SUFFICIENT CONDITIONS FOR THE PROJECTIVE FREENESS OF BANACH ALGEBRAS

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Abstract. Let R be a unital semi-simple commutative complex Banach algebra, and let $M(R)$ denote its maximal ideal space, equipped with the Gelfand topology. Sufficient topological conditions are given on $M(R)$ for R to be a projective free ring, that is, a ring in which every finitely generated projective R-module is free. Several examples are included, notably the Hardy algebra $H^\infty(X)$ of bounded holomorphic functions on a Riemann surface of finite type, and also some algebras of stable transfer functions arising in control theory.

1. Introduction

The aim of this article is to give sufficient conditions on the maximal ideal space of a Banach algebra R which guarantee that R is a projective free ring. (Throughout the paper, we will consider only complex semi-simple commutative unital Banach algebras.) The precise definition of a projective free ring is recalled below.

Definition 1.1. Let R be a commutative ring with identity. The ring R is said to be *projective free* if every finitely generated projective R-module is free. Recall that if M is an R-module, then

(1) M is called *free* if $M \cong R^d$ for some integer $d \geq 0$;
(2) M is called *projective* if there exists an R-module N and an integer $d \geq 0$ such that $M \oplus N \cong R^d$.

In terms of matrices (see [5, Proposition 2.6]), the ring R is projective free iff every square idempotent matrix F is conjugate (by an invertible matrix) to a matrix of the form

$$\operatorname{diag}(I_k, 0) := \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}.$$

From the matricial definition, we see for example that any field $\mathbf{k}$ is projective free, since matrices F satisfying $F^2 = F$ are diagonalizable over $\mathbf{k}$. Quillen and Suslin (see [13]) proved, independently, that the polynomial ring over a projective free ring is again projective free. From this we obtain that the polynomial ring $\mathbf{k}[x_1, \dots, x_n]$ is projective free. Also, if R is any

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projective free ring, then the formal power series ring $R[[x]]$ in a central indeterminate x is again projective free [6, Theorem 7]. So it follows that the ring of formal power series $\mathbf{k}[[x_1, \dots, x_n]]$ is also projective free.

However, very little seems to be known about the projective freeness of topological rings arising in analysis. H. Grauert [9] proved that the ring $H(\mathbb{D}^n)$ of holomorphic functions on the polydisk

$$\mathbb{D}^n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_k| < 1,\ k = 1, \dots, n\}$$

is projective free. In fact, from Grauert's celebrated theorem [10], by the method of the proof of our Theorem 1.2, one obtains that the ring $H(X)$ of holomorphic functions on a connected reduced Stein space X satisfying the property that any holomorphic vector bundle of finite rank on X is topologically trivial, is projective free. For instance, this is the case if the space X is contractible or if it is biholomorphic to a connected noncompact (possibly singular) Riemann surface.

In this article, we formulate certain topological conditions on the maximal ideal space of a Banach algebra R that guarantee the projective freeness of R. Our motivation arises from a result of Lin [14, Theorem 3] (see also Tolokonnikov [22, Theorem 4]), which says that the contractibility of the maximal ideal space $M(R)$ implies that the Banach algebra R is a Hermite ring. The concept of a Hermite ring is a weaker notion than that of a projective free ring. Indeed, a commutative ring R with identity is said to be *Hermite* if every finitely generated stably free R-module is free. Recall that a R-module M is called *stably free* if there exist nonnegative integers n, d such that $M \oplus R^n = R^d$. Clearly every stably free module is projective, and so every projective free ring is Hermite. In this article we prove, in particular, that contractibility of the maximal ideal space of a Banach algebra in fact implies not just Hermiteness, but also projective freeness.

In the subsequent results the maximal ideal space $M(R)$ of a semi-simple commutative unital complex Banach algebra R is considered with the Gelfand topology. Our main results are the following:

Theorem 1.2. *Let R be a semi-simple commutative unital complex Banach algebra. If the Banach algebra $C(M(R))$ of complex continuous functions on $M(R)$ is a projective free ring, then R is a projective free ring.*

Theorem 1.3. *Let R be a semi-simple commutative unital complex Banach algebra. If $M(R)$ is connected and each complex vector bundle of finite rank on $M(R)$ is topologically trivial, then $C(M(R))$ (and so R) is a projective free ring.*

As a corollary we then have the following:

Corollary 1.4. *Let R be a semi-simple commutative unital complex Banach algebra. If $M(R)$ is connected and homotopically equivalent to a compact space of covering dimension ≤ 4, and such that $H^2(M(R), \mathbb{Z})$ and $H^4(M(R), \mathbb{Z})$ are trivial, then $C(M(R))$ (and so R) is a projective free ring.*

In particular, if $M(R)$ is contractible, then $C(M(R))$ (and so R) is a projective free ring.

Here $H^*(X,\mathbb{Z})$ stand for Čech cohomology groups with integer coefficients of X. Also, recall that the *covering dimension* of a normal space X is the least integer n with the property that if $\mathcal{U}$ is a finite open covering of X, there is another open covering $\mathcal{V}$ such that every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$ and every $x \in X$ is in at most $n+1$ elements of $\mathcal{V}$. If there is no such n, we say that the covering dimension of X is infinite.

Several examples are included as applications of the above results. E.g., by the deep result of Suárez [21, Theorem 4.5, Corollary 3.9], the maximal ideal space $M(H^\infty(\mathbb{D}))$ is connected, has covering dimension 2 and $H^2(M(H^\infty(\mathbb{D})),\mathbb{Z}) = 0$. In particular, $H^\infty(\mathbb{D})$ is projective free. The same result was proved in [15] by a different method. Using some results from [3] we can in fact prove a more general result. For its formulation we require the following definitions.

Let $N \Subset M$ be a relatively compact domain in an open Riemann surface M such that
$$\pi_1(N) \cong \pi_1(M).$$
(Here $\pi_1(X)$ stands for the fundamental group of X.) Let R be an unbranched covering of N and $i : U \hookrightarrow R$ be a domain in R. Assume that

the induced homomorphism $i_* : \pi_1(U) \to \pi_1(R)$ is injective.

Then we have:

Theorem 1.5. *The Hardy algebra $H^\infty(U)$ of bounded holomorphic functions on U is projective free.*

As a particular case of this result we obtain that $H^\infty(X)$ is projective free for every Caratheodory hyperbolic (that is, $H^\infty(X)$ separates the points in X) Riemann surface X of finite type. In fact, according to a result of Stout [20], such an X is biholomorphic to a compact Riemann surface with finitely many closed disks and points removed. In Theorem 1.5, taking U as the interior of the closure of the image of X in this Riemann surface, we get the required result from our theorem.

We also list some other Banach algebra examples of projective free rings in Section 3. Some of these algebras of holomorphic functions arise as natural classes of stable transfer functions in Control Theory. We also briefly explain the relevance of projective free rings in solving the stabilization problem in control theory. In fact we answer the open question posed in [16], namely whether or not the rings $W^+(\mathbb{D})$ and $\mathcal{A}$ (defined in Section 3) are projective free.

2. Proof of the main results

Proof of Theorem 1.2: The set V of idempotent $n \times n$ matrices is defined by a system of algebraic equations:
$$V = \{Z \in M_n(\mathbb{C}) \mid Z^2 - Z = 0\}.$$

In particular, V is a closed complex algebraic subvariety of $\mathbb{C}^{n^2} \cong M_n(\mathbb{C})$ and so it is a Stein manifold. Next, V is the disjoint union of connected components V_k consisting of matrices of rank k, $0 \le k \le n$. The component V_k is

$$\{G^{-1}\mathrm{diag}(I_k, 0)G \in M_n(\mathbb{C}) \mid G \in GL_n(\mathbb{C})\}.$$

In fact, each V_k is a complex submanifold of $\mathbb{C}^{n^2}$ and so it is a Stein manifold. To check it, note that V_k is the image of the complex homogeneous manifold $GL_n(\mathbb{C})/G_k$ where G_k is the group of all invertible matrices of the form $\mathrm{diag}(H_1, H_2)$ where H_1 is of size $k \times k$ and H_2 is of size $(n-k) \times (n-k)$. The map $GL_n(\mathbb{C}) \to V_k$, $G \mapsto G^{-1}\mathrm{diag}(I_k, 0)G$, can be taken down to the surjective map of $GL_n(\mathbb{C})/G_k \to V_k$ because elements of G_k commute with $\mathrm{diag}(I_k, 0)$. Also, one can easily check that this map is of maximal rank (computing the Jacobi matrix at a point), and so V_k is smooth.

Let $U \subset V_k$ be an open subset biholomorphic to $\mathbb{D}^l$, $l := \dim_{\mathbb{C}} V_k$. By Z we denote the matrix complex coordinate on $M_n(\mathbb{C}) = \mathbb{C}^{n^2}$. Consider the holomorphic function $Z|_U : U \to M_n(\mathbb{C})$. By the definition, $(Z|_U)^2 = Z|_U$. Thus according to Grauert's theorem [9] and [5, Proposition 2.6], there is a holomorphic matrix function $f_U : U \to GL_n(\mathbb{C})$ such that

$$f_U^{-1}(Z|_U)f_U = \mathrm{diag}(I_k, 0).$$

Let $\mathcal{U}$ be an open cover of V_k by open sets biholomorphic to $\mathbb{D}^l$. For $U, W \in \mathcal{U}$ such that $U \cap W \neq \emptyset$ we define

$$c_{UW} := f_U^{-1} f_W \quad \text{on} \quad U \cap W.$$

By the definition c_{UW} commutes with $\mathrm{diag}(I_k, 0)$, and so it takes its values in the group $G_k = GL_k(\mathbb{C}) \times GL_{n-k}(\mathbb{C})$. Thus we obtain a holomorphic 1-cocycle on $\mathcal{U}$ with values in G_k. We denote by $\pi : E_k \to V_k$ the principal vector bundle on V_k with fibre G_k constructed by the cocycle $\{c_{UW}\}$.

Assume now that $C(M(R))$ is a projective free ring. Then clearly $M(R)$ is connected. An idempotent F with entries in R can be regarded as a continuous map $F : M(R) \to V$, and so by connectedness its image belongs to one of V_k. Consider the principal bundle F^*E_k on $M(R)$ (the pullback of E_k with respect to F). From the projective freeness of $C(M(R))$, it follows that there is a continuous matrix function $h : M(R) \to GL_n(\mathbb{C})$ such that $h^{-1}Fh = \mathrm{diag}(I_k, 0)$. Consider the matrices

$$h_U := h^{-1}(f_U \circ F) \quad \text{on} \quad F^{-1}(U) \quad \text{for} \quad U \cap F(M(R)) \neq \emptyset.$$

By the definition we have

$$h_U^{-1}\mathrm{diag}(I_k, 0)h_U = \mathrm{diag}(I_k, 0).$$

Thus $h_U : F^{-1}(U) \to G_k$ and $h_U^{-1}h_V = c_{UV} \circ F$ on $F^{-1}(U \cap V)$. This shows that the bundle F^*E_k is topologically trivial.

Further, let Γ be a partially ordered (by inclusion) set of all finitely generated closed subalgebras of algebra R having the common unit with R. For each subalgebra $\gamma \in \Gamma$, we denote its maximal ideal space by M_γ. We will naturally identify M_γ with a polynomially convex subset of $\mathbb{C}^{k(\gamma)}$ where $k(\gamma)$ is the number of generators of γ. If $\gamma, \beta \in \Gamma$ and $\gamma \geq \beta$, consider the natural projection $\omega^\gamma_\beta : \mathbb{C}^{k(\gamma)} \to \mathbb{C}^{k(\beta)}$ on the first $k(\beta)$ coordinates. Then ω^γ_β maps M_γ into M_β and the inverse limiting system of compact sets $\{M_\gamma, \omega\}_{\gamma\in\Gamma}$ is defined. Its inverse limit is naturally identified with $M(R)$ and the projection $\omega_\gamma : M(R) \to M_\gamma$ sends each homomorphism $R \to \mathbb{C}$ to its restriction to the subalgebra γ (see for example Royden [18] for background on the theory of function algebras).

Let $\alpha \subset R$ be a closed subalgebra generated by entries of an idempotent F and 1. Clearly the idempotent F is also defined on M_α and so it determines a continuous map $F_\alpha : \mathbb{C}^{k(\alpha)} \to \mathbb{C}^{n^2}$. By definition we have $F = F_\alpha \circ \omega_\alpha$. It is well known that $F_\alpha(M_\alpha)$ coincides with the polynomially convex hull in $\mathbb{C}^{n^2}$ of compact set $F(M(R))$. Since $F(M(R))$ belongs to a closed algebraic subvariety V_k of $\mathbb{C}^{n^2}$, its polynomially convex hull belongs to V_k as well. Thus F_α maps M_α into V_k.

Consider the system of principal vector bundles $F^*_\alpha E_k$ on M_α (the pullback of E_k with respect to F_α). By the definition we have $\omega^*_\alpha(F^*_\alpha E_k) = F^* E_k$. Since F^*E_k is topologically trivial, there is an index α such that $F^*_\alpha E_k$ is also topologically trivial. (It follows for example from basic results on bundles presented in [7] and [12].) Observe that $F^*_\alpha E_k$ is also a well-defined holomorphic bundle on the closed algebraic submanifold $F^{-1}_\alpha(V_k) \subset \mathbb{C}^{k(\alpha)}$. Since $M_\alpha \subset F^{-1}_\alpha(V_k)$ and $F^*_\alpha E_k|_{M_\alpha}$ is topologically trivial, there exists an open neighbourhood $U_\alpha \subset \mathbb{C}^{k(\alpha)}$ of M_α such that $F^*_\alpha E_k|_{U_\alpha}$ is topologically trivial (see for instance [14, Lemma 4]). Since M_α is a holomorphically convex compact subset of the Stein manifold $F^{-1}_\alpha(V_k)$, there exists an analytic open Weil polyhedron $W_\alpha \subset U$ containing M_α. Thus $F^*_\alpha E_k|_{W_\alpha}$ is a topologically trivial fibre bundle whose fibre is a complex Lie group G_k. According to Grauert's theorem [10], $F^*_\alpha E_k|_{W_\alpha}$ is also holomorphically trivial. This implies that there are holomorphic functions $g_U : F^{-1}_\alpha(U) \cap W_\alpha \to G_k$ such that

$$g_U^{-1} g_V = c_{UV} \circ F_\alpha \quad \text{on} \quad F^{-1}_\alpha(U \cap V) \cap W_\alpha.$$

Let us define

$$G := (f_U \circ F_\alpha) g_U^{-1} \quad \text{on} \quad F^{-1}_\alpha(U) \cap W_\alpha.$$

By the definition G is a holomorphic function on W_α with values in $GL_n(\mathbb{C})$ and $G^{-1}(F \circ F_\alpha)G = \operatorname{diag}(I_k, 0)$. Let $\widetilde{G} := G \circ \omega_\alpha$. Then $\widetilde{G}$ is an invertible matrix with entries in R and $\widetilde{G}^{-1} F \widetilde{G} = \operatorname{diag}(I_k, 0)$. This completes the proof of the theorem. □

Proof of Theorem 1.3: The proof follows immediately from the proof of Theorem 1.2 which relies on connectedness of $M(R)$ and the triviality of a certain complex vector bundle of finite rank on $M(R)$. □

Proof of Corollary 1.4: Assume that $M(R)$ is connected and homotopically equivalent to a compact topological space X of covering dimension ≤ 4. From the assumptions of the corollary it follows that $H^2(X, \mathbb{Z}) = H^4(X, \mathbb{Z}) =$

0. Since the covering dimension of X is ≤ 4, by the Freudenthal expansion theorem [8], X can be presented as an inverse limit of a sequence of compact polyhedra $\{Q_j\}_{j\in\mathbb{N}}$ with $\dim Q_j \le 4$. Let $\pi_j : X \to Q_j$, $\pi_i^j : Q_j \to Q_i$ be the continuous projections in the inverse limit construction. Then by a well-known theorem about continuous maps of inverse limits of compact spaces (see for example [7]) and the fact that all complex vector bundles of rank n on X can be obtained as pullbacks of the universal bundle $EU(n)$ on the classifying space $BU(n)$ of the unitary group $U(n) \subset GL_n(\mathbb{C})$ under some continuous maps $X \to BU(n)$ (see for example [12]), for each complex vector bundle E on X of rank n there is $j_0 \in \mathbb{N}$ and a complex vector bundle E_{j_0} of rank n on Q_{j_0} such that the pullback $\pi_{j_0}^* E_{j_0}$ is isomorphic to E. Further, it is well known (see for example [2, Chapter II, Corollary 14.6]) that the injective limit of Čech cohomology groups with coefficients in $\mathbb{Z}$ with respect to the family $\{\pi_i^j : Q_j \to Q_i\}$ gives the corresponding Čech cohomology groups of X. Thus using this and the conditions of the corollary, without loss of generality we may assume that j_0 is so large that $H^2(Q_{j_0}, \mathbb{Z}) = H^4(Q_{j_0}, \mathbb{Z}) = 0$.

Assume first that $n = 1$, that is that the rank of E is one. Then it is well-known that E is topologically trivial if and only if its first Chern class $c_1(E)$ represents 0 in $H^2(X, \mathbb{Z})$. Since the latter is the zero group, this implies that E is topologically trivial. Assume now that $n \ge 2$. Consider the bundle E_{j_0} on Q_{j_0}. Since Q_{j_0} is a CW-complex of dimension ≤ 4, the only obstruction for E_{j_0} to have a nowhere zero continuous section lies in the group $H^4(Q_{j_0}, \mathbb{Z})$ (see for example [12]). Since this group is trivial, E_{j_0} always has such a section. Proceeding by induction on rank n and using that each rank-1 complex vector bundle on Q_{j_0} is trivial (because $H^2(Q_{j_0}, \mathbb{Z}) = 0$) we obtain that E_{j_0} has n orthonormal sections. Thus E_{j_0} is topologically trivial. Since $\pi_{j_0}^* E_{j_0}$ is isomorphic to E, the last bundle is topologically trivial as well.

Thus we proved that every complex vector bundle on X of finite rank is topologically trivial. Since $M(R)$ is homotopically equivalent to X, the same is true for complex vector bundles of finite rank on $M(R)$. Now we apply Theorem 1.3 to get the required result. □

3. Examples and remarks

3.1. $H^\infty(U)$ where U is a Riemann surface satisfying the conditions of Theorem 1.5. By $H_n^\infty(U)$ we denote the $H^\infty(U)$-module consisting of columns $(f_1, \dots, f_n)$, $f_i \in H^\infty(U)$. Any $H^\infty(U)$-invariant subspace of $H_n^\infty(U)$ is called a *submodule*. We say that a submodule $M \subset H_n^\infty(U)$ is *closed in the topology of the pointwise convergence on U* if for any net $\{f_\alpha\} \subset M$ which pointwise converges on U to an $f \in H_n^\infty(U)$, we have $f \in M$. The following result was proved in [3, Theorem 1.7]:

Proposition 3.1. *Let $M \subset H_n^\infty(U)$ be a submodule closed in the topology of the pointwise convergence on U. Then for some $k \le n$ the module M can* *be represented as* $M = H \cdot H_k^\infty(U)$ *where* H *is a* $n \times k$ *matrix with entries in* $H^\infty(U)$*. Moreover, if* $r : \mathbb{D} \to U$ *is the universal covering map, then the pullback matrix* r^*H *with entries in* $H^\infty(\mathbb{D})$ *is left invertible at almost each point of the boundary* $\partial\mathbb{D}$*.*

Let us prove now Theorem 1.5.

Proof of Theorem 1.5: Let F be an idempotent of size $n \times n$ with entries in $H^\infty(U)$. By the definition the matrix F determines a linear operator $H_n^\infty(U) \to H_n^\infty(U)$. The set $M_1 := \mathrm{im}(F)$ is a submodule of $H_n^\infty(U)$ which, by the definition, coincides with $\ker(I_n - F)$. Thus M_1 is a submodule closed in the topology of the pointwise convergence on U. According to the above result $M_1 = H_1 \cdot H_k^\infty(U)$ where H_1 is an $n \times k$ matrix with entries in $H^\infty(U)$. Also, from the second part of Proposition 3.1 we obtain that there exists a point ξ of the maximal ideal space $M(H^\infty(U))$ such that $H_1(\xi)$ is left invertible. (Here we naturally extend functions from $H^\infty(U)$ to $M(H^\infty(U))$ by the Gelfand transform.) Such ξ belongs to the image of the Šilov boundary of $M(H^\infty(\mathbb{D}))$ under the map $M(H^\infty(\mathbb{D})) \to M(H^\infty(U))$ generated by the map $r : \mathbb{D} \to U$. Since F has the same rank at each point of the maximal ideal space $M(H^\infty(U))$ (because $M(H^\infty(U))$ is connected), the invertibility of $H_1(\xi)$ implies that k equals the rank of F. In particular, H_1 is left invertible at each $\xi \in M(H^\infty(U))$.

Similarly $M_2 = \ker F = \mathrm{im}(I - F)$ is a submodule of $H_n^\infty(U)$ closed in the topology of the pointwise convergence on U. Thus $M_2 = H_2 \cdot H_{n-k}^\infty(U)$ where H_2 is a left invertible $n \times (n-k)$ matrix with entries in $H^\infty(U)$. From the fact $M_1 \cap M_2 = \{0\}$, it easily follows that the matrix $H = (H_1, H_2)$ is invertible with entries in $H^\infty(U)$ and H^{-1} also has entries in $H^\infty(U)$. Moreover, $H^{-1}FH = \mathrm{diag}(I_k, 0)$. This completes the proof of Theorem 1.5. $\square$

Let us present an example of a (flat) Riemann surface U satisfying conditions of Theorem 1.5.

Consider the standard action of the group $\mathbb{Z} + i\mathbb{Z}$ on $\mathbb{C}$ by translations. The fundamental domain of this action is the square

$$R := \{z = x + iy \in \mathbb{C} \mid \max\{|x|, |y|\} \le 1\}.$$

By R_t we denote the square similar to R with side length t. Let O be the orbit of $0 \in \mathbb{C}$ with respect to the action of $\mathbb{Z} + i\mathbb{Z}$. For any $x \in O$ we will choose some $t(x) \in [\frac{1}{2}, \frac{3}{4}]$ and consider the square $R(x) := x + R_{t(x)}$ centered at x. Let $V \subset \mathbb{C}$ be a simply connected domain satisfying the property:

there exists a subset $\{x_i\}_{i \in I} \subset O$ *such that* $V \cap \left(\bigcup_{x \in O} R(x)\right) = \bigcup_{i \in I} R(x_i).$

We set $U := V \setminus (\cup_{i \in I} R(x_i))$. Then U satisfies the required conditions. In fact, the quotient space $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ is a torus $\mathbb{CT}$. Let S be the image of $R_{1/3}$ in $\mathbb{CT}$. Then U belongs to the covering C of $\mathbb{CT} \setminus S$ with covering group $\mathbb{Z} + i\mathbb{Z}$. The condition that the embedding $U \hookrightarrow C$ induces an injective homomorphism of fundamental groups follows from the construction of U.

3.2. **Other Banach algebras of holomorphic functions.** Let $U \subset \mathbb{C}^n$ be the direct product of bounded domains $U_j \subset \mathbb{C}^{n_j}$, where $1 \leq j \leq k$, $n = n_1 + \cdots + n_k$, and the U_j are either convex or strictly pseudoconvex. Assume that the closure $\overline{U}$ of U is homotopically equivalent to a 4-dimensional compact CW-complex X with $H^2(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ equal to 0. Then the maximal ideal space of the Banach algebra $A(U)$ of holomorphic functions on U continuous on $\overline{U}$ (with pointwise operations and the supremum norm) can be identified with $\overline{U}$ (this can be seen using approximation theorems from [11]). Applying Corollary 1.4, it follows that $A(U)$ is a projective free ring. In particular, the polydisk algebra $A(\mathbb{D}^n)$ is a projective free ring.

Corollary 3.2. *With the notation above, $A(U)$ is a projective free ring.*

Let $W^+(\mathbb{D})$ denote the Wiener algebra,

$$W^+(\mathbb{D}) := \left\{ f : \overline{\mathbb{D}} \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic in } \mathbb{D} \text{ and } \sum_{k=0}^{\infty} |a_k| < \infty, \\ \text{where } f(z) = \sum_{k=0}^{\infty} a_k z^k \;\; (z \in \mathbb{D}) \end{array} \right\}.$$

The set $W^+(\mathbb{D})$, equipped with pointwise operations, and the norm

$$\|f\|_1 = \sum_{k=0}^{\infty} |a_k|, \text{ where } f = \sum_{k=0}^{\infty} a_k z^k \in W^+(\mathbb{D}),$$

is a Banach algebra, whose maximal ideal space can be identified with $\overline{\mathbb{D}}$. Applying Corollary 1.4, $W^+(\mathbb{D})$ is also a projective free ring. This answers the open question posed in [16].

Corollary 3.3. $W^+(\mathbb{D})$ *is a projective free ring.*

3.3. **Algebras of almost periodic functions.** The algebra AP^n of complex valued (uniformly) almost periodic functions is, by definition, the smallest closed subalgebra of $L^\infty(\mathbb{R}^n)$ that contains all the functions $e_\lambda := e^{i\langle \lambda, x\rangle}$. Here the variable $x = (x_1, \ldots, x_n)$ and the parameter $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, and

$$\langle \lambda, x \rangle := \sum_{k=1}^{n} \lambda_k x_k.$$

For any $f \in AP^n$, its *Bohr-Fourier series* is defined by the formal sum

$$\sum_{\lambda} f_\lambda e^{i\langle \lambda, x\rangle}, \quad x \in \mathbb{R}^n, \tag{1}$$

where

$$f_\lambda := \lim_{N \to \infty} \frac{1}{(2N)^n} \int_{[-N,N]^n} e^{-i\langle \lambda, x\rangle} a(x) dx, \quad \lambda \in \mathbb{R}^n,$$

and the sum in (1) is taken over the set $\sigma(f) := \{\lambda \in \mathbb{R}^n \mid f_\lambda \neq 0\}$, called the *Bohr-Fourier spectrum* of f. The Bohr-Fourier spectrum of every $f \in AP^n$ is at most a countable set.

The *almost periodic Wiener algebra* APW^n is defined as the set of all AP^n such that the Bohr-Fourier series (1) of f converges absolutely. The almost periodic Wiener algebra is a Banach algebra with pointwise operations and the norm $\|f\| := \sum_{\lambda\in\mathbb{R}^n} |f_\lambda|$. Let Δ be a nonempty subset of $\mathbb{R}^n$. Denote

$$\begin{aligned} AP^n_\Delta &= \{f \in AP^n \mid \sigma(f) \subset \Delta\} \\ APW^n_\Delta &= \{f \in APW^n \mid \sigma(f) \subset \Delta\}. \end{aligned}$$

If Δ is an additive subset of $\mathbb{R}^n$, then AP^n_Δ (respectively APW^n_Δ) is a Banach subalgebra of AP^n (respectively APW^n). Moreover, if $0 \in \Delta$, then AP^n_Δ and APW^n_Δ are also unital.

A subset S of $\mathbb{R}^n$ is said to be a *half-space* if it has the following properties:

(1) $\mathbb{R}^n = S \cup (-S)$;
(2) $S \cap (-S) = \{0\}$;
(3) if $\lambda, \mu \in S$, then $\lambda + \mu \in S$;
(4) if $\lambda \in S$ and α is a nonnegative real number, then $\alpha\lambda \in S$.

An example of a half-space in $\mathbb{R}^2$ is the union of the open right half-plane $\{(x, y) \mid x > 0\}$ and the ray $\{(0, y) \mid y \geq 0\}$.

Let $S \subset \mathbb{R}^n$ be a half-space, and let $\Sigma \subset S$ be an *additive semigroup* (if $\lambda, \mu \in \Sigma$, then $\lambda + \mu \in \Sigma$) and suppose $0 \in \Sigma$.

The following result was shown in [17] (which is a multidimensional extension of the result from [4]):

Proposition 3.4. *The maximal ideal space $M(AP^n_\Sigma)$ and $M(APW^n_\Sigma)$ are contractible.*

Thus we now have the following:

Corollary 3.5. *AP^n_Σ and APW^n_Σ are projective free rings.*

3.4. **Algebras of measures with support in $[0, +\infty)$.** Let $\mathcal{M}_+$ denote the set of all complex Borel measures with support contained in $[0, +\infty)$. Then $\mathcal{M}_+$ is a complex vector space with addition and scalar multiplication defined as usual, and it becomes a complex algebra if we take convolution of measures as the operation of multiplication. With the norm of μ taken as the total variation of μ, $\mathcal{M}_+$ is a Banach algebra. Recall that the *total variation* $\|\mu\|$ *of* μ is defined by

$$\|\mu\| = \sup \sum_{n=1}^{\infty} |\mu(E_n)|,$$

the supremum being taken over all *partitions of* $[0, +\infty)$, that is over all countable collections $(E_n)_{n\in\mathbb{N}}$ of Borel subsets of $[0, +\infty)$ such that $E_n \bigcap E_m = \emptyset$ whenever $m \neq n$ and $[0, +\infty) = \bigcup_{n\in\mathbb{N}} E_n$. The identity with respect to convolution in $\mathcal{M}_+$ is the *Dirac measure* δ, given by

$$\delta(E) = \begin{cases} 1 & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E. \end{cases}$$

If $\mu \in \mathcal{M}_+$ and $\theta \in [0, 1)$, then we define the complex Borel measure μ_θ as follows:

$$\mu_\theta(E) = \int_E (1 - \theta)^t d\mu(t),$$

where E is a Borel subset of $[0,+\infty)$. If $\theta = 1$, then we define $\mu_1 = \mu(\{0\})\delta$. The following was shown in [19]:

Proposition 3.6. *Suppose that R is a Banach subalgebra of $\mathcal{M}_+$, such that it has the property:*

$$\text{(P) For all } \mu \in R \text{ and for all } \theta \in [0,1],\ \mu_\theta \in R.$$

Then the maximal ideal space $M(R)$ is contractible.

The following are examples of R satisfying the property (P):

(1) The subalgebra of $\mathcal{M}_+$ consisting of all complex Borel measures of the type $\mu_a + \alpha\delta$, where μ_a is absolutely continuous (with respect to the Lebesgue measure) and $\alpha \in \mathbb{C}$.
(2) The subalgebra $\mathcal{A}$ of $\mathcal{M}_+$, consisting of all complex Borel measures that do not have a singular non-atomic part, also possesses the property (P).

Thus we now have the following:

Corollary 3.7. *If R is a Banach subalgebra of $\mathcal{M}_+$, such that it has the property* (P) *from Proposition 3.6, then R is a projective free ring. In particular $\mathcal{A}$ is a projective free ring.*

This answers the open question posed in [16].

3.5. **Relevance of projective free rings in Control Theory.** In the factorization approach to control theory [23], one starts with a normed unital commutative complex algebra R, thought of as the class of 'stable transfer functions of control linear systems'. (So depending on the class of systems under consideration, R could be $H^\infty(\mathbb{D})$, or $W^+(\mathbb{D})$ or the set $\widehat{\mathcal{A}}$ of Laplace transforms of elements of $\mathcal{A}$, or some other ring.) Let Q_R denote the field of fractions of R. The *stabilization problem* is the following:

Given $P \in Q_R^{p\times m}$ (called the *plant*), find $C \in Q_R^{m\times p}$ (called a *controller*) such that

$$\begin{bmatrix} I & -P \\ -C & I \end{bmatrix}^{-1} \in R^{(p+m)\times(p+m)}.$$

If such a C exists for a given P, then P is called *stabilizable*. It turns out that this stabilization problem can be solved if $P \in Q_R^{p\times m}$ has a *doubly coprime factorization*, that is, there exist $(N, D) \in R^{p\times m} \times R^{m\times m}$ and $(\widetilde{N}, \widetilde{D}) \in R^{p\times m} \times R^{p\times p}$ such that

(1) $\det D \neq 0$ and $\det \widetilde{N} \neq 0$,
(2) $P = ND^{-1}$ and $P = \widetilde{D}^{-1}\widetilde{N}$,
(3) there exist $(X, Y) \in R^{m\times p} \times R^{m\times m}$ and $(\widetilde{X}, \widetilde{Y}) \in R^{m\times p} \times R^{p\times p}$ such that $XN + YD = I$ and $\widetilde{N}\widetilde{X} + \widetilde{D}\widetilde{Y} = I$.

In fact, in the case the plant P has a doubly coprime factorization, one can characterize all stabilizing controllers C of the plant, and explicit formulae for C can be given in terms of the solutions $X, Y, \widetilde{X}, \widetilde{Y}$ of the equations in (3) above; see [23].

The following result [15, Theorem 6.3] explains the relevance of projective free rings in control theory.

Proposition 3.8. *If R is a projective free ring, then every stabilizable plant $P \in Q_R^{p\times m}$ has a doubly coprime factorization.*

In light of the above, it is then natural to ask the question if the standard classes of stable transfer functions used in control theory are projective free or not. The examples collected in this section show that all standard classes of stable transfer functions are indeed projective free. In particular, we have shown that $W^+(\mathbb{D})$ and $\mathcal{A}$ are projective free rings, hence answering the open question posed in [16].

Thus it was hitherto not known whether plants failing to possess a doubly coprime factorization over $W^+(\mathbb{D})$ or over $\mathcal{A}$ are stabilizable or not. But from our result, we can conclude that the plants that do not have a doubly coprime factorization over $W^+(\mathbb{D})$ or $\mathcal{A}$ are not stabilizable. For example, if we consider the plant $p = nd^{-1} \in Q_{W^+(\mathbb{D})}$, where

$$\begin{aligned} n &= (1-z)^3 e^{-\frac{1+z}{1-z}} \in W^+(\mathbb{D}), \\ d &= (1-z)^3 \in W^+(\mathbb{D}), \end{aligned}$$

then p does not have a doubly coprime factorization. Owing to the fact that $W^+(\mathbb{D})$ is projective free, we can conclude that p is not stabilizable.

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