in \mathbb{C}^m$,

$$\left\langle P_- P_{\mathcal{M}} P_- \left(\sum_{j=1}^N k_{w_j} x_j \right), \sum_{i=1}^N k_{w_i} x_i \right\rangle_{H_2(\mathbb{D},\mathbb{C}^m)}$$
$$= -\sigma^2 \sum_{i,j=1}^N \left\langle \left(M_{\Theta_{21}} P_{H_2} M_{\Theta_{21}^*} - M_{\Theta_{22}} P_{H_2} M_{\Theta_{22}^*} \right) k_{w_j} x_j, k_{w_i} x_i \right\rangle_{H_2(\mathbb{D},\mathbb{C}^m)}$$
$$= -\sigma^2 \sum_{i,j=1}^N x_i^* \frac{\Theta_{21}(w_i)\Theta_{21}(w_j)^* - \Theta_{22}(w_i)\Theta_{22}(w_j)^*}{1 - w_i \overline{w_j}} x_j.$$

By the density of the span of the kernel functions $\{k_w x \colon w \in \mathbb{D},\ x \in \mathbb{C}^m\}$, the formula (2.21) for the number of negative eigenvalues for $P_-^* P_{\mathcal{M}} P_-$ now follows.

Finally, from the (J, J_σ)-unitary property of Θ we know that $\Theta(\zeta)^{-1} = J_1 \Theta(\zeta)^* J_\sigma$ for almost all $\zeta \in \mathbb{T}$, and hence

$$\Theta(\zeta) J_1 \Theta(\zeta)^* = J_{\sigma^{-1}}.$$

In particular,

$$\Theta_{21}(\zeta)\Theta_{21}(\zeta)^* - \Theta_{22}(\zeta)\Theta_{22}(\zeta)^* = -\sigma^{-2} I_m$$

or

$$\Theta_{22}(\zeta)\Theta_{22}(\zeta)^* = \Theta_{21}(\zeta)\Theta_{21}(\zeta)^* + \sigma^{-2} I_m \geq \sigma^{-2} I_m \tag{2.25}$$

for almost all $\zeta \in \mathbb{T}$. Hence, for all such ζ, $\Theta_{22}(\zeta)$ is invertible and

$$0 \leq \Theta_{22}(\zeta)^{-1}\Theta_{21}(\zeta)\Theta_{21}(\zeta)^*\Theta_{22}(\zeta)^{*-1} = I_m - \sigma^{-2}\Theta_{22}(\zeta)^{-1}\Theta_{22}(\zeta)^{*-1} \leq I_m. \tag{2.26}$$

We conclude that $\Theta_{22}^{-1}\Theta_{21} \in L_\infty(\mathbb{T}, \mathbb{C}^{m \times p})$ with $\|\Theta_{22}^{-1}\Theta_{21}\| \leq 1$. Moreover, by conjugating the kernel in (2.21) by Θ_{22}^{-1} (multiplying by $\Theta_{22}(z)^{-1}$ on the left and by $\Theta_{22}(w)^{*-1}$ on the right for the generic sets of z and w for which these are defined), we see that the kernel

$$\frac{I_m - (\Theta_{22}^{-1}\Theta_{21})(z)(\Theta_{22}^{-1}\Theta_{21})(w)^*}{1 - z\overline{w}}$$

also has k negative squares on $\mathbb{D} \times \mathbb{D}$, i.e., $\Theta_{22}^{-1}\Theta_{21} \in H_{\infty,k}(\mathbb{D}, \mathbb{C}^{m \times p})$ with $\|\Theta_{22}^{-1}\Theta_{21}\|_\infty \leq 1$. Thus $\Theta_{22}^{-1}\Theta_{21}$ has a left Kreĭn–Langer factorization $\Theta_{22}^{-1}\Theta_{21} = B^{-1}F$ with B an $m \times m$ Blaschke–Potapov product of degree k and $F \in H_\infty(\mathbb{D}, \mathbb{C}^{m \times p})$. From the fact that $\mathcal{M} = L_2$-closure of $\Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{m+p})$ and the fact that $\begin{bmatrix} 0 & I_m \end{bmatrix} \mathcal{M} \subset H_2(\mathbb{D}, \mathbb{C}^m)$, we see that the matrix entries of both Θ_{21} and Θ_{22} are in H_2. From $\Theta_{22}^{-1}\Theta_{21} = B^{-1}F$ we conclude that Θ_{22} must have outer-inner factorization of the form $\Theta_{22} = \Theta_{22,o} \cdot B$ with $\Theta_{22,o} \in H_2(\mathbb{D}, \mathbb{C}^{m \times m})$ outer and $B \in H_\infty(\mathbb{D}, \mathbb{C}^{m \times m})$ a Blaschke–Potapov product of degree k. This completes the proof of Proposition 2.18. ☐

Winding number proof of Theorem 2.2. Suppose that $Q \in H_\infty(\mathbb{D}, \mathbb{C}^{p \times m})$ with $\|Q\|_\infty \leq 1$. From (2.25) and (2.26) we see that $\Theta_{22}(\zeta)$ is invertible and that $\|(\Theta_{22}^{-1}\Theta_{21})(\zeta)\| < 1$ for almost all $\zeta \in \mathbb{T}$. Hence the quantity

$$\Theta_{21}(\zeta)Q(\zeta) + \Theta_{22}(\zeta) = \Theta_{22}(\zeta)(I_m + \Theta_{22}(\zeta)^{-1}\Theta_{21}(\zeta)Q(\zeta))$$

is invertible for almost all $\zeta \in \mathbb{T}$. We may then define a $p \times m$ matrix-valued function K on $\mathbb{T}$ by

$$K = (V_{11}Q + V_{12})(\Theta_{21}Q + \Theta_{22})^{-1}. \tag{2.27}$$

We verify that $K \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ and in fact that $\|G + K\|_\infty \leq \sigma$ as follows. Note that

$$\begin{aligned}\begin{bmatrix} G+K \\ I_m \end{bmatrix}(\Theta_{21}Q+\Theta_{22}) &= \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix}\begin{bmatrix} K \\ I_m \end{bmatrix}(\Theta_{21}Q+\Theta_{22}) = \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix}\begin{bmatrix} V_{11}Q+V_{12} \\ \Theta_{21}Q+\Theta_{22} \end{bmatrix} \\ &= \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} V \begin{bmatrix} Q \\ I_m \end{bmatrix} = \Theta \begin{bmatrix} Q \\ I_m \end{bmatrix}. \end{aligned} \tag{2.28}$$

(Here we use that $\left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right] \cdot V = \Theta$ and thus also $\begin{bmatrix} V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} \Theta_{21} & \Theta_{22} \end{bmatrix}$.) Consequently, considering the various expressions below as functions on $\mathbb{T}$, we have

$$\begin{aligned}(G+K)^*(G+K) - \sigma^2 I_m &= \begin{bmatrix} (G+K)^* & I_m \end{bmatrix} J_\sigma \begin{bmatrix} G+K \\ I_m \end{bmatrix} \\ &= (\Theta_{21}Q+\Theta_{22})^{*-1}\begin{bmatrix} Q^* & I_m \end{bmatrix} \Theta^* J_\sigma \Theta \begin{bmatrix} Q \\ I_M \end{bmatrix} (\Theta_{21}Q+\Theta_{22})^{-1} \\ &= (\Theta_{21}Q+\Theta_{22})^{*-1}(Q^*Q - I_m)(\Theta_{21}Q+\Theta_{22})^{-1} \leq 0,\end{aligned}$$

where the last step follows from the assumption that $\|Q\|_\infty \leq 1$. In particular, it follows that

$$\|K\|_\infty \leq \sigma + \|G\|_\infty \text{ whenever } K = (V_{11}Q+V_{12})(\Theta_{21}Q+\Theta_{22})^{-1} \text{ with } \|Q\|_\infty \leq 1. \tag{2.29}$$

Moreover, from (2.28) we see that $G + K$ is given in terms of Q, as in (2.3).

For the discussion in this paragraph we consider the special case $\|Q\|_\infty < 1$; in the end we shall use this special case to arrive at the general case by an approximation argument. We observed at the end of the proof of Proposition 2.18 that the matrix entries of Θ_{21} and Θ_{22} are all in H_2, and by Proposition 2.18 we know that Θ_{22} has outer-inner factorization $\Theta_{22} = F \cdot B$ with F outer and the inner factor B equal to a Blaschke–Potapov product of degree k. For any function f analytic on the disk $\mathbb{D}$ (with possibly finitely many exceptional points), set $f_r(z) = f(rz)$ for each $r < 1$. Then $\Theta_{22,r}$ still has the form $F'_r \cdot B'_r$ with F'_r outer and B'_r a Blaschke–Potapov product of degree k, as long as we take $r < 1$ sufficiently close to 1. Moreover, by Proposition 2.6, we know that there is an $r_0 < 1$ such that, for all r subject to $r_0 \leq r < 1$, we have $\|\Theta_{22,r}^{-1}\Theta_{21,r}\|_\infty \leq \frac{1}{2}(1 + \|Q\|_\infty^{-1})$, with the consequence that

$$\begin{aligned}\|\Theta_{22,r}^{-1}\Theta_{21,r}Q_r\|_\infty &\leq \frac{1}{2}\|Q_r\|_\infty(1+\|Q\|_\infty^{-1}) \\ &\leq \frac{1}{2}(\|Q\|_\infty + 1) < 1 \quad \text{for all } r_0 \leq r < 1.\end{aligned} \tag{2.30}$$

By the Neumann series estimate, it follows that $(I + \Theta_{22,r}^{-1}\Theta_{21,r}Q_r)$ is invertible in $L_\infty(\mathbb{T}, \mathbb{C}^{m\times m})$ with

$$\|(I + \Theta_{22,r}^{-1}\Theta_{21,r}Q_r)^{-1}\|_\infty \leq \frac{1}{1 - \frac{1}{2}(1+\|Q\|_\infty)} = \frac{2}{1 - \|Q\|_\infty}. \tag{2.31}$$

Another consequence of the estimate (2.30) is that the determinant of $(I+\Theta_{22,r}^{-1}\Theta_{21,r} Q_r)$ has winding number around the unit circle equal to zero. As $\det \Theta_{22,r} = \det(F_r')\cdot \det B_r'$ has winding number equal to k (since F_r' is outer and B_r' is a matrix Blaschke–Potapov product of degree k), it follows that the determinant of

$$\Theta_{21,r}Q_r + \Theta_{22,r} = \Theta_{22,r}\left(\Theta_{22,r}^{-1}\Theta_{21,r}Q_r + I_m\right)$$

has winding number equal to k around the unit circle. As $\Theta_{21,r}Q_r+\Theta_{22,r}$ is in the disk algebra (analytic on the open disk and continuous on the closed disk), we conclude that $(\Theta_{21,r}Q_r + \Theta_{22,r})^{-1}$ is in $H_{\infty,k}(\mathbb{D},\mathbb{C}^{m\times m})$.

We now return to the case of a general $Q \in H_\infty(\mathbb{D},\mathbb{C}^{p\times m})$ with $\|Q\| \le 1$. Let s be a number with $0 < s < 1$. Then the above analysis applies to the situation where we have $s\cdot Q$ in place of Q. Thus

$$\|(\Theta_{22,r}^{-1}\Theta_{21,r}(sQ_r) + I)^{-1}\|_\infty \le \frac{2}{1-s\|Q\|_\infty} \le \frac{2}{1-s} \tag{2.32}$$

for all $r < 1$ sufficiently close to 1. Also, from (2.26) we read off that $\|\Theta_{22}^{-1}\|_\infty \le \sigma$; by Proposition 2.6 we then have $\|\Theta_{22,r}^{-1}\|_\infty \le \sigma+\epsilon$ for any given $\epsilon > 0$ as long as we take $r < 1$ sufficiently close to 1. Hence, for all $r < 1$ but sufficiently close to 1, we have

$$\begin{aligned}\|(\Theta_{21,r}(sQ_r) + \Theta_{22,r})^{-1}\|_\infty &= \|(\Theta_{22,r}^{-1}\Theta_{21,r}(sQ_r) + I)^{-1}\Theta_{22,r}^{-1}\|_\infty \\ &\le \left(\frac{2}{1-s}\right)\cdot(\sigma+\epsilon) < \infty.\end{aligned}$$

We conclude that $(\Theta_{21,r}(sQ_r) + \Theta_{22,r})^{-1}$ converges pointwise boundedly, and hence, by part (2) of Proposition 2.3, also in the $L_\infty(\mathbb{T},\mathbb{C}^{m\times m})$-weak-$*$ topology, to $(\Theta_{21}(sQ) +\Theta_{22})^{-1}$ as $r\to 1$. As we have seen above, each $(\Theta_{21,r}(sQ_r)+\Theta_{22,r})^{-1}$ is in $H_{\infty,k}(\mathbb{D}, \mathbb{C}^{m\times m})$. By Proposition 2.4, we conclude that $(\Theta_{21}(sQ)+\Theta_{22})^{-1} \in H_{\infty,k'}(\mathbb{D},\mathbb{C}^{m\times m})$ with $k' \le k$. But then $K_s := (V_{11}(sQ) + V_{12})(\Theta_{21}(sQ) + \Theta_{22})^{-1}$ is in $(H_2(\mathbb{D},\mathbb{C}^{p\times m})\cdot H_{\infty,k'}) \cap L_\infty$, and hence is in fact in $H_{\infty,k'}(\mathbb{D},\mathbb{C}^{p\times m})$ for some $k'\le k$. By (2.29), we know that $\|K_s\|_\infty \le \sigma + ||G||_\infty$ for all $s < 1$. Hence, by another application of part (2) of Proposition 2.3 combined with Proposition 2.4, we may let $s\to 1$ and conclude that $K = (V_{11}Q + V_{12})(\Theta_{21}Q + \Theta_{22})^{-1}$ is in $H_{\infty,k'}(\mathbb{D},\mathbb{C}^{p\times m})$ for some $k'\le k$. Since we have already verified that $\|G + K\|_\infty \le \sigma$ and we know by Proposition 2.7 that k is the smallest possible index for a solution to the Nehari–Takagi problem to exist for level σ if $\sigma_k > \sigma > \sigma_{k+1}$, we conclude that necessarily $k' = k$. We have now verified that the formula (2.3) provides a solution K to the Nehari–Takagi problem as asserted in Theorem 2.2.

Conversely, suppose that $K \in H_{\infty,k}(\mathbb{D},\mathbb{C}^{p\times m})$ provides a solution of the Nehari–Takagi problem. Then K has a Kreĭn–Langer factorization $K = F'B'^{-1}$, where $F' \in H_\infty(\mathbb{D},\mathbb{C}^{p+m})$ and $B' \in H_\infty(\mathbb{D},\mathbb{C}^{m\times m})$ is a Blaschke–Potapov product of degree k. Then

$$\begin{aligned}\begin{bmatrix} G+K \\ I_m\end{bmatrix} B' &= \begin{bmatrix} I_p & G \\ 0 & I_m\end{bmatrix}\begin{bmatrix} K \\ I_m\end{bmatrix} B' \\ &= \Theta\cdot\Lambda\cdot\begin{bmatrix} F' \\ B'\end{bmatrix} \\ &= \Theta\cdot\begin{bmatrix}\Lambda_{11}F' + \Lambda_{12}B' \\ \Lambda_{21}F' + \Lambda_{22}B'\end{bmatrix} =: \Theta\cdot\begin{bmatrix} Q_1 \\ Q_2\end{bmatrix},\end{aligned} \tag{2.33}$$

where $Q_1 := \Lambda_{11}F' + \Lambda_{12}B' \in H_2(\mathbb{D}, \mathbb{C}^{p\times m})$ and $Q_2 := \Lambda_{21}F' + \Lambda_{22}B' \in H_2(\mathbb{D}, \mathbb{C}^{m\times m})$. Since $\|G+K\|_\infty \le \sigma$ by assumption,

$$\begin{aligned}
0 &\ge B'^* \left((G+K)^*(G+K) - \sigma^2 I_m\right) B' \\
&= B'^* \begin{bmatrix} (G+K)^* & I_m \end{bmatrix} J_\sigma \begin{bmatrix} G+K \\ I_m \end{bmatrix} B' \\
&= \begin{bmatrix} Q_1^* & Q_2^* \end{bmatrix} \Theta^* J_\sigma \Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\
&= Q_1^* Q_1 - Q_2^* Q_2 \qquad (2.34)
\end{aligned}$$

a.e. on $\mathbb{T}$. We conclude that

$$Q_2(\zeta)x(\zeta) = 0 \Rightarrow Q_1(\zeta)x(\zeta) = 0. \qquad (2.35)$$

From the definition of Q_1 and Q_2 in (2.33) we see that

$$B' = \Theta_{21}Q_1 + \Theta_{22}Q_2. \qquad (2.36)$$

Hence (2.35) forces $B'(\zeta)x(\zeta) = 0$ as well, and hence $x(\zeta) = 0$ for almost all $\zeta \in \mathbb{T}$. We conclude that $Q_2(\zeta)$ is invertible a.e. on $\mathbb{T}$ and $Q(\zeta) := Q_1(\zeta)Q_2(\zeta)^{-1}$ makes sense. The calculation (2.34) then implies that $\|Q\|_\infty \le 1$, while (2.33) shows that we recover $G+K$ from Q as in the representation (2.3).

It remains to show that $Q \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$. For this piece of the argument we borrow some ideas from the Grassmannian approach. If $Q_2 H_\infty(\mathbb{C}, \mathbb{C}^m)$ is not dense in $H_2(\mathbb{D}, \mathbb{C}^m)$, we may choose a nonzero $h_0 \in H_\infty(\mathbb{D}, \mathbb{C}^m)$ lying in $H_2(\mathbb{D}, \mathbb{C}^m) \ominus \overline{Q_2 H_\infty(\mathbb{D}, \mathbb{C}^m)}$. Then

$$\begin{bmatrix} 0 \\ h_0 \end{bmatrix} \perp_{J_1} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} H_\infty(\mathbb{D}, \mathbb{C}^m).$$

Since $\Theta^* J_\sigma \Theta = J_1$ on $\mathbb{T}$, it then follows that

$$\Theta \begin{bmatrix} 0 \\ h_0 \end{bmatrix} = \begin{bmatrix} \Theta_{12}h_0 \\ \Theta_{22}h_0 \end{bmatrix} \perp_{J_\sigma} \Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} H_\infty(\mathbb{D}, \mathbb{C}^m) = \begin{bmatrix} G+K \\ I_m \end{bmatrix} B' H_\infty(\mathbb{D}, \mathbb{C}^m) \text{ (by (2.33))}.$$

Hence

$$\|\Theta_{12}h_0 + (G+K)B'h\|_2^2 \le \|\Theta_{22}h_0 + B'h\|_2^2$$

for all $h \in H_2(\mathbb{D}, \mathbb{C}^m)$. Therefore there is a contraction operator X from $\mathcal{D}_0 := \operatorname{span}\{\Theta_{22}h_0\} + B'H_2(\mathbb{D}, \mathbb{C}^m)$ into $L_2(\mathbb{T}, \mathbb{C}^p)$ such that

$$\begin{bmatrix} X \\ I_m \end{bmatrix} (h_0 + B'h) = \begin{bmatrix} \Theta_{12}h_0 \\ \Theta_{22}h_0 \end{bmatrix} + \begin{bmatrix} G+K \\ I_m \end{bmatrix} B'h \in \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} H_2(\mathbb{D}, \mathbb{C}^{p+m}) \qquad (2.37)$$

for all $h \in H_2(\mathbb{D}, \mathbb{C}^m)$. Note that $\left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right] H_2(\mathbb{D}, \mathbb{C}^{p+m})$ has a J_σ-orthogonal splitting

$$\begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} H_2(\mathbb{D}, \mathbb{C}^{p+m}) = \begin{bmatrix} H_2(\mathbb{D}, \mathbb{C}^p) \\ \{0\} \end{bmatrix} \oplus_{J_\sigma} \begin{bmatrix} H_G \\ I_m \end{bmatrix} \mathcal{E}_+ \oplus_{J_\sigma} \begin{bmatrix} H_G \\ I_m \end{bmatrix} \mathcal{E}_-, \qquad (2.38)$$

where $\mathcal{E}_+ = \operatorname{im} E((\sigma, +\infty))$ and $\mathcal{E}_- = \operatorname{im} E([0, \sigma))$ and where we have set $E(\cdot)$ equal to the spectral projection for the self-adjoint operator $(H_G^* H_G)^{1/2}$. Note that the first two direct summands in (2.38) are uniformly J_σ-positive, while the last is uniformly J_σ-negative. As $\|X\| \le 1$, the equality (2.37) forces the existence of a subspace $\mathcal{E}_-^0$ of $\mathcal{E}_-$ so that

$$\begin{bmatrix} X \\ I_m \end{bmatrix} \mathcal{D}_0 = \left\{ \begin{bmatrix} Y_2 e \\ 0 \end{bmatrix} + \begin{bmatrix} H_G \\ I_m \end{bmatrix} Y_1 e + \begin{bmatrix} H_G \\ I_m \end{bmatrix} e \colon e \in \mathcal{E}_-^0 \right\} \tag{2.39}$$

for operators $Y_1 \colon \mathcal{E}_-^0 \mapsto \mathcal{E}_+$ and $Y_2 \colon \mathcal{E}_-^0 \mapsto H_2(\mathbb{D}, \mathbb{C}^p)$. In particular,

$$\mathcal{D}_0 = \{Y_1 e + e \colon e \in \mathcal{E}_-^0\}. \tag{2.40}$$

But the subspace $\mathcal{D}_0 = \operatorname{span}\{\Theta_{22} h_0\} + B' H_2(\mathbb{D}, \mathbb{C}^m)$ has codimension $k-1$ in $H_2(\mathbb{D}, \mathbb{C}^m)$, while the subspace on the right in (2.40) has the same codimension in $H_2(\mathbb{D}, \mathbb{C}^m)$ as does $\mathcal{E}_-^0$. As $\mathcal{E}_-^0$ is a subspace of $\mathcal{E}_-$ which has codimension k in $H_2(\mathbb{D}, \mathbb{C}^m)$, we conclude that the right-hand side of (2.40) has codimension at most k in $H_2(\mathbb{D}, \mathbb{C}^m)$. In this way we arrive at a contradiction and conclude that necessarily Q_2 is outer. It now follows that $Q = Q_1 Q_2^{-1}$ is of bounded type with no inner factor in the denominator. This together with $Q \in L_\infty(\mathbb{T}, \mathbb{C}^{p\times m})$ gives us finally that $Q \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$, as wanted. ∎

Remark 2.19. *The band method.* A very flexible method for solving a variety of interpolation and extension problems which has evolved into increasing levels of sophistication over the past two decades is the so-called *band method* (see [22] for an excellent overview and [35] for one of the latest variations). Recent work (see [26]) enhances this abstract scheme to handle the Nehari–Takagi problem ($\sigma_{k+1} < \sigma < \sigma_k$ with $k \ge 1$) rather than merely the suboptimal Nehari problem ($\sigma_1 < \sigma$). However, the core of the method involves solving equations in a Wiener-like algebra; this limitation forces the spectral factor Λ and its inverse $\Lambda^{-1} = V$ (in the discrete-time setting) to be in $H_\infty(\mathbb{D}, \mathbb{C}^{(p+m)\times(p+m)})$ rather than merely in $H_2(\mathbb{D}, \mathbb{C}^{(p+m)\times(p+m)})$. A remaining open issue appears to be the extension of this abstract framework to include the situation studied in this paper.

3. State-space solutions. Let X be an arbitrary Hilbert space, and let A be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$. Let $B \in \mathcal{L}(\mathbb{C}^m, X)$, $C \in \mathcal{L}(X, \mathbb{C}^p)$. Assume that the triple (A, B, C) satisfies

A1. $B^*(\cdot I - A^*)^{-1} x \in H_2(\mathbb{C}_+, \mathbb{C}^m)$ (input stable),
A2. $C(\cdot I - A)^{-1} x \in H_2(\mathbb{C}_+, \mathbb{C}^p)$ (output stable),
A3. $C(\cdot I - A)^{-1} B u \in H_\infty(\mathbb{C}_+, \mathbb{C}^p)$ (input-output stable)

for all $x \in X$, $u \in \mathbb{C}^m$. Condition A3 holds if and only if $\mathcal{D} \in \mathcal{L}(L^2(\mathbb{R}_+; \mathbb{C}^m), L^2(\mathbb{R}_+; \mathbb{C}^p))$, where

$$(\mathcal{D}u)(t) = C \int_0^t T(t-s) B u(s)\, ds \quad (u \in L^2(\mathbb{R}_+; \mathbb{C}^m)). \tag{3.1}$$

Equation (3.1) is equivalent to $\widehat{\mathcal{D}u} = G\widehat{u}$, where $G(s) := C(sI - A)^{-1} B$ and $\widehat{u}$ denotes the Laplace transform of u ($\widehat{u}(s) := \int_0^\infty \mathrm{e}^{-st} u(t)\, du$). It is well known that $\|\mathcal{D}\| = \|G\|_{H_\infty}$. By Plancherel's theorem (and the closed graph theorem), A2 means that $\mathcal{C} : X \to L^2(\mathbb{R}_+; \mathbb{C}^p)$ is bounded, where $(\mathcal{C}x)(t) := CT(t)x$, $t \ge 0$. Hence it also follows that $L_C := \mathcal{C}^*\mathcal{C} \in \mathcal{L}(X)$. Similarly, A1 implies that $\mathcal{B}^{\mathrm{d}} : X \to L^2(\mathbb{R}_+; \mathbb{C}^m)$ is bounded, where $(\mathcal{B}^{\mathrm{d}} x)(t) := B^* T(t)^* x$, $t \ge 0$. Thus, $L_B := \mathcal{B}\mathcal{B}^* \in \mathcal{L}(X)$, where $\mathcal{B}^* := \mathcal{R}\mathcal{B}^{\mathrm{d}}$, $(\mathcal{R}f)(t) := f(-t)$ (the reflection). (See, for instance, Curtain and Zwart [16] or Mikkola [30] for details.)

It is easy to see that if the system is *exponentially stable* (that is, there are $\epsilon > 0$, $M < \infty$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq M\mathrm{e}^{-\epsilon t}$ for all $t > 0$), then A1–A3 are satisfied (and $\overline{\mathbb{C}_+} \subset \rho(A)$). However, there are several important systems that are not exponentially stable but for which A1–A3 hold. In this section we shall derive the state-space formulas for the factors Λ and V for such systems; we use the additional assumption that the open right half-plane $\mathbb{C}_+$ is contained in the resolvent set $\rho(A)$, but this assumption can be relaxed (for example, a zero-measurable spectrum on each vertical line on $\mathbb{C}_+$ is not a problem; see Remark 3.4 below).

In Lemma 3.1 below we show that if A1–A3 hold, then the systems $(A, -, B^*L_C)$ and $(A^*, -, CL_B)$ are output stable, that is, $B^*L_CT \in \mathcal{L}(X, L^2(\mathbb{R}_+; \mathbb{C}^m))$, $CL_BT^* \in \mathcal{L}(X, L^2(\mathbb{R}_+; \mathbb{C}^p))$. We use the following notation:

$$(\pi_+ f)(t) := \begin{cases} f(t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \quad \text{and} \quad \mathcal{D}^{\mathrm{d}} := \mathcal{R}\mathcal{D}^*\mathcal{R}$$

is the input-output map of (A^*, C^*, B^*) (see [30, Lemma 6.2.9(b)]).

LEMMA 3.1. *If* A1, A2, *and* A3 *hold, then* $\pi_+\mathcal{D}^*\mathcal{C}x = B^*L_CT(\cdot)x$ *and* $\mathcal{R}\pi_-\mathcal{D}\mathcal{B}^*x = \pi_+(\mathcal{D}^{\mathrm{d}})^*\mathcal{B}^{\mathrm{d}}x = CL_BT(\cdot)^*x$ *for each* $x \in X$. *In particular, there is* $M < \infty$ *such that* $\|B^*L_C(\cdot I - A)^{-1}x\|_{H_2(\mathbb{C}_+, \mathbb{C}^m)} \leq M\|x\|_X$ *and* $\|CL_B(\cdot I - A^*)^{-1}x\|_{H_2(\mathbb{C}_+, \mathbb{C}^p)} \leq M\|x\|_X$ *for all* $x \in X$.

Proof. By Lemma 4.2.6 of [33], we have $\pi_+\mathcal{D}^*\mathcal{C}x = B^*L_CTx$ (everywhere, by continuity). The first inequality is obtained from Plancherel's theorem with $M := \|\mathcal{D}\| \max\{\|\mathcal{C}\|, \|\mathcal{B}\|\}$. Applying the above to (A^*, C^*, B^*), we obtain the second equality and inequality (because $\|(\mathcal{D}^{\mathrm{d}})^*\| = \|\mathcal{D}^{\mathrm{d}}\| = \|B^*(\cdot I - A^*)^{-1}C^*\|_\infty = \|G^*\|_\infty = \|G\|_\infty = \|\mathcal{D}\|$). ◻

Now we are ready to give the state-space formulas for the factors Λ and V. The case where $\overline{\mathbb{C}_+} \subset \rho(A)$ is simple, and the general case will be reduced to that by using the results given in section 6.

LEMMA 3.2. *Assume that the triple* (A, B, C) *satisfies* A1, A2, A3 *and that* $\mathbb{C}_+ \subset \rho(A)$. *Let* $G(s) = C(sI - A)^{-1}B$ *be the associated transfer function, with associated Hankel singular values* $\sigma_1 \geq \sigma_2 \geq \cdots$, *and let* σ *be such that* $\sigma_{k+1} < \sigma < \sigma_k$. *Let* Λ *be defined as follows:*

(3.2)
$$\Lambda(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \frac{1}{\sigma^2} \begin{bmatrix} -CL_B \\ \sigma B^* \end{bmatrix} \left(I - \frac{1}{\sigma^2} L_C L_B \right)^{-1} (sI + A^*)^{-1} \begin{bmatrix} C^* & L_C B \end{bmatrix},$$

$s \in \mathbb{C}_-$. *Then* Λ *has the following properties:*

(1) $\Lambda(i\omega)^* \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right] \Lambda(i\omega) = \left[\begin{smallmatrix} I_p & G(i\omega) \\ 0 & I_m \end{smallmatrix}\right]^* \left[\begin{smallmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{smallmatrix}\right] \left[\begin{smallmatrix} I_p & G(i\omega) \\ 0 & I_m \end{smallmatrix}\right]$ *for almost all* $\omega \in \mathbb{R}$.

(2) $\Lambda(s)$ *is invertible for each* $s \in \mathbb{C}_-$, *and its inverse is given by*

(3.3)
$$V(s) = \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix} - \frac{1}{\sigma^2} \begin{bmatrix} -CL_B \\ \sigma B^* \end{bmatrix} (sI + A^*)^{-1} \left(I - \frac{1}{\sigma^2} L_C L_B \right)^{-1} \begin{bmatrix} C^* & \frac{1}{\sigma} L_C B \end{bmatrix},$$

$s \in \mathbb{C}_-$.

(3) $\Lambda(\cdot) - \left[\begin{smallmatrix} I_p & 0 \\ 0 & \sigma I_m \end{smallmatrix}\right] \in H_2(\mathbb{C}_-, \mathbb{C}^{(p+m)\times(p+m)})$.

(4) $V(\cdot) - \left[\begin{smallmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{smallmatrix}\right] \in H_2(\mathbb{C}_-, \mathbb{C}^{(p+m)\times(p+m)})$.

Proof. 1° *Case* $\overline{\mathbb{C}_+} \subset \rho(A)$*:* The proofs of parts (1) and (2) go in a way similar to the suboptimal Nehari problem addressed in Curtain and Zwart [16, section 8.3].

The new part is to show that parts (3) and (4) hold. Set

$$\widehat{g}(s) := \begin{bmatrix} C \\ B^* L_C \end{bmatrix} (sI - A)^{-1} \quad (s \in \rho(A)). \tag{3.4}$$

By Lemma 3.1, we have $\widehat{g}x \in H_2(\mathbb{C}_+, \mathbb{C}^{p+m})$ for all $x \in X$, and so

$$\widehat{f} := \widehat{g} \left(I - \frac{1}{\sigma^2} L_B L_C \right)^{-1} \begin{bmatrix} -L_B C^* \\ \sigma B \end{bmatrix} \tag{3.5}$$

satisfies $\widehat{f}z \in H_2(\mathbb{C}_+, \mathbb{C}^{p+m})$ for all $z \in \mathbb{C}^{p+m}$. Thus $\widehat{f} \in H_2(\mathbb{C}_+, \mathbb{C}^{(p+m)\times(p+m)})$. Since $-\widehat{f}(-\bar{s})^* = \Lambda(s) - \left[\begin{smallmatrix} I_p & 0 \\ 0 & \sigma I_m \end{smallmatrix}\right]$, we obtain that (3) holds.

Part (4) can be proved in an analogous way.

2° *The general case* $\mathbb{C}_+ \subset \rho(A)$*:* The proof in 1° establishes (2)–(4). However, (1) is more complicated: Now (3.2) defines Λ on $\mathbb{C}_-$ only, and on $i\mathbb{R}$ it is defined a.e. as the radial (or nontangential) limit or, equivalently, as the Fourier (Laplace) transform of the inverse Laplace transform of Λ. This follows from (3) (see below).

Nevertheless, the triple $(A-\epsilon, B, C)$ satisfies the assumptions of 1° (cf. Lemma 6.1). Therefore, the corresponding functions Λ_ϵ and G_ϵ satisfy (1) in place of Λ and G. (Note that $G_\epsilon(i\omega) := C(i\omega I - (A-\epsilon))^{-1}B = G(i\omega + \epsilon)$.) Repeat (3.4) and (3.5) with $\widehat{g}_\epsilon, \widehat{f}_\epsilon, A-\epsilon, L_{C,\epsilon}, L_{B,\epsilon}$ in place of $\widehat{g}, \widehat{f}, A, L_C, L_B$, respectively.

By (2) and (3) of Lemma 6.1 (and Lemma A.3.1(j3) of Mikkola [30]), we have $g_\epsilon x := \pi_+ \left[\begin{smallmatrix} I \\ \mathcal{D}_\epsilon^* \end{smallmatrix}\right] \mathcal{C}_\epsilon x \to \pi_+ \left[\begin{smallmatrix} I \\ \mathcal{D}^* \end{smallmatrix}\right] \mathcal{C}x$ in $L^2(\mathbb{R}_+; \mathbb{C}^{p+m})$, and so $\widehat{g}_\epsilon x \to \widehat{g}x$ in $L^2(i\mathbb{R}; \mathbb{C}^{p+m})$, as $\epsilon \to 0+$, for all $x \in X$. Therefore, $\widehat{f}_\epsilon z \to \widehat{f}z$ in $L^2(i\mathbb{R}; \mathbb{C}^{p+m})$ for all $z \in \mathbb{C}^{p+m}$ (here we also need (7) and (4) of Lemma 6.1); hence a subsequence converges a.e. on $i\mathbb{R}$. But, similarly, $G_\epsilon(i\omega)z = G(i\omega+\epsilon)z \to G(i\omega)z$, as $\epsilon \to 0+$, for almost every $\omega \in \mathbb{R}$, for each $z \in \mathbb{C}^{p+m}$ (use the standard H^∞ boundary function result, such as Theorem 3.3.1(c1) of Mikkola [30]).

We already noted above that $\langle \Lambda_\epsilon(i\omega)\tilde{z}, \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right] \Lambda_\epsilon(i\omega)z \rangle = \big\langle \left[\begin{smallmatrix} I_p & G_\epsilon(i\omega) \\ 0 & I_m \end{smallmatrix}\right] \tilde{z}, \left[\begin{smallmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{smallmatrix}\right] \left[\begin{smallmatrix} I_p & G_\epsilon(i\omega) \\ 0 & I_m \end{smallmatrix}\right] z \big\rangle$ for almost every $\omega \in \mathbb{R}$, for any $z, \tilde{z} \in \mathbb{C}^{p+m}$. By the above, we can remove the ϵ's (just let $\epsilon \to 0$). Since $\mathbb{C}^{p+m}$ has a finite basis, (1) holds. ◻

In light of Lemma 3.2, we now obtain our main theorem by invoking the key frequency-domain result, namely, Theorem 2.1. The following theorem gives explicit formulas (in terms of the state-space parameters) for all solutions to the suboptimal Hankel norm approximation problem in the case of infinite-dimensional systems which do not necessarily have an exponentially stable semigroup.

THEOREM 3.3. *Assume that the triple* (A, B, C) *satisfies* A1, A2, A3 *and that* $\mathbb{C}_+ \subset \rho(A)$*. Let* $G(s) = C(sI - A)^{-1}B$ *be the associated transfer function, and let the Hankel singular values be denoted by* $\sigma_1 \geq \sigma_2 \geq \cdots$*. Suppose that* σ *is such that* $\sigma_{k+1} < \sigma < \sigma_k$*, and let* V *be given by* (3.3)*.*

Then K *is such that* $K(\cdot) \in H_{\infty,k}(\mathbb{C}_-, \mathbb{C}^{p\times m})$ *and* $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ *if and only if* K *is given by* $K(i\omega) = (V_{11}(i\omega)Q(i\omega) + V_{12}(i\omega))(V_{21}(i\omega)Q(i\omega) + V_{22}(i\omega))^{-1}, \omega \in \mathbb{R}$*, for some* $Q \in H_\infty(\mathbb{C}_-, \mathbb{C}^{p\times m})$ *such that* $\|Q\|_\infty \leq 1$*.*

This follows from Theorem 2.1 and Lemma 3.2.

Remark 3.4.

(a) The assumption $\mathbb{C}_+ \subset \rho(A)$ can be weakened in all our results, including the above. Indeed, it suffices that, for instance, the Lebesgue measure of $\{r+\omega i \in \sigma(A) : \omega \in \mathbb{R}\}$ is zero for all small $r > 0$, as one can verify from the proofs.

(b) Finally, we remark that in Chapter 6 of Sasane [40], using another approach, state-space formulas were given in the nonexponentially stable case. However, these were in terms of the parameters of the shifted system Σ_ϵ and only guaranteed that, for a small enough shift, they are also solutions to the original system. Also, while only the existence of some solutions was demonstrated in [40], here we give a complete parameterization of *all* solutions.

4. An application to the case of well-posed linear systems. Finally, in this last section we give an application of Theorem 3.3 to obtain state-space formulas for the suboptimal Hankel norm approximation problem for *well-posed* linear systems. This was done using the idea of reciprocal systems in Curtain and Sasane [15], but there, instead of Theorem 3.3, a weaker result from Chapter 6 of Sasane [40] (which was mentioned in Remark 3.4) was used. Here, using Theorem 3.3, we obtain a different solution to the problem, where, as opposed to Curtain and Sasane [15], we now obtain a parameterization of the set of *all* solutions to the suboptimal Hankel norm approximation problem for well-posed linear systems.

We consider the suboptimal Hankel norm approximation problem for a well-posed linear system Σ on a Hilbert space X, with input space $\mathbb{C}^m$, output space $\mathbb{C}^p$, generating operators A, B, C, semigroup $\{T(t)\}_{t\geq 0}$, and transfer function G, under the following assumptions:

B1. $0 \in \rho(A)$ and $\mathbb{C}_+ \subset \rho(A)$.
B2. Σ is input-stable.
B3. Σ is output-stable.
B4. $G \in H_\infty(\mathbb{C}_+, \mathbb{C}^{p\times m})$.

(Condition B1 can be relaxed; for example, it suffices to assume that $0 \in \rho(A)$ and $\sigma(A) \cap \mathbb{C}_+$ is at most countable (see Remark 3.4(a)). Moreover, instead of $0 \in \rho(A)$ it suffices to assume that $ir \in \rho(A)$ for some $r \in \mathbb{R}$, but then one must replace A by $A - ir$ in the formulas, so that the new G equals the old $G(ir + \cdot)$.)

The *reciprocal system* of such a well-posed linear system is defined as the well-posed linear system Σ_r with the bounded generating operators $A^{-1}, A^{-1}B, -CA^{-1}$. In Curtain and Sasane [15], it was established that if Σ satisfies B1–B4 above, then its reciprocal system is such that

1. A1, A2, A3 from the previous section are satisfied;
2. $\mathbb{C}_+ \subset \rho(A^{-1})$;
3. the controllability and observability Gramians of Σ_r are the same as the controllability and observability Gramians of Σ;
4. $K_r \in H_{\infty,k}(\mathbb{C}_-; \mathbb{C}^{p\times m})$ is a solution to the suboptimal Hankel norm approximation problem of the reciprocal system Σ_r if and only if

$$K(s) := K_r\left(\frac{1}{s}\right) - G(0) \quad \text{for } s \in \mathbb{C}_- \tag{4.1}$$

is a solution[1] to the suboptimal Hankel norm approximation problem of the original system Σ.

[1]Note that from equation (4.1), it follows that $K \in H_{\infty,k}(C_-, \mathbb{C}^{p\times m})$ if and only if $K_r \in H_{\infty,k}(C_-, \mathbb{C}^{p\times m})$.

In light of these remarks, we have thus proved the following theorem.

THEOREM 4.1. *Suppose that the well-posed linear system* Σ *with transfer function* G *satisfies assumptions* B1–B4. *Let* σ *be such that* $\sigma_{k+1} < \sigma < \sigma_k$, *where* $\sigma_1 \geq \sigma_2 \geq \cdots$ *are the Hankel singular values of* G. *Let* V *be given by*

$$V(s) = \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix} - \frac{1}{\sigma^2} \begin{bmatrix} CA^{-1}L_B \\ \sigma(A^{-1}B)^* \end{bmatrix} \left(s + (A^{-1})^*\right)^{-1} \left(I - \frac{1}{\sigma^2} L_C L_B\right)^{-1} \cdot \begin{bmatrix} -(CA^{-1})^* & \frac{1}{\sigma} L_C B A^{-1} \end{bmatrix},$$

$s \in \mathbb{C}_-$, *where* L_B *and* L_C *denote the controllability Gramian and the observability Gramian, respectively, of the system* Σ, *and* $N_\sigma := (I - \frac{1}{\sigma^2} L_B L_C)^{-1}$. *Then* $K \in H_{\infty,k}(\mathbb{C}_-, \mathbb{C}^{p\times m})$ *satisfies* $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ *if and only if*

$$K(s) = K_r\left(\frac{1}{s}\right) - G(0),$$

where $K_r(i\omega) = (V_{11}(i\omega)Q(i\omega) + V_{12}(i\omega))(V_{21}(i\omega)Q(i\omega) + V_{22}(i\omega))^{-1}$, $\omega \in \mathbb{R}$, *for some* $Q \in H_\infty(\mathbb{C}_-, \mathbb{C}^{p\times m})$ *such that* $\|Q\|_\infty \leq 1$.

This solves the suboptimal Hankel norm approximation problem for well-posed linear systems.

5. Appendix A. In this appendix we present the proofs that were deferred in section 2.

5.1. Proof of Proposition 2.14.

Proof. Suppose that the matrix-valued function K has a Kreĭn–Langer factorization $K = F \cdot B^{-1}$ with $F \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ and with $B \in H_\infty(\mathbb{D}, \mathbb{C}^{m\times m})$ a Blaschke–Potapov function of degree k. Then the graph of the multiplication operator M_{G+K} restricted to the subspace $B \cdot H_2(\mathbb{D}, \mathbb{C}^m)$ satisfies

$$\begin{aligned} \mathcal{G}_{M_{G+K}} &:= \begin{bmatrix} G+K \\ I \end{bmatrix} BH_2(\mathbb{D}, \mathbb{C}^m) \\ &= \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} BH_2(\mathbb{D}, \mathbb{C}^m) \\ &\subset \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} H_2(\mathbb{D}, \mathbb{C}^{m+p}) =: \mathcal{M}. \end{aligned} \tag{5.1}$$

If also $\|G+K\|_\infty \leq \sigma$, then we see that $\mathcal{G}_{G+K}$ is a *negative subspace* in the Kreĭn-space inner product

$$\langle f, g\rangle_{J_\sigma} = \frac{1}{2\pi}\int_{\mathbb{T}} \langle J_\sigma f(\zeta), g(\zeta)\rangle_{\mathbb{C}^{m+p}}\ |d\zeta|$$

on $L_2(\mathbb{T}, \mathbb{C}^{m+p})$; i.e., each function $f \in \mathcal{G}_{M_{G+K}}$ has negative J_σ-self-inner product

$$\langle f, f\rangle_{J_\sigma} \leq 0 \quad \text{for } f \in \mathcal{G}_{M_{G+K}}. \tag{5.2}$$

Since $B \cdot H_2(\mathbb{D}, \mathbb{C}^m)$ has codimension k in $H_2(\mathbb{D}, \mathbb{C}^m)$, we see in addition that $\mathcal{G}_{M_{G+K}}$ *has codimension* k *in a maximal negative subspace of the Kreĭn space* $\mathcal{K} := \left(\left[\begin{smallmatrix} L_2(\mathbb{T},\mathbb{C}^p) \\ H_2(\mathbb{D},\mathbb{C}^m)\end{smallmatrix}\right], \langle\cdot,\cdot\rangle_{L_\sigma}\right)$. In addition, since $\mathcal{G} := \mathcal{G}_{G+K}$ is the graph of a multiplication operator M_{G+K}, we see that $\mathcal{G}$ *is invariant for the shift operator* $M_\zeta\colon f(\zeta) \mapsto \zeta f(\zeta)$. We have thus verified the following: *If $K = FB^{-1}$ is a solution of the Nehari–Takagi problem, then the subspace $\mathcal{G} = \mathcal{G}_{G+K} := \left[\begin{smallmatrix} G+K \\ I \end{smallmatrix}\right] \cdot B \cdot H_2(\mathbb{D}, \mathbb{C}^m)$ (where B is the Blaschke–Potapov product of degree k chosen so that $K \cdot B \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$) satisfies conditions* (1)–(3) *in the statement of Proposition* 2.14.

Conversely, if $\mathcal{G}$ is a subspace of $\mathcal{K}$ which satisfies conditions (1)–(3) in Proposition 2.14, one can reverse the steps and come up with a $K \in H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p+m})$ ($k' \le k$) which solves the Nehari–Takagi problem as follows. Since $\mathcal{G}$ is a negative subspace in the J_σ-inner product, $\mathcal{G}$ necessarily has the form of a graph space

$$\mathcal{G} = \begin{bmatrix} X \\ I \end{bmatrix} \mathcal{D}(X),$$

where the angle operator $X \colon \mathcal{D}(X) \mapsto L_2(\mathbb{T}, \mathbb{C}^p)$ has domain $\mathcal{D}(X) \subset H_2(\mathbb{D}, \mathbb{C}^m)$ and norm $\|X\| \le \sigma$. Since $\mathcal{G}$ has codimension k in a maximal negative subspace, necessarily $\dim H_2(\mathbb{D}, \mathbb{C}^m) \ominus \mathcal{D}(X) = k$. Since $\mathcal{G}$ is shift invariant, we have

$$\begin{bmatrix} M_\zeta X \\ M_\zeta \end{bmatrix} \mathcal{D}(X) \subset \begin{bmatrix} X \\ I \end{bmatrix} \mathcal{D}(X).$$

Hence $\mathcal{D}(X)$ is shift invariant, and

$$M_\zeta X x = X M_\zeta x \quad \text{for } x \in \mathcal{D}(X).$$

But then, by the Beurling–Lax theorem, $\mathcal{D}(X)$ has the form $\mathcal{D}(X) = B \cdot H_2(\mathbb{D}, \mathbb{C}^m)$ for a Blaschke–Potapov factor of degree k, and the rule

$$X \colon \zeta^{-n} B h \mapsto \zeta^{-n} X(Bh)$$

(for $h \in H_2(\mathbb{D}, \mathbb{C}^m)$ and $n = 0, 1, 2, \dots$) extends X to an operator, still called X, defined on the dense subset $\cup_{n=0}^\infty \zeta^{-n} B H_2(\mathbb{D}, \mathbb{C}^m)$ of $L_2(\mathbb{T}, \mathbb{C}^m)$, still with norm $\|X\| \le \sigma$, such that $XM_\zeta = M_\zeta X$. This forces X to be a multiplication operator $X = M_{G+K}$ for some matrix function $K \in L_\infty(\mathbb{T}, \mathbb{C}^{p+m})$ with $\|G+K\|_\infty \le \sigma$. From the fact that $\mathcal{G} \subset \mathcal{M}$, we have

$$\begin{bmatrix} G+K \\ I \end{bmatrix} B H_2(\mathbb{D}, \mathbb{C}^m) \subset \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} H_2(\mathbb{D}, \mathbb{C}^m),$$

i.e.,

$$\begin{bmatrix} I & G \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} G+K \\ I \end{bmatrix} B H_2(\mathbb{D}, \mathbb{C}^m) = \begin{bmatrix} K \\ I \end{bmatrix} B H_2(\mathbb{D}, \mathbb{C}^m) \subset H_2(\mathbb{D}, \mathbb{C}^{p+m}).$$

In particular, $K \cdot B$ maps $H_2(\mathbb{D}, \mathbb{C}^m)$ into $H_2(\mathbb{D}, \mathbb{C}^p)$, and we see that $F \colon = K \cdot B \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$. But then $K = F \cdot B^{-1}$ has the Kreĭn–Langer factorization form required to be in the class $H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$ for k' at most k. Proposition 2.14 now follows. □

5.2. Proof of Proposition 2.16.

Proof. By Proposition 2.7, since $\sigma_k > \sigma > \sigma_{k+1}$ we know that the existence of a solution $K \in H_{\infty,k'}(\mathbb{D}, \mathbb{C}^{p\times m})$ with $k' \le k$ forces $k' = k$. This combined with the result in Proposition 2.14 implies that the only content to be added by Proposition 2.16 is that, under the hypothesis that $\sigma_k > \sigma > \sigma_{k+1}$, *a subspace $\mathcal{G}$ of $\mathcal{M}$ is $\mathcal{M}$-maximal negative* (i.e., maximal as a negative subspace contained in $\mathcal{M}$) *if and only* *if $\mathcal{G} \subset \mathcal{M}$ has codimension k in a $\mathcal{K}$-maximal negative subspace $\widetilde{\mathcal{G}}$ of $\mathcal{K}$.* One can see this general principle as follows. As $\mathcal{M}$ is regular, $\mathcal{M}$ has a fundamental decomposition $\mathcal{M} = \mathcal{M}_+ \dot{+} \mathcal{M}_-$, where $\mathcal{M}_+$ is a uniformly positive subspace and $\mathcal{M}_-$ is a uniformly negative subspace in the Kreĭn-space inner product $\langle \cdot, \cdot \rangle_{J_\sigma}$. As $\mathcal{M}^{[\perp]}$ is also regular, $\mathcal{M}^{[\perp]}$ also has a fundamental decomposition as $\mathcal{M}^{[\perp]} = \mathcal{P} \dot{+} \mathcal{N}$, where $\mathcal{P}$ is uniformly positive and $\mathcal{N}$ is uniformly negative. We note also that, as a consequence of Proposition 2.15, $\dim \mathcal{N} = k$. Then $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$ is a fundamental decomposition for $\mathcal{K}$, where

$$\mathcal{K}_+ = \mathcal{M}_+ \dot{+} \mathcal{P}, \qquad \mathcal{K}_- = \mathcal{M}_- \dot{+} \mathcal{N}.$$

By the angle-operator–graph correspondence, $\mathcal{M}$-maximal negative subspaces of $\mathcal{M}$ are of the form

$$\mathcal{G} = \{Xx + x \colon x \in \mathcal{M}_-\},$$

where X is a Hilbert space contraction operator from $(\mathcal{M}_-, -\langle \cdot, \cdot \rangle_{J_\sigma})$ into $(\mathcal{M}_+, \langle \cdot, \cdot \rangle_{J_\sigma})$. Similarly, $\mathcal{K}$-maximal negative subspaces of $\mathcal{K}$ are of the form

$$\widetilde{\mathcal{G}} = \{\widetilde{X}x + x \colon x \in \mathcal{K}_- = \mathcal{M}_- \dot{+} \mathcal{N}\},$$

where $\widetilde{X}$ is a contraction operator from $(\mathcal{K}_-, -\langle \cdot, \cdot \rangle_{J_\sigma})$ into $(\mathcal{K}_+ = \mathcal{M}_+ \dot{+} \mathcal{P}, \langle \cdot, \cdot \rangle_{J_\sigma})$. From this model, it is clear that $\mathcal{M}$-maximal negative subspaces of $\mathcal{M}$ match up exactly with those subspaces of $\mathcal{M}$ which have codimension k in a $\mathcal{K}$-maximal negative subspace of $\mathcal{K}$. This completes the proof of Proposition 2.16. ◻

5.3. Proof of Proposition 2.17. The proof of Proposition 2.17 requires a preliminary lemma.

LEMMA 5.1. *Suppose that $R \in H_2(\mathbb{D}, \mathbb{C}^{p+m})$ is outer and that $\mathcal{G}$ is a closed, shift-invariant subspace of $H_2(\mathbb{D}, \mathbb{C}^{p+m})$. Then $\mathcal{G} \cap R \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$ is dense in $\mathcal{G}$.*

Proof. Let $g \in \mathcal{G}$. For $n = 1, 2, \ldots$ choose scalar outer functions r_n so that

$$|r_n(\zeta)| = \min\left\{\frac{n}{\|R(\zeta)^{-1}g(\zeta)\|}, 1\right\} \quad \text{for almost all } \zeta \in \mathbb{T}.$$

Then $g_n := r_n \cdot g \in \mathcal{G}$ since $\mathcal{G}$ is shift invariant. Since $\|R^{-1}(\zeta)g_n(\zeta)\| \leq n$ for almost all $\zeta \in \mathbb{T}$, by construction, we see that $g_n \in R \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+n})$. Finally, since $|r_n(\zeta)| \leq 1$ for almost all $\zeta \in \mathbb{T}$ and $g \in H_2(\mathbb{D}, \mathbb{C}^{p+m})$, we see that $\{g_n\}$ converges to g as $n \to \infty$ in $H_2(\mathbb{D}, \mathbb{C}^{p+m})$, and the lemma follows. ◻

Proof of Proposition 2.17. Suppose first that $\mathcal{G} \subset \mathcal{M}$ is maximal negative as a subspace of $\mathcal{M}$ in the J_σ-inner product. Then $\mathcal{G}$ has the form $\mathcal{G} = \left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right] \cdot \mathcal{G}'$, where $\mathcal{G}'$ is a closed shift-invariant subspace of $H_2(\mathbb{D}, \mathbb{C}^{p+m})$. By Lemma 5.1 we know that $\mathcal{G}' \cap V \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$ is dense in $\mathcal{G}'$. Multiplication by $\left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right]$ then gives that $\mathcal{G} \cap \Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$ is dense in $\mathcal{G}$. We may write $\mathcal{G} \cap \Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$ in the form

$$\mathcal{G} \cap \Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m}) = \Theta \cdot \mathcal{G}_1,$$

where $\mathcal{G}_1 \subset H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$.

We assert that $\mathcal{G}_1$ *is weak-$*$ closed in* $H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$. By part (1) of Proposition 2.3, it suffices to consider a sequence $\{h_n\}_{n=1,2,\ldots}$ of elements of $\mathcal{G}_1$ convergent in the weak-$*$ topology to an element h of $L_\infty(\mathbb{D}, \mathbb{C}^{p+m})$ and prove that in fact $h \in \mathcal{G}_1$, i.e., that $\Theta h \in \mathcal{G} \cap \Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$. From the characterization of $H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$

as that subspace of $L_\infty(\mathbb{T}, \mathbb{C}^{m+p})$ consisting of functions F for which all the Fourier coefficients of negative index vanish,

$$\frac{1}{2\pi}\int_{\mathbb{T}} F(\zeta)\overline{\zeta}^n |d\zeta| = 0 \quad \text{for } n = -1, -2, \dots,$$

it is easily seen that $H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$ is weak-$*$ closed in $L_\infty(\mathbb{T}, \mathbb{C}^{p+m})$ and hence $h \in H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$. It therefore remains only to show that $\Theta h \in \mathcal{G}$. For this purpose note that, for any $k \in L_2(\mathbb{T}, \mathbb{C}^{p+m})$,

$$\langle \Theta h_n, k\rangle_{L_2} = \frac{1}{2\pi}\int_{\mathbb{T}} k(\zeta)^* \Theta h_n(\zeta) k(\zeta)|d\zeta|. \tag{5.3}$$

As $k^*\Theta \in L_2(\mathbb{T}, \mathbb{C}^{1\times(m+n)}) \subset L_1(\mathbb{T}, \mathbb{C}^{1\times(m+n)})$ and h_n is assumed to converge to h in the weak-$*$ topology of $L_\infty(\mathbb{T}, \mathbb{C}^{(m+n)\times 1})$, we may take limits in (5.3) to get

$$\lim_{n\to\infty} \langle \Theta h_n, k\rangle_{L_2} = \langle \Theta h, k\rangle_{L_2} \quad \text{for each } k \in L_2(\mathbb{T}, \mathbb{C}^{p+m}); \tag{5.4}$$

i.e., Θh_n converges to Θh in the weak topology on $L_2(\mathbb{T}, \mathbb{C}^{p+m})$. As $\Theta h_n \in \mathcal{G}$ for each n and as norm-closed subspaces of a Hilbert space are also closed in the weak topology (see [42, Theorem 6.3, page 158]), it follows that $\Theta h \in \mathcal{G}$ as wanted. We conclude that $\mathcal{G}_1$ *is weak-$*$ closed as asserted.*

By the Beurling–Lax theorem for weak-$*$ closed subspaces of $H_\infty(\mathbb{D}, \mathbb{C}^{m+p})$ (see, e.g., [41] or [25, page 25] for the scalar case), there is a matrix inner function $\psi = \left[\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right]$ in $H_\infty(\mathbb{D}, \mathbb{C}^{(p+m)\times m_1})$ (for some $m_1 \le m+p$) so that

$$\mathcal{G}_1 = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} H_\infty(\mathbb{D}, \mathbb{C}^{m_1}). \tag{5.5}$$

The inner property of ψ means that

$$\psi_1(\zeta)^*\psi_1(\zeta) + \psi_2(\zeta)^*\psi_2(\zeta) = I_{m'} \quad \text{for almost all } \zeta \in \mathbb{T}. \tag{5.6}$$

From the fact that $\mathcal{G}_1$ is J_1-negative we then also get

$$\psi_1(\zeta)^*\psi_1(\zeta) - \psi_2(\zeta)^*\psi_2(\zeta) \le 0 \quad \text{for almost all } \zeta \in \mathbb{T}. \tag{5.7}$$

Hence, if we set $Q(\zeta) = \psi_2(\zeta)\psi_1(\zeta)^\ddagger$, where $\psi_2(\zeta)^\ddagger$ is the left Moore–Penrose generalized inverse of $\psi_2(\zeta)$,

$$\psi_2(\zeta)^\ddagger \colon c \mapsto \begin{cases} 0 & \text{if } c \perp \operatorname{im}\psi_2(\zeta),\\ c' & \text{if } c = \psi_2(\zeta)c',\end{cases} \tag{5.8}$$

then $Q(\zeta)$ defines a contraction operator from $\mathbb{C}^m$ into $\mathbb{C}^p$ for almost all $\zeta \in \mathbb{T}$, and we can rewrite (5.5) as

$$\mathcal{G}_1 = \begin{bmatrix} Q\\ I_m\end{bmatrix} \psi_2 H_\infty(\mathbb{D}, \mathbb{C}^{m_1}). \tag{5.9}$$

We next argue that $\psi_2 H_\infty(\mathbb{D}, \mathbb{C}^{m_1})$ *is weak-$*$ closed in* $H_\infty(\mathbb{D}, \mathbb{C}^m)$. Indeed, suppose that $\psi_2 h_n$ converges in the L_∞-weak-$*$ topology to an element $k \in L_\infty$. Then the computation (for each $g \in L_1(\mathbb{T}, \mathbb{C}^p)$)

$$[\psi_1 h_n, g] = [Q\psi_2 h_n, g] = [\psi_2 h_n, Q^* g] \to [k, Q^* g] = [Qk, g]$$

shows that $\psi_1 h_n$ tends weak-$*$ to Qk. (Here we let $[\cdot,\cdot]$ denote the duality pairing

$$[F, f] = \frac{1}{2\pi}\int_{\mathbb{T}} f(\zeta)^* F(\zeta)|d\zeta| \quad \text{for } F \in L_\infty(\mathbb{T}, \mathbb{C}^{m'}) \text{ and } f \in L_1(\mathbb{T}, \mathbb{C}^{m'})$$

between $L_\infty(\mathbb{T}, \mathbb{C}^{m'})$ and $L_1(\mathbb{T}, \mathbb{C}^{m'})$ for any fixed choice of m', and we use that $Q^* \cdot g \in L_1(\mathbb{T}, \mathbb{C}^m)$ for any $g \in L_1(\mathbb{T}, \mathbb{C}^p)$ since $Q(\zeta)$ is contractive a.e.) We conclude that

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} h_n \to \begin{bmatrix} Q \\ I_m \end{bmatrix} k$$

in the weak-$*$ topology. As $\mathcal{G}_1$ is closed in the weak-$*$ topology, it follows that $\left[\begin{smallmatrix} Q \\ I_m \end{smallmatrix}\right] k \in \mathcal{G}_1$, and hence, in particular, $k \in \psi_2 H_\infty(\mathbb{D}, \mathbb{C}^{m_1})$. We conclude that $\psi_2 H_\infty(\mathbb{D}, \mathbb{C}^{m_1})$ *is closed in the weak-$*$ topology as wanted.*

We next argue that in fact

$$\psi_2 H_\infty(\mathbb{D}, \mathbb{C}^{m'}) = H_\infty(\mathbb{D}, \mathbb{C}^m). \tag{5.10}$$

Indeed, via a second application of the Beurling–Lax theorem for weak-$*$ closed subspaces of vector-valued H_∞, by the fact established in the previous paragraph it follows that there is an $m \times m'$ matrix inner function ϕ so that $\psi_2 H_\infty(\mathbb{D}, \mathbb{C}^{m'}) = \phi H_\infty(\mathbb{D}, \mathbb{C}^{m'})$. If $m' < m$ (or more generally, if $\phi H_\infty(\mathbb{D}, \mathbb{C}^{m'})$ does not fill up all of $H_\infty(\mathbb{D}, \mathbb{C}^m)$), then we may choose a nonzero vector $f \in (H_2(\mathbb{D}, \mathbb{C}^m) \ominus \psi H_2(\mathbb{D}, \mathbb{C}^{m'})) \cap H_\infty(\mathbb{D}, \mathbb{C}^m)$ so that the L_2-closure of $M_\Theta \cdot \left(\operatorname{span} \left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right] + \mathcal{G}_1\right)$ is a larger negative subspace of $\mathcal{M}$ which includes $\mathcal{G}$ as a subspace. As $\mathcal{G}$ is assumed to be maximal negative in $\mathcal{M}$, this leads to a contradiction, and we conclude that $m' = m$ and ϕ is a unitary constant. We have now arrived at

$$\mathcal{G} \cap \Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m}) = \Theta \begin{bmatrix} Q \\ I \end{bmatrix} \cdot H_\infty(\mathbb{D}, \mathbb{C}^m).$$

Taking closures in this identity gives the representation (2.15) for the shift-invariant subspace $\mathcal{G}$ assumed to be maximal negative in $\mathcal{M}$.

Conversely, suppose that $Q \in H_\infty(\mathbb{D}, \mathbb{C}^{p\times m})$ with $\|Q\|_\infty \leq 1$ and we define $\mathcal{G} \subset L_2(\mathbb{T}, \mathbb{C}^{p\times m})$ by (2.15). From the factorization $\left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right] = \Theta \cdot \Lambda$, where Λ and $V = \Lambda^{-1}$ are in $H_2(\mathbb{D}, \mathbb{C}^{(p+m)\times(p+m)})$, it is clear that $\mathcal{G} \subset \mathcal{M} := \left[\begin{smallmatrix} I_p & G \\ 0 & I_m \end{smallmatrix}\right] \cdot H_2(\mathbb{D}, \mathbb{C}^{p+m})$. Since $\|Q\|_\infty \leq 1$ and Θ is (J_1, J_σ)-unitary on $\mathbb{T}$, we see that necessarily $\mathcal{G}$ is negative in the J_σ-inner product. If $\mathcal{G}'$ is a J_σ-negative subspace with $\mathcal{G} \subset \mathcal{G}' \subset \mathcal{M}$, then by the same argument as in the first part of the proof we know that

$$\mathcal{G}' \cap \Theta \cdot H_\infty(\mathbb{D}, \mathbb{C}^{p+m}) = \mathcal{G}'_1$$

for some weak-$*$ closed subspace $\mathcal{G}'_1$ of $H_\infty(\mathbb{D}, \mathbb{C}^{p+m})$. The (J_1, J_σ)-unitary property of Θ and the fact that $\mathcal{G}'$ is J_σ-negative then force $\mathcal{G}'_1$ to be J_1-negative. Hence $\mathcal{G}'_1$ can contain no elements of the form $\left[\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right]$ with $h \in H_\infty(\mathbb{D}, \mathbb{C}^p)$ nonzero. We conclude that $\mathcal{G}'_1$ is a graph space; i.e., there is an operator X mapping some domain $\mathcal{D}(X) \subset H_\infty(\mathbb{D}, \mathbb{C}^m)$ into $H_\infty(\mathbb{D}, \mathbb{C}^p)$ so that $\mathcal{G}'_1 = \left[\begin{smallmatrix} X \\ I_m \end{smallmatrix}\right] \mathcal{D}(X)$. Since $\mathcal{G}' \supset \mathcal{G}$, we see next that $\mathcal{G}'_1 \supset \mathcal{G}_1$, i.e.,

$$\begin{bmatrix} X \\ I_m \end{bmatrix} \mathcal{D}(X) \supset \begin{bmatrix} Q \\ I_m \end{bmatrix} H_\infty(\mathbb{D}, \mathbb{C}^m).$$

As $\mathcal{D}(X) \subset H_\infty(\mathbb{D}, \mathbb{C}^m)$, we must have $\mathcal{D}(X) = H_\infty(\mathbb{D}, \mathbb{C}^m)$, X is the operator of multiplication by Q, and $\mathcal{G} = \mathcal{G}'$ is $\mathcal{M}$-maximal negative. This concludes the proof of Proposition 2.17. $\square$

6. Appendix B. In this appendix we present the proofs that were deferred in section 3.

To prove Lemma 3.2(1) we want to study how the operators determined by "shifted" triple $(A-\epsilon, B, C)$ converge to those determined by (A, B, C), as $\epsilon \to 0+$. For the shifted system, the semigroup is denoted by $\{T_\epsilon(t)\}_{t\geq 0}$, the controllability map is denoted by $\mathcal{C}_\epsilon$, and the controllability Gramian $\mathcal{C}_\epsilon^*\mathcal{C}_\epsilon$ is abbreviated by $L_{C,\epsilon}$; similarly, one uses the notation $\mathcal{B}_\epsilon^*, L_{B,\epsilon}, \mathcal{D}_\epsilon$, etc. Since the functions in A1, A2, and A3 are shifted to the left ($C(sI-(A-\epsilon I))^{-1}x = C((s+\epsilon)I-A)^{-1}x$, etc.), the assumptions A1–A3 hold a fortiori. Some further claims are straightforward, while others are more complicated.

LEMMA 6.1. *Assume that the triple (A, B, C) satisfies* A1, A2, *and* A3. *Then, with the above notation, as $\epsilon \to 0+$, we have the following for all $k \in \{1, 2, 3, \dots\}$, $t \geq 0$, $x \in X$, $u \in L^2(\mathbb{R}_+; \mathbb{C}^m)$, $y \in L^2(\mathbb{R}_+; \mathbb{C}^p)$:*

(1) $T_\epsilon(t) = \mathrm{e}^{-\epsilon t}T(t)$, $\mathcal{C}_\epsilon x = \mathrm{e}^{-\epsilon\cdot}\mathcal{C}x$, *and* $\mathcal{D}_\epsilon u = \mathrm{e}^{-\epsilon\cdot}\mathcal{D}\mathrm{e}^{\epsilon\cdot}u$.

(2) $\|\mathcal{C}_\epsilon\| \leq \|\mathcal{C}\|$ *and* $\|\mathcal{C}_\epsilon x_\epsilon - \mathcal{C}x\|_2 \to 0$ *whenever* $\|x_\epsilon - x\|_X \to 0$.

(3) $\|\mathcal{D}_\epsilon\| \leq \|\mathcal{D}\|$ *and* $\|\mathcal{D}_\epsilon^* y_\epsilon - \mathcal{D}^* y\|_2 \to 0$ *whenever* $\|y_\epsilon - y\|_2 \to 0$.

(4) $\|L_{C,\epsilon}\| \leq \|L_C\|$, $\|L_{B,\epsilon}\| \leq \|L_B\|$, $L_{C,\epsilon}x \to L_C x$, *and* $L_{B,\epsilon}x \to L_B x$.

(5) σ_k *is the kth singular value of* $\Gamma := \mathcal{C}\mathcal{B}$.

(6) $\sigma_{k,\epsilon} \to \sigma_k-$, *where* $\sigma_{k,\epsilon}$ *is the kth singular value of* $\Gamma_\epsilon := \mathcal{C}_\epsilon\mathcal{B}_\epsilon$.

(7) *Let* $\sigma_k > \sigma > \sigma_{k+1}$. *Then there are* $\epsilon_0 > 0$ *and* $M_0 < \infty$ *such that* $N_{\sigma,\epsilon} := (I - \sigma^{-2}L_{B,\epsilon}L_{C,\epsilon})^{-1}$ *exists and* $\|N_{\sigma,\epsilon}\| \leq M_0$ *for all* $\epsilon \in (0, \epsilon_0)$. *Moreover,* $N_{\sigma,\epsilon}x \to N_\sigma x$.

Proof. (1) This is straightforward.

(2) By (1), we have $\|\mathcal{C}_\epsilon\| \leq \|\mathcal{C}\|$. Obviously, $\|\mathcal{C}_\epsilon x - \mathcal{C}x\|_2 \to 0$. Since $\mathcal{C}_\epsilon x_\epsilon - \mathcal{C}x = \mathcal{C}_\epsilon(x_\epsilon - x) + (\mathcal{C}_\epsilon - \mathcal{C})x$, we obtain that (2) holds.

(3) We have $\widehat{\mathcal{D}_\epsilon u}(s) = (G(\cdot)u(\cdot - \epsilon))(s+\epsilon) = G(\epsilon + s)u(s)$, i.e., $G_\epsilon = G(\epsilon + \cdot)$. Therefore, $\|G_\epsilon\|_\infty \leq \|G\|_\infty$ and $G_\epsilon^* y_0 \to G^* y_0$ a.e. on $i\mathbb{R}$, for any $y_0 \in \mathbb{C}^p$ (see, e.g., Theorem 3.3.1(c1) of [30]). Consequently, $G_\epsilon^*\widehat{y} \to G^*\widehat{y}$ in $L^2(i\mathbb{R}; \mathbb{C}^m)$ for any $\widehat{y} \in L^2(i\mathbb{R}; \mathbb{C}^p)$, by the dominated convergence theorem. By Plancherel's theorem, this means that $\mathcal{D}_\epsilon^* y \to \mathcal{D}^* y$ for any $y \in L^2(\mathbb{R}; \mathbb{C}^m)$. Because the functions $\mathcal{D}_\epsilon^*$ are uniformly bounded, (3) also holds.

(4) We have

$$\|L_{C,\epsilon}\| = \sup_{\|x\|\leq 1}\langle x, L_{C,\epsilon}x\rangle = \sup_{\|x\|\leq 1}\|\mathcal{C}_\epsilon x\|_{L_2}^2 \leq \sup_{\|x\|\leq 1}\|\mathcal{C}x\|_{L_2}^2 = \sup_{\|x\|\leq 1}\langle x, L_C x\rangle = \|L_C\|.$$

Moreover, $\langle x, (L_C - L_{C,\epsilon})x\rangle = \int_0^\infty (1 - \mathrm{e}^{-2\epsilon t})|(\mathcal{C}x)(t)|^2\,dt \to 0$. By duality (i.e., (A^*, C^*, B^*) in place of (A, B, C)), we obtain the claims for L_B.

(5) By Plancherel's theorem, H_G and Γ are isomorphic (see [16, Lemma 8.2.3(c), page 397]).

(6) Define $(S_\epsilon f)(t) = \mathrm{e}^{-\epsilon t}f$ ($\epsilon \in \mathbb{R}$). For any $f \in L^2(\mathbb{R}; \mathbb{C}^n)$, $n \geq 1$, we have $S_\epsilon f \to f$ in L^2 as $\epsilon \to 0$. Moreover, $\|S_\epsilon f\| \leq \|f\|$ (respectively, $\|S_{-\epsilon}f\| \leq \|f\|$) if $f = 0$ on $\mathbb{R}_-$ (respectively, $\mathbb{R}_+$). Therefore, $\Gamma_\epsilon^*\Gamma_\epsilon u \to \Gamma^*\Gamma u$ for each $u \in L^2(\mathbb{R}_-; \mathbb{C}^m)$ (see Mikkola [30, Lemma A.3.1(j3)] and note that $\mathcal{C}_\epsilon = S_\epsilon\mathcal{C}$, $\mathcal{B}_\epsilon = \mathcal{B}S_\epsilon$). Thus, we get $\liminf_{\epsilon\to 0+}\sigma_{k,\epsilon} \geq \sigma_k$ from Lemma 6.3.

Conversely, if $\operatorname{rank} K \leq k-1$, then $\operatorname{rank} K_\epsilon \leq k-1$, where $K_\epsilon := S_\epsilon K S_{-\epsilon}$, and $\|\Gamma_\epsilon - K_\epsilon\| = \|S_\epsilon(\Gamma - K)S_{-\epsilon}\| \leq \|\Gamma - K\|$. Hence $\sigma_{k+1,\epsilon} \leq \sigma_{k+1}$. Similarly, we observe that $\sigma_{k+1,\epsilon} \leq \sigma_{k+1,\epsilon'}$ when $\epsilon > \epsilon' > 0$.

(7) Let δ be as in Lemma 6.2(2), and choose ϵ_0 so that $\sigma_{l,\epsilon}^2 - \sigma_l^2 < \delta/2$ for $l = k$, $k+1$, and $\epsilon \in (0, \epsilon_0)$ (use (6)). Then Lemma 6.2(2) implies that $\|(\sigma^2 I - \Gamma_\epsilon^*\Gamma_\epsilon)^{-1}\| \leq 2/\delta$,

that is, that $\|(I - \sigma^{-2}\Gamma_\epsilon^*\Gamma_\epsilon)^{-1}\| \le 2\sigma^2/\delta$, for $\epsilon \in (0, \epsilon_0)$. Apply $(I - ST)^{-1} = I + S(I - TS)^{-1}T$ to $T = \sigma^{-2}\mathcal{B}_\epsilon^*$, $S := \mathcal{C}_\epsilon^*\mathcal{C}_\epsilon\mathcal{B}_\epsilon$ to obtain the inequality in (7) for $M_0 := 1 + 2\sigma^2\|\mathcal{C}\|\|\mathcal{B}\|/\delta$ (use (2) and its dual). The last claim follows from the others and (4) (see (j3)–(j5) of Lemma A.3.1 of Mikkola [30]). $\square$

In the above we have used the following two lemmas. In the first one we present some sort of a singular value decomposition with k largest singular values on the diagonal and a small operator on the bottom-right corner.

LEMMA 6.2 (partial singular value decomposition). *Assume that $\{\sigma_k\}$ are the singular values of $S \in \mathcal{L}(X, Y)$ and X, Y are Hilbert spaces.*

(1) *For any $k \in \{0, 1, 2, \dots\}$, there is a k-dimensional subspace $X_k \subset X$ such that $S^*S = \operatorname{diag}(\sigma_1^2, \dots, \sigma_k^2; T)$ on $X_k \times X_k^\perp = X$, $\|T\| = \sigma_{k+1}^2$.*

(2) *We have $\sigma^2 \in \rho(S^*S)$ and $\|(\sigma^2 - S^*S)^{-1}\| \le \delta^{-1}$, where $\delta := \min\{\sigma_k^2 - \sigma^2, \sigma^2 - \sigma_{k+1}^2\}$.*

Claim (1) follows from pp. 212–213 of [21], alternatively, by using a resolution of the identity of S^*S. Claim (2) follows, because $(\sigma^2 - S^*S)^{-1} = \operatorname{diag}((\sigma_1^2 - \sigma^2)^{-1}, \dots, (\sigma_k^2 - \sigma^2)^{-1}; (\sigma^2 - T)^{-1})$. Recall that $\sigma_k := \inf\{\|S - K\| : K \in \mathcal{L}(X, Y),\ \operatorname{rank} K \le k - 1\} = \inf_{\dim M \le k-1} \|SP_{M^\perp}\|$, where $P_{M^\perp}$ is the orthogonal projection $X \to M^\perp$.

LEMMA 6.3 ($\liminf_n \sigma_{k,n} \ge \sigma_k$). *Let $S_n, S \in \mathcal{L}(X, Y)$ for all n, and let $S_n^*S_nx \to S^*Sx$ for all $x \in X$, where X, Y are Hilbert spaces. Then* $\liminf_{n\to\infty} \sigma_{k,n} \ge \sigma_k$, *where $\sigma_{k,n}$ is the kth singular value of S_n ($n \in \mathbb{N}$).*

Proof. Given $\epsilon > 0$, choose N such that $\|(S_n^*S_n - S^*S)P\| < \epsilon$ for all $n \ge N$, where $P : X \to X_{k-1}$ is the orthogonal projection and X_{k-1} is as in Lemma 6.2.

We obviously have $\sigma_k = \inf_{\dim M \le k-1} \|SP_{M^\perp}\|$, where $P_{M^\perp}$ is the orthogonal projection $X \to M^\perp$. But if $\dim M \le k - 2$, then there is $x \in M^\perp \cap X_{k-1}$ such that $\|x\| = 1$ (otherwise we would have $X_{k-1} \subset \operatorname{Ker}(P_{M^\perp}) = M$). Then $\|S_nx\| = \langle Px, S_n^*S_nPx\rangle \ge \langle x, S^*Sx\rangle - \epsilon \ge \sigma_k - \epsilon$. $\square$

Acknowledgments. The authors would like to thank Professor Ruth Curtain of the Department of Mathematics, University of Groningen, The Netherlands, and the anonymous referees for useful comments on the previous draft, which led to improvements in the exposition.

REFERENCES

[1] V. M. ADAMJAN, D. Z. AROV, AND M. G. KREĬN, *Infinite Hankel block matrices and related problems of extension*, Amer. Math. Soc. Transl. Ser. 2, 111 (1978), pp. 133–156 (in English); Izv. Akad. Nauk. Armjan. SSR Ser. Mat., 6 (1971), pp. 87–112 (in Russian).

[2] J. A. BALL, *Nevanlinna-Pick interpolation: Generalizations and applications*, in Surveys of Some Recent Results in Operator Theory I, J. B. Conway and B. B. Morrell, eds., Pitman, Boston, 1988, pp. 51–94.

[3] J. A. BALL, I. GOHBERG, AND L. RODMAN, *Interpolation of Rational Matrix Functions*, Oper. Theory Adv. Appl. 45, Birkhäuser-Verlag, Basel, Switzerland, 1990.

[4] J. A. BALL AND J. W. HELTON, *A Beurling-Lax theorem for the Lie group $U(m, n)$ which contains most classical interpolation theory*, J. Operator Theory, 9 (1983), pp. 107–142.

[5] J. A. BALL AND J. W. HELTON, *Factorization results related to shifts in an indefinite metric*, Integral Equations Operator Theory, 5 (1982), pp. 632–658.

[6] J. A. BALL AND J. W. HELTON, *Shift invariant subspaces, passivity, reproducing kernels and H_∞-optimization*, in Contributions to Operator Theory and Its Applications, I. Gohberg, J. W. Helton, and L. Rodman, eds., Oper. Theory Adv. Appl. 35, Birkhäuser-Verlag, Basel, Switzerland, 1988, pp. 265–310.

[7] J. A. BALL AND A. C. M. RAN, *Optimal Hankel norm model reductions and Wiener–Hopf factorization* I: *The canonical case*, SIAM J. Control Optim., 25 (1987), pp. 362–382.

[8] J. S. Baras and R. W. Brockett, *H^2-functions and infinite-dimensional realization theory*, SIAM J. Control Optim., 13 (1975), pp. 221–241.

[9] J. Bognár, *Indefinite Inner Product Spaces*, Springer-Verlag, New York, Heidelberg, Berlin, 1974.

[10] K. F. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*, Oper. Theory Adv. Appl. 3, Birkhäuser-Verlag, Basel, Boston, 1981.

[11] R. F. Curtain and J. C. Oostveen, *The Nehari problem for nonexponentially stable systems*, Integral Equations Operator Theory, 31 (1998), pp. 307–320.

[12] R. F. Curtain and A. C. M. Ran, *Explicit formulas for Hankel norm approximations of infinite-dimensional systems*, Integral Equations Operator Theory, 12 (1989), pp. 455–469.

[13] R. F. Curtain and A. J. Sasane, *Sub-optimal Hankel norm approximation for the analytic class of infinite-dimensional systems*, Integral Equations Operator Theory, 43 (2002), pp. 356–377.

[14] R. F. Curtain and A. J. Sasane, *Sub-optimal Hankel norm approximation for the Pritchard–Salamon class of infinite-dimensional systems*, Integral Equations Operator Theory, 39 (2001), pp. 98–126.

[15] R. F. Curtain and A. J. Sasane, *Hankel norm approximation for well-posed linear systems*, Systems Control Lett., 48 (2003), pp. 407–414.

[16] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Texts Appl. Math. 21, Springer-Verlag, New York, 1995.

[17] J. C. Doyle, K. Glover, and K. Zhou, *Robust and Optimal Control*, Prentice–Hall, Englewood Cliffs, NJ, 1996.

[18] N. Dunford and L. T. Schwartz, *Linear Operators Part* I: *General Theory*, John Wiley & Sons, New York, 1958.

[19] K. Glover, *All optimal Hankel-norm approximations of linear multivariable systems and their L_∞ error bounds*, Internat. J. Control, 39 (1984), pp. 1115–1193.

[20] K. Glover, R. F. Curtain, and J. R. Partington, *Realization and approximation of linear infinite-dimensional systems with error bounds*, SIAM J. Control Optim., 26 (1988), pp. 863–898.

[21] I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators*, Vol. I, Oper. Theory Adv. Appl. 49, Birkhäuser-Verlag, Basel, Switzerland, 1990.

[22] I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators*, Vol. II, Oper. Theory Adv. Appl. 63, Birkhäuser-Verlag, Basel, Switzerland, 1993.

[23] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Grad. Texts in Math. 19, Springer-Verlag, New York, Berlin, 1982.

[24] S. Hansen and G. Weiss, *New results on the operator Carleson measure criterion*, IMA J. Math. Control Inform., 14 (1997), pp. 3–32.

[25] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, London, 1964.

[26] O. Iftime, M. A. Kaashoek, H. Sandberg, and A. Sasane, *Grassmannian Approach to the Hankel Norm Approximation Problem*, preprint, Mittag–Leffler Institute, Stockholm, Sweden, 2003.

[27] M. G. Kreĭn and H. Langer, *Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume Π_κ*, Colloq. Math. Soc. János Bolyai, 5 (1972), pp. 353–399.

[28] M. G. Kreĭn and H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen.* I. *Einige Funktionenklassen und ihre Darstellungen*, Math. Nachr., 77 (1977), pp. 187–236.

[29] S. Y. Kung and D. W. Lin, *A state-space formulation for optimal Hankel-norm approximations*, IEEE Trans. Automat. Control, 26 (1981), pp. 942–946.

[30] K. M. Mikkola, *Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations*, Ph.D. thesis, Helsinki University of Technology, Helsinki, Finland, 2002; available online at http://www.math.hut.fi/~kmikkola/research/thesis/.

[31] A. M. Nikolaičuk and I. M. Spitkovsky, *Factorization of Hermitian matrix-functions and their applications to boundary value problems*, Ukrainian Math. J., 27 (1975), pp. 767–779.

[32] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, American Elsevier, New York, 1970.

[33] J. Oostveen, *Strongly Stabilizable Distributed Parameter Systems*, Frontiers Appl. Math. 20, SIAM, Philadelphia, 2000.

[34] V. V. Peller, *Hankel Operators and Their Applications*, Springer Monogr. Math., Springer-Verlag, New York, 2003.

[35] L. Rodman, I. M. Spitkovsky, and H. J. Woerdeman, *Abstract Band Method via Factorization, Positive and Band Extensions of Multivariable Almost Periodic Matrix Functions, and Spectral Estimation*, Mem. Amer. Math. Soc. 762, AMS, Providence, RI, 2002.

[36] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford Math. Monogr., Oxford University Press, New York, 1985.
[37] H. L. Royden, *Real Analysis*, 3rd ed., Prentice–Hall, Englewood Cliffs, NJ, 1988.
[38] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc., 127 (1967), pp. 179–201.
[39] D. Sarason, *Operator-theoretic aspects of the Nevanlinna-Pick interpolation problem*, in Operators and Function Theory (Lancaster, 1984), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 153, S. C. Power, ed., D. Reidel, Dordrecht, The Netherlands, 1985, pp. 279–314.
[40] A. J. Sasane, *Hankel Norm Approximation for Infinite-Dimensional Systems*, Lecture Notes in Control and Inform. Sci. 277, Springer-Verlag, Berlin, 2002.
[41] T. P. Srinivasan, *Simply invariant subspaces*, Bull. Amer. Math. Soc., 69 (1963), pp. 706–709.
[42] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., John Wiley & Sons, New York, 1980.