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Complex Analysis and Operator Theory

Stable Ranks of Banach Algebras of Operator-Valued Analytic Functions

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Abstract. Let E be a separable infinite-dimensional Hilbert space, and let $H(\mathbb{D};\mathcal{L}(E))$ denote the algebra of all functions $f:\mathbb{D}\to\mathcal{L}(E)$ that are holomorphic. If $\mathcal{A}$ is a subalgebra of $H(\mathbb{D};\mathcal{L}(E))$, then using an algebraic result of Corach and Larotonda, we derive that under some conditions, the Bass stable rank of $\mathcal{A}$ is infinite. In particular, we deduce that the Bass (and hence topological stable ranks) of the Hardy algebra $H^\infty(\mathbb{D};\mathcal{L}(E))$, the disk algebra $A(\mathbb{D};\mathcal{L}(E))$ and the Wiener algebra $W_+(\mathbb{D};\mathcal{L}(E))$ are all infinite.

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1. Introduction

In this paper, we prove that the Bass and topological stable ranks of several common Banach algebras of operator-valued holomorphic functions are all infinite when the underlying vector space E is a separable infinite-dimensional Hilbert space.

1.1. Stable ranks

The notions of Bass/topological stable ranks play important roles in algebraic/topological K-theory (see [1] and [6]), but they also have applications in the control-theoretic problem of stabilization via a factorization approach [5]. We recall the definition of Bass stable rank and topological stable rank below.

Definition 1.1. Let $\mathcal{A}$ be a ring with identity element denoted by 1 and $n\in\mathbb{N}$. An element

$$a=(a_1,\dots,a_n)\in\mathcal{A}^n:=\underbrace{\mathcal{A}\times\dots\times\mathcal{A}}_{n\text{ times}}$$

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is called (*left*) *unimodular* if there exists $b = (b_1, \dots, b_n) \in \mathcal{A}^n$ such that

$$\sum_{k=1}^{n} b_k a_k = 1 \,. \tag{1.1}$$

We denote the set of unimodular elements of $\mathcal{A}^n$ by $U_n(\mathcal{A})$.

An element $a \in U_{n+1}(\mathcal{A})$ is called (*left*) *reducible* if there exists $x = (x_1, \dots, x_n) \in \mathcal{A}^n$ such that

$$(a_1 + x_1 a_{n+1}, \dots, a_n + x_n a_{n+1}) \in U_n(\mathcal{A}) \,. \tag{1.2}$$

The (*left*) *Bass stable rank of* $\mathcal{A}$, denoted by bsr $\mathcal{A}$, is the least integer $n \geq 1$ such that every $a \in U_{n+1}(\mathcal{A})$ is reducible, and it is infinite if no such integer n exists.

Now let $\mathcal{A}$ denote a Banach algebra. (By a Banach algebra we mean a complex Banach algebra with a unit element 1; we do not assume commutativity.) The (*left*) *topological stable rank of* $\mathcal{A}$, denoted by tsr $\mathcal{A}$, is the least integer $n \geq 1$ such that $U_n(\mathcal{A})$ is dense in $\mathcal{A}^n$, and it is infinite if no such integer exists.

Remark 1.2. Analogously one can define a *right* Bass/topological stable rank, by changing the multiplication order in (1.1) and (1.2).

It turns out that for any ring $\mathcal{A}$ the left Bass stable rank is always equal to the right Bass stable rank (see [10]).

Moreover, it is known, see [6, Proposition 1.6], that the left and right topological stable ranks are equal for a Banach algebra with a continuous involution $\star$, and so in this case one can unambiguously talk about *the* topological stable rank. In our case of subalgebras of $H(\mathbb{D}; \mathcal{L}(E))$ listed below in Definition 1.4, we use the involution $\cdot^\star$ defined as follows: $f^\star = (f(\cdot^*))^*$, and $\cdot^*$ denotes the complex conjugate and the adjoint of a bounded linear operator.

We will study the Bass and topological stable ranks of several Banach algebras of operator-valued holomorphic functions, and these are introduced in Definition 1.4 below.

1.2. Definitions and notation

Throughout the article, we will denote the complex conjugate of $z \in \mathbb{C}$ by z^*, and the closure of a set S by $\overline{S}$.

We will denote the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ by $\mathbb{D}$, the closed unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ by $\overline{\mathbb{D}}$, and the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ by $\mathbb{T}$.

Throughout the article, if $\mathcal{A}$ is a ring and $n \in \mathbb{N}$, then $\mathcal{A}^n$ will denote the set of all n-tuples with entries from $\mathcal{A}$. In Proposition 2.1 (and its proof), we will consider $\mathcal{A}^n$ as a left $\mathcal{A}$-module with componentwise addition and scalar multiplication (from the left):

$$\alpha \cdot (a_1, \dots, a_n) = (\alpha a_1, \dots, \alpha a_n) \,, \quad \alpha \in \mathcal{A} \,, \quad (a_1, \dots, a_n) \in \mathcal{A}^n \,.$$

Notation 1.3. Throughout this article, unless otherwise stated, E always denotes an infinite-dimensional separable complex Hilbert space with a fixed orthonormal basis $(e_n)_{n\in\mathbb{N}}$. This allows us to identify any $A \in \mathcal{L}(E)$ with the corresponding infinite matrix with respect to the orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Here $\mathcal{L}(E)$ denotes the complex Banach space of bounded linear operators from E to E, equipped with the operator norm. We will use $H(\mathbb{D};\mathcal{L}(E))$ to denote the algebra of functions $f : \mathbb{D} \to \mathcal{L}(E)$ that are holomorphic with pointwise operations of vector addition, scalar multiplication, and multiplication in the algebra.

The norm in E and $\mathcal{L}(E)$ will be denoted by $\| \cdot \|$ in order to distinguish it easily from the norms in the various subalgebras of $H(\mathbb{D};\mathcal{L}(E))$ considered in the article. The latter will be denoted by $\|\cdot\|$, with an appropriate subscript, as we will see in the definitions below.

Definition 1.4.

1. $H^\infty(\mathbb{D};\mathcal{L}(E))$ denotes the subalgebra of $H(\mathbb{D};\mathcal{L}(E))$ consisting of all functions that are bounded in $\mathbb{D}$. Equipped with the supremum norm, $\|f\|_\infty := \sup_{z\in\mathbb{D}} \| f(z) \|$, $H^\infty(\mathbb{D};\mathcal{L}(E))$ is a Banach algebra.
2. $A(\mathbb{D};\mathcal{L}(E))$ denotes the subalgebra of functions from $H^\infty(\mathbb{D};\mathcal{L}(E))$ that have a continuous extension to $\overline{\mathbb{D}}$. Endowed with the supremum norm $\|\cdot\|_\infty$, $A(\mathbb{D};\mathcal{L}(E))$ forms a Banach algebra.
3. More generally, if S be an open subset of $\mathbb{T}$, then $A_S(\mathbb{D};\mathcal{L}(E))$ is the subalgebra of functions from $H(\mathbb{D};\mathcal{L}(E))$ that are bounded in $\mathbb{D}$ and have a continuous extension to $\mathbb{D}\cup S$. With the supremum norm $\|\cdot\|_\infty$, $A_S(\mathbb{D};\mathcal{L}(E))$ is a Banach algebra. (Note that if $S = \emptyset$, then $A_S(\mathbb{D};\mathcal{L}(E)) = H^\infty(\mathbb{D};\mathcal{L}(E))$, while if $S = \mathbb{T}$, then $A_S(\mathbb{D};\mathcal{L}(E)) = A(\mathbb{D};\mathcal{L}(E))$.)
4. The *Wiener algebra* $W_+(\mathbb{D};\mathcal{L}(E))$ is the subalgebra of functions $f \in H(\mathbb{D};\mathcal{L}(E))$ with $\|f\|_1 := \sum_{n=0}^\infty \| f_n \| < \infty$, where $f = \sum_{n=0}^\infty z^n f_n$ ($z \in \mathbb{D}$, $f_n \in \mathcal{L}(E)$). Equipped with the norm given by $\|\cdot\|_1$, $W_+(\mathbb{D};\mathcal{L}(E))$ is a Banach algebra.
5. If $E = \mathbb{C}$, then we denote $H^\infty(\mathbb{D};\mathcal{L}(E))$, $A(\mathbb{D};\mathcal{L}(E))$, $W_+(\mathbb{D};\mathcal{L}(E))$ by H^∞, A, W_+, respectively.

1.3. Known results

Sergei Treil [9] proved that bsr $H^\infty = 1$, and Daniel Suárez [8] showed that tsr $H^\infty = 2$. The Bass stable rank of the ring of all finite square matrices of size n with entries from the ring $\mathcal{A}$ is related to the Bass stable rank of $\mathcal{A}$ [10, Theorem 3]:

$$\text{bsr } \mathcal{A}^{n\times n} = \lfloor -(\text{bsr } \mathcal{A} - 1)/n \rfloor + 1\,. \tag{1.3}$$

Here for $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the largest integer less than or equal to r. So when E is finite-dimensional, bsr $H^\infty(\mathbb{D};\mathcal{L}(E)) = 1$. There is also a similar relation relating the topological stable ranks when $\mathcal{A}$ is a Banach algebra with a continuous involution (see the proof of [6, Theorem 6.1]):

$$\text{tsr } \mathcal{A}^{n\times n} = \lceil (\text{tsr } \mathcal{A} - 1)/n \rceil + 1\,. \tag{1.4}$$

Here $\lceil r \rceil$ denotes the least integer greater than r. Hence tsr $H^\infty(\mathbb{D}; \mathcal{L}(E)) = 2$ when E is finite-dimensional.

The above results are also known for the disk algebra A and the Wiener algebra W_+: bsr $A = 1$ [3], tsr $A = 2$ [6], bsr $W_+ = 1$ [7], tsr $W_+ = 2$ [4]. Using (1.3) and (1.4), we also have bsr $A(\mathbb{D}; \mathcal{L}(E)) = 1$, tsr $A(\mathbb{D}; \mathcal{L}(E)) = 2$, bsr $W_+(\mathbb{D}; \mathcal{L}(E)) = 1$, tsr $W_+(\mathbb{D}; \mathcal{L}(E)) = 2$ when E is finite-dimensional.

1.4. Main result

Our main result is the following, which is proved in Section 3:

Theorem 1.5. *Let E be an infinite-dimensional separable Hilbert space. With the notation from Definition* 1.4*, the Bass stable ranks of the following rings is infinite:*

1. $H^\infty(\mathbb{D}; \mathcal{L}(E))$
2. $A(\mathbb{D}; \mathcal{L}(E))$
3. $A_S(\mathbb{D}; \mathcal{L}(E))$*, where* $S \subset \mathbb{T}$*,* S *open in* $\mathbb{T}$
4. $W_+(\mathbb{D}; \mathcal{L}(E))$.

It is known that for any Banach algebra $\mathcal{A}$, the Bass stable rank of $\mathcal{A}$ is bounded above by the minimum of the left and right topological stable ranks of $\mathcal{A}$ [6, Corollary 2.4]. So we have the following:

Corollary 1.6. *Let E be an infinite-dimensional separable Hilbert space. With the notation from Definition* 1.4*, the* (*left/right*) *topological stable ranks of the following Banach algebras is infinite:*

1. $H^\infty(\mathbb{D}; \mathcal{L}(E))$
2. $A(\mathbb{D}; \mathcal{L}(E))$
3. $A_S(\mathbb{D}; \mathcal{L}(E))$*, where* $S \subset \mathbb{T}$*,* S *open in* $\mathbb{T}$
4. $W_+(\mathbb{D}; \mathcal{L}(E))$.

From the proof of Lemma 3.1 and Corollary 1.6 above given below in Section 3, one can see that similar results hold for the analogues of the above algebras of operator-valued holomorphic functions of *several* complex variables as well.

2. An algebraic result of Corach and Larotonda

We recall the following result from Corach and Larotonda [2]. Since the theorem plays an important role in this article, we include its self-contained proof here for the sake of completeness.

Proposition 2.1 (Corach and Larotonda). *Let $\mathcal{A}$ be a ring with identity such that $\mathcal{A}$ and $\mathcal{A}^2$ are left $\mathcal{A}$-module isomorphic. Then the Bass stable rank of $\mathcal{A}$ is infinite.*

Proof. Suppose on the contrary that the Bass stable rank of $\mathcal{A}$ is finite, say r. Since $\mathcal{A}$ and $\mathcal{A}^2$ are left $\mathcal{A}$-module isomorphic, it is easy to see by induction that then also $\mathcal{A}$ and $\mathcal{A}^{r+1}$ are isomorphic as left $\mathcal{A}$-modules. Let $\tau : \mathcal{A}^{r+1} \to \mathcal{A}$ be a left $\mathcal{A}$-module isomorphism. We have for all $(\alpha_1, \dots, \alpha_{r+1}) \in \mathcal{A}^{r+1}$:

$$\begin{aligned}\tau(\alpha_1, \dots, \alpha_{r+1}) &= \alpha_1 \cdot \underbrace{\tau\big((1,0,\dots,0)\big)}_{=:a_1} + \dots + \alpha_{r+1} \cdot \underbrace{\tau\big((0,\dots,0,1)\big)}_{=:a_{r+1}} \\ &= \alpha_1 \cdot a_1 + \dots + \alpha_{r+1} \cdot a_{r+1} .\end{aligned}$$

Since τ is surjective, it follows that $(a_1, \dots, a_{r+1}) \in U_{r+1}(\mathcal{A})$. But the Bass stable rank of $\mathcal{A}$ is r, and so there must exist elements $b_1, \dots, b_r \in \mathcal{A}$ such that

$$(a_1 + b_1 a_{r+1}, \dots, a_r + b_r a_{r+1}) \in U_r(\mathcal{A}) . \tag{2.1}$$

Now consider the left $\mathcal{A}$-module homomorphism $\tau' : \mathcal{A}^r \to \mathcal{A}$ given by

$$\tau'(\alpha_1, \dots, \alpha_r) := \alpha_1 \cdot (a_1 + b_1 a_{r+1}) + \dots + \alpha_r \cdot (a_r + b_r a_{r+1}) , \quad (\alpha_1, \dots, \alpha_r) \in \mathcal{A}^r .$$

From (2.1), it follows that τ' is surjective. Now define the left $\mathcal{A}$-module homomorphism $\mu : \mathcal{A}^r \to \mathcal{A}^{r+1}$ given by

$$\mu(\alpha_1, \dots, \alpha_r) := (\alpha_1, \dots, \alpha_r, \alpha_1 b_1 + \dots + \alpha_r b_r) , \quad (\alpha_1, \dots, \alpha_r) \in \mathcal{A}^r .$$

It can then be verified that $\tau \circ \mu = \tau'$. But since τ is an isomorphism and τ' is surjective, it now follows that μ is also surjective. Hence $(0, \dots, 0, 1) \in \mathcal{A}^{r+1}$ must belong to the image of μ, and so there exists an element $(\alpha_1, \dots, \alpha_r) \in \mathcal{A}^r$ such that

$$\mu(\alpha_1, \dots, \alpha_r) = (\alpha_1, \dots, \alpha_r, \alpha_1 b_1 + \dots + \alpha_r b_r) = (0, \dots, 0, 1) .$$

Thus $\alpha_1, \dots, \alpha_r$ must be zero, and we have the contradiction that $0 = \alpha_1 b_1 + \dots + \alpha_r b_r = 1$. □

3. Proof of Theorem 1.5

In this section we will prove our main result (Theorem 1.5). The following lemma gives a homomorphism that enables the application of Proposition 2.1.

Lemma 3.1. *Let E be an infinite-dimensional separable Hilbert space. With the Notation 1.3, the map $\varphi : H(\mathbb{D}; \mathcal{L}(E)) \to (H(\mathbb{D}; \mathcal{L}(E)))^2$ given by*

$$f = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots \\ f_{21} & f_{22} & f_{23} & \cdots \\ f_{31} & f_{32} & f_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mapsto \varphi(f) = \left(\underbrace{\begin{bmatrix} f_{11} & f_{13} & f_{15} & \cdots \\ f_{21} & f_{23} & f_{25} & \cdots \\ f_{31} & f_{33} & f_{35} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\varphi_1(f)} , \underbrace{\begin{bmatrix} f_{12} & f_{14} & f_{16} & \cdots \\ f_{22} & f_{24} & f_{26} & \cdots \\ f_{32} & f_{34} & f_{36} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\varphi_2(f)} \right) ,$$

for $f \in H(\mathbb{D};\mathcal{L}(E))$ *is well-defined, and is a left* $H(\mathbb{D};\mathcal{L}(E))$*-module isomorphism.*

Proof. If $U_1, U_2 \in \mathcal{L}(E)$ are the isometries given by

$$U_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} := \begin{bmatrix} x_1 \\ 0 \\ x_2 \\ 0 \\ x_3 \\ 0 \\ \vdots \end{bmatrix} \quad \text{and} \quad U_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} := \begin{bmatrix} 0 \\ x_1 \\ 0 \\ x_2 \\ 0 \\ x_3 \\ \vdots \end{bmatrix},$$

then it is clear that

$$\forall z \in \mathbb{D}, \quad \big(\varphi_1(f)\big)(z) = f(z)U_1 \quad \text{and} \quad \big(\varphi_2(f)\big)(z) = f(z)U_2 . \tag{3.1}$$

Thus for all $z \in \mathbb{D}$, $(\varphi_1(f))(z), (\varphi_2(f))(z) \in \mathcal{L}(E)$. Moreover, since $f_{jk} \in H(\mathbb{D})$ for all $j, k \in \mathbb{N}$, it follows that $\varphi_1(f), \varphi_2(f)$ are holomorphic. So the map φ is well-defined. Moreover, if $f, g \in H(\mathbb{D};\mathcal{L}(E))$, then

$$\varphi(f+g) = \varphi(f) + \varphi(g) \quad \text{and} \quad \varphi(gf) = g\varphi(f) ,$$

and so φ is a left $H(\mathbb{D};\mathcal{L}(E))$-module homomorphism. The homomorphism φ is injective, since $\varphi(f) = 0$ implies $f_{jk} = 0$ for all $j, k \in \mathbb{N}$, and so $f = 0$. Moreover, given

$$(f, g) = \left(\begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots \\ f_{21} & f_{22} & f_{23} & \cdots \\ f_{31} & f_{32} & f_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots \\ g_{21} & g_{22} & g_{23} & \cdots \\ g_{31} & g_{32} & g_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \right),$$

we define

$$h = \begin{bmatrix} f_{11} & g_{11} & f_{12} & g_{12} & f_{13} & g_{13} & \cdots \\ f_{21} & g_{21} & f_{22} & g_{22} & f_{23} & g_{23} & \cdots \\ f_{31} & g_{31} & f_{32} & g_{32} & f_{33} & g_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then for all $z \in \mathbb{D}$,

$$h(z)x = f(z) \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \end{bmatrix} + g(z) \begin{bmatrix} x_2 \\ x_4 \\ x_6 \\ \vdots \end{bmatrix}, \tag{3.2}$$

and so

$$\| h(z)x \| \leq (\| f(z) \| + \| g(z) \|) \| x \| . \tag{3.3}$$

Thus $h(z) \in \mathcal{L}(E)$ for all $z \in \mathbb{D}$, and h is also holomorphic since f_{jk}, g_{jk} are in $H(\mathbb{D})$ for all $j, k \in \mathbb{N}$. We have $\varphi(h) = (f, g)$, and so φ is surjective as well. □

Theorem 3.2. *Suppose that E is an infinite-dimensional separable Hilbert space. We use Notation* 1.3. *Let φ be as in the statement of Lemma* 3.1. *Suppose that $\mathcal{A}$ is a subalgebra of $H(\mathbb{D};\mathcal{L}(E))$ with the following properties:*

1. $\varphi(\mathcal{A}) \subset \mathcal{A}^2$
2. $\varphi^{-1}(\mathcal{A}^2) \subset \mathcal{A}$.

Then the Bass stable rank of $\mathcal{A}$ is infinite.

Proof. Using Lemma 3.1 and the hypothesis, we see that the restriction of φ to $\mathcal{A}$ gives a left $\mathcal{A}$-module isomorphism from $\mathcal{A}$ to $\mathcal{A}^2$, and so by Theorem 2.1, the claim follows. □

In particular, with $\mathcal{A} = H(\mathbb{D};\mathcal{L}(E))$, we have the following:

Corollary 3.3. *Let E be an infinite-dimensional separable Hilbert space. Then the Bass stable rank of $H(\mathbb{D};\mathcal{L}(E))$ is infinite.*

Proof of Theorem 1.5. The verification of $\varphi(\mathcal{A}) \subset \mathcal{A}^2$ for each of the algebras follows readily from (3.1):

Recall that $A_S(\mathbb{D};\mathcal{L}(E))$ handles simultaneously the cases of $A(\mathbb{D};\mathcal{L}(E))$ (when $S = \mathbb{T}$) and $H^\infty(\mathbb{D};\mathcal{L}(E))$ (when $S = \emptyset$). If $f \in A_S(\mathbb{D};\mathcal{L}(E))$, then from (3.1), we see that for all $z \in \mathbb{D}$, $\|(\varphi_k(f))(z)\| \leq \|f\|_\infty \|U_k\|$ $(k = 1, 2)$, and so $\|\varphi_k(f)\|_\infty \leq \|f\|_\infty < \infty$. Moreover, if $f(z)$ has a limit L in $\mathcal{L}(E)$ as $z \to \zeta \in \mathbb{T}$, then (3.1) shows that also $(\varphi_k(f))(z)$ has the limit LU_k, $k = 1, 2$. This shows that $\varphi(f) \in (A_S(\mathbb{D};\mathcal{L}(E)))^2$.

If $f = \sum_{n=0}^\infty z^n f_n \in W_+(\mathbb{D};\mathcal{L}(E))$, then (3.1) gives $\varphi_k(f) = \sum_{n=0}^\infty z^n(f_n U_k)$, $k = 1, 2$. Since $\sum_{n=0}^\infty \|f_n U_k\| \leq \sum_{n=0}^\infty \|f_n\| \cdot 1 = \|f\|_1 < \infty$, it follows that $\varphi(f) \in (W_+(\mathbb{D};\mathcal{L}(E)))^2$.

That $\varphi^{-1}(\mathcal{A}^2) \subset \mathcal{A}$ can be checked using (3.2) and (3.3):

If $(f, g) \in (A_S(\mathbb{D};\mathcal{L}(E)))^2$, then from (3.3), we see that for all $z \in \mathbb{D}$, $\|h(z)\| \leq \|f\|_\infty + \|g\|_\infty$, and so $\|h\|_\infty < \infty$. Moreover, if $f(z), g(z)$ have limits in $\mathcal{L}(E)$ as $z \to \zeta \in \mathbb{T}$, then (3.2) shows that for every $x \in E$, $h(z)x$ has a limit in E as $z \to \zeta$. By the Banach-Steinhaus theorem, $h(z)$ has a limit in $\mathcal{L}(E)$ as $z \to \zeta$. This shows that $\varphi^{-1}(f, g) \in A_S(\mathbb{D};\mathcal{L}(E))$.

If $f, g \in W_+(\mathbb{D};\mathcal{L}(E))$, then (3.2) gives for all $n \geq 0$ that

$$h^{(n)}(0)x = f^{(n)}(0)\begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \end{bmatrix} + g^{(n)}(0)\begin{bmatrix} x_2 \\ x_4 \\ x_6 \\ \vdots \end{bmatrix},$$

and so $\|h^{(n)}(0)\| \leq \|f^{(n)}(0)\| + \|g^{(n)}(0)\|$. But we have $\|f\|_1 = \sum_{n=0}^\infty \frac{1}{n!}\|f^{(n)}(0)\| < \infty$ and also $\|g\|_1 = \sum_{n=0}^\infty \frac{1}{n!}\|g^{(n)}(0)\| < \infty$. So it follows that $\|h\|_1 = \sum_{n=0}^\infty \frac{1}{n!}\|h^{(n)}(0)\| < \infty$. Thus $\varphi^{-1}(f, g) \in W_+(\mathbb{D};\mathcal{L}(E))$. □

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