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Complex Analysis and Operator Theory

A Spectrally Minimal Realization Formula for $H^\infty(\mathbb{D})$

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Abstract. In this article we prove a representation theorem for $H^{\infty}(\mathbb{D})$ functions, such that the realization formula is *spectrally minimal* in the following sense: the spectrum of the main operator in the realization intersects the unit circle precisely at those points where the given function has no holomorphic extension. We also extend this result to operator-valued H^{∞} functions.

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1. Introduction

It is known (see for example [2, Theorem 6.5], [6, Theorem 2-1]) that every function in $H^{\infty}(\mathbb{D})$ can be represented as a "transfer function", that is, there exists a Hilbert space H and a bounded operator

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} H \\ \mathbb{C} \end{bmatrix} \to \begin{bmatrix} H \\ \mathbb{C} \end{bmatrix}$$
(1.1)

such that

$$g(z) = D + Cz(I - zA)^{-1}B, \quad z \in \mathbb{D}.$$
 (1.2)

In the right-hand side of the realization formula (1.2), it is understood that for every nonzero $z \in \mathbb{D}$, I-zA is invertible as an element of $\mathcal{L}(H)$, that is, $z^{-1} \in \rho(A)$, the resolvent set of A. The background to the realization formula (1.2) arises from relating g to the input-output map (the *transfer function*) of a discrete-time linear time-invariant system

$$\begin{cases} x(n+1) = Ax(n) + Bu(n), \\ y(n) = Cx(n) + Du(n). \end{cases}$$
(1.3)

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Here $n \in \mathbb{Z}$ can be interpreted as a time variable, u(n), x(n), y(n) have interpretations as input signal, state vector, and output signal, respectively, at time n. Application of the Z-transform $(x(n))_n \mapsto \hat{x}(z) = \sum_{n \in \mathbb{Z}} x(n)z^n$ to all quantities in (1.3) formally leads to $\hat{y}(z) = g(z)\hat{u}(z)$, where g is as in (1.2); see [6] and [4].

However, the same transfer function can be realized by generators A with widely differing spectra. From an intuitive point of view, the question then arises if we can have a generator which reflects the singularities of the transfer function in a faithful manner, that is, we would like to have an A with the smallest possible spectrum required to model the singularities of the given transfer function. We make this precise below.

Definition 1.1. Let V as in (1.1) be a realization of g, so that (1.2) holds. Let

 $S := \{ z \in \mathbb{T} \mid g \text{ is holomorphic across } z \}.$

We call the realization (1.1) for $g \in H^{\infty}(\mathbb{D})$ spectrally minimal if

$$S^{-1} := \{ z^{-1} \mid z \in S \} = \rho(A) \cap \mathbb{T}.$$

Our main result is that a spectrally minimal realization always exists.

In the literature, there are many canonical methods for constructing a "minimal" realization (not minimal in the sense of spectrum) for a given function. However, we know no earlier methods that would always lead to a spectrally minimal realization. For example, if a function g is cyclic, then both the canonical left-shift realization [6] and the minimal optimal passive scattering realization [1] of this function have the whole closed unit disc as the spectrum of the main operator. Moreover, then the latter realization coincides with the deBranges–Rovnyak realization. Thus, if $g(z) = ((z+1)/2)^{1/2}$, then none of these realizations is spectrally minimal, because then g is holomorphic on $\mathbb{T} \setminus \{-1\}$ and cyclic.

It has been shown earlier (for instance in [6, §4], [5], [10, §9.8]), by different methods, that in certain special cases of g's from $H^{\infty}(\mathbb{D})$, a realization can be chosen so that the component of $\rho(A)$ containing the origin is a maximal holomorphic domain of g. We cover the general case when g is arbitrary in $H^{\infty}(\mathbb{D})$, but we study the spectrum in the closed unit disk only. Our main result is the following:

Theorem 1.2. Let S be an open subset of \mathbb{T} , and let $g \in H^{\infty}(\mathbb{D})$ have a holomorphic extension across S. Then g admits a spectrally minimal realization of the form (1.2).

The proof will be presented in Section 2. An operator-valued version of this result is given in Section 3.

We will use the following standard notation.

 $\mathbb{D}, \overline{\mathbb{D}}, \mathbb{T}$ the open unit disk, the closed unit disk, and the unit circle, respectively:

 $\mathbb{D} := \{ z \in \mathbb{C} \mid |z| < 1 \}, \ \overline{\mathbb{D}} := \{ z \in \mathbb{C} \mid |z| \le 1 \}, \ \mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \};$

 $H^{\infty}(\mathbb{D}; X)$ If X is a Banach space, then $H^{\infty}(\mathbb{D}; X)$ denotes the space of X-valued bounded holomorphic functions on \mathbb{D} , equipped with the

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supremum norm. The space $H^{\infty}(\mathbb{D};\mathbb{C})$ will be denoted simply by $H^{\infty}(\mathbb{D})$.

- $\mathcal{L}(X,Y)$ If X, Y are Hilbert spaces, then $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y, equipped with the operator norm;
- $\rho(A)$ If $A \in \mathcal{L}(X)$ is a linear transformation on a Hilbert space X, then $\rho(A)$ denotes the resolvent set of A.

2. A spectrally minimal realization formula for $H^{\infty}(\mathbb{D})$

In this section we will prove our main result on the existence of spectrally minimal realizations in Theorem 1.2 below.

Before giving the proof, we explain the main idea behind it: assuming for the moment that g(0) = 0, if g were holomorphic in a domain containing $\overline{\mathbb{D}}$, then by Cauchy integral formula, we would have

$$\frac{g(z)}{z} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(e^{i\theta})e^{-i\theta}}{e^{i\theta} - z} i e^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} (1 - ze^{-i\theta})^{-1} \frac{g(e^{i\theta})}{e^{i\theta}} d\theta \,,$$

and we view this integral as $C(I - zA)^{-1}B$, where $C \in \mathcal{L}(L^2(\mathbb{T}); \mathbb{C}), A \in \mathcal{L}(L^2(\mathbb{T}), L^2(\mathbb{T})), B \in \mathcal{L}(\mathbb{C}; L^2(\mathbb{T}))$ are given by:

$$\begin{split} C &= \frac{1}{2\pi} \int_0^{2\pi} \bullet \, d\theta \,, \\ A &= \text{multiplication operator on } L^2(\mathbb{T}) \text{ by } e^{-i\theta} \,, \\ B &= \left(u \mapsto \frac{g(e^{i \cdot})}{e^{i \cdot}} u \right). \end{split}$$

However, our g need not be holomorphic across every point of \mathbb{T} , and so we will modify the above construction. First of all, in order to get the correct spectrum, we will replace the contour above (which was simply the circle), now to a suitable curve γ , such that γ matches the circle precisely at those points where g does not have a holomorphic extension and otherwise it passes through the region outside the circle where g is holomorphic, and take A as the multiplication operator on $L^2(\mathbb{T})$ by the function $\theta \mapsto 1/(R(\theta)e^{i\theta})$, where $R(\theta)e^{i\theta}$ gives the polar representation of γ . Moreover, we cannot directly apply Cauchy integral formula with this new γ , since g is not holomorphic across γ at the points where γ matches with the circle. Hence we will first work with a scaled version of γ by $r \in (0, 1)$, and subsequently pass the limit as r increases to 1.

In order to construct γ in Theorem 1.2, we will need the following technical lemma.

Lemma 2.1. Let O be an open set in \mathbb{R}^2 containing the segment (0,1). Then there exists a $f \in C^{\infty}(\mathbb{R};\mathbb{R})$ such that: (a) f(x) > 0 for all $x \in (0,1)$;

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- (b) f(x) = 0 for all $x \notin (0,1)$;
- (c) $(x, y) \in O$ for all $x \in (0, 1)$ and all $y \in [0, f(x)]$.

Proof. We first define f on (0, 1/2). For $n \geq 3$, let $g_n \in C^{\infty}(\mathbb{R})$ be such that

$$g_n(x) = \begin{cases} 0 & x \le \frac{1}{n} \\ 1 & x \ge \frac{1}{n-1} \\ \in (0,1) & x \in (\frac{1}{n-1}, \frac{1}{n}) \end{cases}.$$

For $n \ge 3$, choose $a_n > 0$ such that $a_n ||g_n||_{C^n} < 1/2^n$. For $n \ge 3$, pick $r_n > 0$ such that $(x, y) \in O$ for $x \in [1/n, 1/2]$ and $y \in [0, r_n]$, also ensuring that the sequence of r_n 's satisfies $r_{n+1} < r_n$ for all $n \ge 3$ and $r_n \to 0$ as $n \to \infty$. For $n \ge 3$, define $b_n = \min\{a_n, r_n - r_{n+1}\}$. Let

$$f_1 = \sum_{n=3}^{\infty} b_n g_n$$

For any $N < \infty$, the series converges in C^N :

$$\|f_1\|_{C^N} \le \sum_{n=3}^{\infty} b_n \|g_n\|_{C^N} \le \sum_{n=3}^{\infty} a_n \|g_n\|_{C^N} < \sum_{n=3}^{N-1} a_n \|g_n\|_{C^N} + \sum_{n=N}^{\infty} \frac{1}{2^n} < +\infty.$$
(2.1)

So $f_1 \in C^{\infty}$. Let now $x \in (0, 1/2]$. Then $x \in (1/(n+1), 1/n]$ for some $n \ge 2$, and hence $g_k(x) = 0$ for $k \le n$ (we define $g_k := 0$ for k < 3), so

$$0 < b_{n+1}g_{n+1}(x) < f_1(x) \le \sum_{k=n+1}^{\infty} b_k \le \sum_{k=n+1}^{\infty} (r_k - r_{k+1}) = r_{n+1}.$$

If $y \in [0, f_1(x)]$, then $y \in [0, r_{n+1}]$ and hence then $(x, y) \in O$.

From (2.1), we also have that $f_1(x) < 1$ for all $x \in (0,1)$ (even $||f_1||_{C^0} \leq \sum_{k=3}^{\infty} 2^{-k} = 1/4$).

Similarly, we can construct a $f_2 \in C^{\infty}$ such that for all $x \in [1/2, 1)$ and all $y \in [0, f_2(x)]$, we have $(x, y) \in O$, and furthermore, $1 > f_2(x) > 0$ for $x \in (0, 1)$. Defining $f = f_1 f_2$, we are done. \square

We are now ready to prove our main result.

Proof of Theorem 1.2. We assume that g(0) = 0 (set D = g(0) in the general case).

Let Ω be a simply connected domain containing $\mathbb{D} \cup S$ such that $\partial \Omega \cap \mathbb{T} = \mathbb{T} \setminus S$, and such that g is holomorphic and bounded in Ω .

We note that it can be arranged that the boundary of Ω is smooth, that is C^1 , and it has a C^1 parameterization $\theta \mapsto R(\theta)e^{i\theta}$. Indeed, first of all, we observe that S can be written as a disjoint union $\cup_{k=1}^{\infty} I_k$ of open arcs

$$I_k = \left\{ e^{i\theta} \mid \theta \in (\alpha_k, \beta_k) \right\}, \quad k \in \mathbb{N}$$

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(see for instance [11, Theorem 1.3]). Now let $R(\theta) = 1$ if $e^{i\theta} \in \mathbb{T} \setminus S$, while if $e^{i\theta} \in S$, say $e^{i\theta} \in I_k$, then take

$$R(\theta) = 1 + f_k \left(\frac{\theta - \alpha_k}{\beta_k - \alpha_k}\right),$$

where f_k is a function constructed as in Lemma 2.1 (then actually $R \in C^{\infty}$).

The map $z \mapsto g(z)/z$ is holomorphic in \mathbb{D} . Let $H = L^2(\mathbb{T})$. Suppose that $z \in \mathbb{D}$. Let r be such that |z| < r < 1.

Define $A_r \in \mathcal{L}(H)$ to be the multiplication operator by $1/(rR(\theta)e^{i\theta})$:

$$(A_r f)(e^{i\theta}) = f(e^{i\theta})/(rR(\theta)e^{i\theta}), \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}).$$

The spectrum of A_r is the range of $\theta \mapsto 1/(rR(\theta)e^{i\theta})$. Since |z| < r, it follows that $1 \in \rho(zA_r)$, and

$$\left((I-zA_r)^{-1}f\right)(e^{i\theta}) = \frac{f(e^{i\theta})}{1-z/\left(rR(\theta)e^{i\theta}\right)}, \quad \theta \in [0,2\pi), \quad f \in L^2(\mathbb{T}).$$

Let $B_r \in \mathcal{L}(\mathbb{C}; H)$ be defined by

$$(B_r u)(e^{i\theta}) = \frac{g(rR(\theta)e^{i\theta})(R(\theta) - iR'(\theta))}{r(R(\theta))^2 e^{i\theta}} u, \quad \theta \in [0, 2\pi), \quad u \in \mathbb{C}.$$

Let $C \in \mathcal{L}(H; \mathbb{C})$ be defined by

$$Cf = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta \,, \quad f \in L^2(\mathbb{T}) \,.$$

Let γ denote the curve $\theta:[0,2\pi]\to rR(\theta)e^{i\theta}.$ By the Cauchy integral formula, we have

$$\frac{g(z)}{z} = \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)/(w)}{w-z} dw$$

$$= \frac{1}{2\pi} \int_{\gamma} \frac{1}{1-z/w} \frac{-ig(w)\frac{dw}{d\theta}/re^{i\theta}}{w^2} dw$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{1-z/(rR(\theta)e^{i\theta})} \frac{g(rR(\theta)e^{i\theta})(R(\theta)-iR'(\theta))}{r(R(\theta))^2 e^{i\theta}} d\theta$$

$$= C(I-zA_r)^{-1}B_r.$$
(2.2)

Note that with a fixed z, the above is true for every r satisfying |z| < r < 1.

Define $A \in \mathcal{L}(H)$ to be the multiplication operator given by:

$$(Af)(e^{i\theta}) = f(e^{i\theta}) / (R(\theta)e^{i\theta}), \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T})$$

We note that $zA_r \to zA$ as $r \nearrow 1$ in the operator norm of $\mathcal{L}(H)$. As $1 \in \rho(zA_r)$ (|z| < r < 1) and $1 \in \rho(zA)$, it follows that $(I - zA_r)^{-1} \to (I - zA)^{-1}$ in $\mathcal{L}(H)$ as $r \nearrow 1$.

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Furthermore, define $B \in \mathcal{L}(\mathbb{C}, H)$ by

$$(Bu)(e^{i\theta}) = \frac{g(R(\theta)e^{i\theta})(R(\theta) - iR'(\theta))}{(R(\theta))^2 e^{i\theta}}u, \quad \theta \in [0, 2\pi), \quad u \in \mathbb{C}.$$

We also note that $B_r \to B$ in $\mathcal{L}(\mathbb{C}; H)$ as $r \nearrow 1$. To see this, it is enough to prove that $g(rR(\cdot)e^{i\cdot}) \to g(R(\cdot)e^{i\cdot})$ as $r \nearrow 1$ in $L^2(\mathbb{T})$. But this follows from the Lebesgue dominated convergence theorem, if we prove that the functions converge pointwise almost everywhere. If $R(\theta) = 1$, then this follows from the fact that the radial limits exist almost everywhere for functions in $H^{\infty}(\mathbb{D})$ [9, Theorem 17.11]. If $R(\theta) > 1$, then this follows from the fact that g is continuous at $R(\theta)e^{i\theta}$ (this can be ensured by having chosen Ω suitably at the outset, that is, by shrinking it enough so that its boundary outside \mathbb{D} lies in the region where g is holomorphic and so continuous to the boundary).

Consequently, from (2.2) we obtain that

$$\frac{g(z)}{z} = C(I - zA)^{-1}B.$$
(2.3)

Since the choice of $z \in \mathbb{D}$ was arbitrary, (2.3) holds for all $z \in \mathbb{D}$. Defining D = 0, we obtain (1.2).

Finally, we observe that $\sigma(A)$ is the range of $1/(R(\theta)e^{i\theta})$. Since $\partial\Omega \cap \mathbb{T} = \mathbb{T} \setminus S$, we have $S^{-1} = \mathbb{T} \cap \rho(A)$.

Remark 2.2. We remark that the spectrally minimal realization given in Theorem 1.2 is not unique, since there is some freedom in the choice of the γ .

3. Operator-valued case

Theorem 3.1. Let U, Y be Hilbert spaces, S be an open subset of \mathbb{T} , and $g \in H^{\infty}(\mathbb{D}; \mathcal{L}(U, Y))$ have a holomorphic extension across S. Then there exists a Hilbert space H and a bounded operator

$$V = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] : \left[\begin{array}{c} H \\ U \end{array} \right] \to \left[\begin{array}{c} H \\ Y \end{array} \right]$$

such that $S^{-1} = \mathbb{T} \cap \rho(A)$ and

$$g(z) = D + Cz(I - zA)^{-1}B, \quad z \in \mathbb{D}.$$
 (3.1)

Proof. Let Ω and $R(\theta)$ be constructed as in the proof of Theorem 1.2, and let $H = L^2(\mathbb{T}; Y)$. Define $B \in \mathcal{L}(U, H)$ by

$$(Bu)(e^{i\theta}) = \frac{g(R(\theta)e^{i\theta})(R(\theta) - iR'(\theta))}{(R(\theta))^2 e^{i\theta}}u, \quad \theta \in [0, 2\pi), \quad u \in U,$$

Let $A \in \mathcal{L}(H)$ be the multiplication operator given by

$$(Af)(e^{i\theta}) = f(e^{i\theta}) / (R(\theta)e^{i\theta}) \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}; Y) \,.$$

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Define $C \in \mathcal{L}(H; Y)$ by

$$Cf = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta \,, \quad f \in L^2(\mathbb{T}; Y) \,.$$

Finally, let

$$D := q(0) \in \mathcal{L}(U, Y) \,.$$

Let $\Lambda \in Y^*$ and $u \in U$. By repeating the argument from the proof of Theorem 1.2 to $\Lambda g(\cdot)u \in H^{\infty}(\mathbb{D})$, we see that

$$\Lambda g(z)u = \Lambda Du + \Lambda C z (I - zA)^{-1} B u = \Lambda \left(D + C z (I - zA)^{-1} B \right) u, \quad z \in \mathbb{D},$$

and that $S^{-1} = \mathbb{T} \cap \rho(A)$. Since the choice of $\Lambda \in Y^*$ and $u \in U$ was arbitrary, it follows that (3.1) holds. \Box

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