Research Article

Noncoherence of a Causal Wiener Algebra Used in Control Theory

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Let $\mathbb{C}_{\geq 0} := \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 0\}$, and let \mathcal{W}^+ denote the ring of all functions $f : \mathbb{C}_{\geq 0} \to \mathbb{C}$ such that $f(s) = f_a(s) + \sum_{k=0}^{\infty} f_k e^{-st_k}$ ($s \in \mathbb{C}_{\geq 0}$), where $f_a \in L^1(0, \infty)$, $(f_k)_{k\geq 0} \in \ell^1$, and $0 = t_0 < t_1 < t_2 < \cdots$ equipped with pointwise operations. (Here $\hat{\cdot}$ denotes the Laplace transform.) It is shown that the ring \mathcal{W}^+ is not coherent, answering a question of Alban Quadrat. In fact, we present two principal ideals in the domain \mathcal{W}^+ whose intersection is not finitely generated.

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1. Introduction

The aim of this paper is to show that the ring $\mathcal{W}^{\scriptscriptstyle +}$ (defined below) is notcoherent.

We first recall the notion of a coherent ring.

Definition 1.1. Let *R* be a commutative ring with identity element 1, and let $R^m = R \times \cdots \times R$ (*m* times). Suppose that $f = (f_1, \dots, f_m) \in R^m$.

(1) An element $(g_1, \ldots, g_m) \in \mathbb{R}^m$ is called a *relation on* f if

$$g_1 f_1 + \dots + g_m f_m = 0. \tag{1.1}$$

- (2) Let f^{\perp} denote the set of all relations on $f \in \mathbb{R}^m$. (Then f^{\perp} is an R-submodule of the R-module \mathbb{R}^m .)
- (3) The ring *R* is called *coherent* if for all $m \in \mathbb{N}$ and all $f \in R^m$, f^{\perp} is finitely generated, that is, there exists a $d \in \mathbb{N}$ and there exist $g_j \in f^{\perp}$, $j \in \{1, ..., d\}$, such that for all $g \in f^{\perp}$, there exist $r_j \in R$, $j \in \{1, ..., d\}$ such that $g = r_1g_1 + \cdots + r_dg_d$.

An integral domain is coherent if and only if the intersection of any two finitely generated ideals of the ring is again finitely generated; see [1, Theorem 2.3.2, page 45].

The coherence of some rings of analytic functions has been investigated in earlier works. For example, McVoy and Rubel [2] showed that the Hardy algebra $H^{\infty}(\mathbb{D})$ is coherent, while the disc algebra $A(\mathbb{D})$ is not. Mortini and von Renteln proved that the Wiener algebra $W^+(\mathbb{D})$ (of all absolutely convergent Taylor series in the open unit disc) is not coherent [3]. In this article, we will show that the ring \mathcal{W}^+ (defined below, and which is useful in control theory) is not coherent.

Notation 1. Throughout the article, we will use the following notation:

$$\mathbb{C}_{\geq 0} := \left\{ s \in \mathbb{C} \mid \operatorname{Re}(s) \ge 0 \right\}.$$

$$(1.2)$$

Definition 1.2. Let \mathcal{W}^+ denote the Banach algebra

$$\mathcal{W}^{+} = \left\{ f: \mathbb{C}_{\geq 0} \longrightarrow \mathbb{C} \middle| \begin{array}{l} f(s) = \widehat{f_{a}}(s) + \sum_{k=0}^{\infty} f_{k} e^{-st_{k}} \ (s \in \mathbb{C}_{\geq 0}), \\ f_{a}: (0, \infty) \longrightarrow \mathbb{C}, \ f_{a} \in L^{1}(0, \infty), \\ \forall k \geq 0, \ f_{k} \in \mathbb{C}, \ (f_{k})_{k \geq 0} \in \ell^{1}, \\ \forall k \geq 0, \ t_{k} \in \mathbb{R}, \ 0 = t_{0} < t_{1} < t_{2} < \cdots \end{array} \right\}$$
(1.3)

equipped with pointwise operations and the norm

$$\|f\|_{\mathcal{W}^{+}} \coloneqq \|f_{a}\|_{L^{1}} + \|(f_{k})_{k>0}\|_{\ell^{1}}.$$
(1.4)

Here $\widehat{f_a}$ denotes the Laplace transform of f_a , given by

$$\widehat{f_a}(s) = \int_0^\infty e^{-st} f_a(t) dt, \quad s \in \mathbb{C}_{\ge 0}.$$
(1.5)

The above algebra arises as a natural class of transfer functions of stable distributed parameter systems in control theory; see [4, 5].

Our main result is the following.

Theorem 1.3. *The ring* \mathcal{W}^+ *is not coherent.*

The relevance of the coherence property in control theory can be found in [6, 7]. We will prove Theorem 1.3 following the same method as in the proof of the noncoherence of $W^+(\mathbb{D})$ given by Mortini and von Renteln in [3].

In Section 3, we will give the proof of Theorem 1.3. But before doing that, in Section 2, we first prove a few technical results needed in the sequel.

2. Preliminaries

We first recall the definition of the Hardy algebra H^{∞} of the open right half plane.

Definition 2.1. Let H^{∞} denote the Hardy space of all bounded analytic functions in the open right half plane equipped with the norm

$$\|\varphi\|_{\infty} := \sup_{\operatorname{Re}(s)>0} |\varphi(s)|, \quad \varphi \in H^{\infty}.$$
(2.1)

In order to prove our main result (Theorem 1.3), we will use the relation between the convergence in H^{∞} versus that in \mathcal{W}^+ .

Lemma 2.2. If $f \in \mathcal{W}^+$, then $f \in H^{\infty}$ and $||f||_{\infty} \leq ||f||_{\mathcal{W}^+}$.

Proof. Let

$$f(s) = \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in \mathbb{C}_{\geq 0}).$$

$$(2.2)$$

For $s \in \mathbb{C}_{\geq 0}$, we have

$$\left|\widehat{f_{a}}(s)\right| = \left|\int_{0}^{\infty} e^{-st} f_{a}(t) dt\right| \le \int_{0}^{\infty} e^{-\operatorname{Re}(s)t} \left|f_{a}(t)\right| dt \le \int_{0}^{\infty} 1 \cdot \left|f_{a}(t)\right| dt = \left\|f_{a}\right\|_{L^{1}},$$
(2.3)

and moreover,

$$\left|\sum_{k=0}^{\infty} f_k e^{-st_k}\right| \le \sum_{k=0}^{\infty} |f_k| e^{-\operatorname{Re}(s)t_k} \le \sum_{k=0}^{\infty} |f_k| \cdot 1 = \|(f_k)_k\|_{\ell^1}.$$
(2.4)

So the result follows.

The maximal ideal \mathfrak{m}_0 (defined below) of \mathcal{W}^+ will play an important role in the remainder of this article.

Notation 2. Let \mathfrak{m}_0 denote the kernel of the complex algebra homomorphism $f \mapsto f(0) : \mathcal{W}^+ \to \mathbb{C}$, that is,

$$\mathfrak{m}_0 := \{ f \in \mathcal{W}^+ \mid f(0) = 0 \}.$$

Then \mathfrak{m}_0 is a maximal ideal of \mathcal{W}^+ , and this maximal ideal plays an important role in the proof of our main result in the next section. We will prove a few technical results about \mathfrak{m}_0 in this section, which will be used in the sequel. The following result is analogous to [3, Lemma 1].

Lemma 2.3. Let $L \neq (0)$ be an ideal in \mathcal{W}^+ contained in the maximal ideal \mathfrak{m}_0 . If $L = L\mathfrak{m}_0$, that is, if every function $f \in L$ can be factorized in a product f = hg of two functions $h \in L$ and $g \in \mathfrak{m}_0$, then L cannot be finitely generated.

Proof. Suppose that

$$L = (f_1, \dots, f_N) \neq (0) \tag{2.5}$$

is a finitely generated ideal in \mathcal{W}^+ contained in the maximal ideal \mathfrak{m}_0 . By our assumption, there are functions $h_n \in L$, $g_n \in \mathfrak{m}_0$ with

$$f_n = h_n g_n \quad (n = 1, \dots, N).$$
 (2.6)

Since $h_n \in L$, there exist functions $q_k^{(n)} \in \mathcal{W}^+$ with

$$h_n = \sum_{k=1}^{N} q_k^{(n)} f_k \quad (n = 1, \dots, N; \ k = 1, \dots, N).$$
(2.7)

From this it follows that

$$\sum_{n=1}^{N} |h_n| \le NC \sum_{n=1}^{N} |f_n| = NC \sum_{n=1}^{N} |h_n g_n| \quad \text{in } \mathbb{C}_{\ge 0},$$
(2.8)

where C is a constant chosen so that

$$\|q_k^{(n)}\|_{\infty} \le C, \quad \forall k \text{ and } n.$$
(2.9)

(Here $\|\cdot\|_{\infty}$ denotes the supnorm over $\mathbb{C}_{\geq 0}$.) This implies together with the Cauchy-Schwarz inequality that

$$\sum_{n=1}^{N} |h_n|^2 \le \left(\sum_{n=1}^{N} |h_n|\right)^2 \le N^2 C^2 \left(\sum_{n=1}^{N} |h_n g_n|\right)^2 \le N^2 C^2 \left(\sum_{n=1}^{N} |h_n|^2\right) \left(\sum_{n=1}^{N} |g_n|^2\right).$$
(2.10)

This inequality holds for all $s \in \mathbb{C}_{\geq 0}$. With $\delta := 1/(N^2C^2)$, we obtain the inequality

$$\delta \le \sum_{n=1}^{N} \left| g_n(s) \right|^2 \tag{2.11}$$

for all points $s \in E$, where

$$E := \left\{ s \in \mathbb{C}_{\geq 0} \left| \sum_{n=1}^{N} |h_n(s)|^2 > 0 \right\}.$$
 (2.12)

Since $L \neq (0)$, *E* is a dense subset of $\mathbb{C}_{\geq 0}$ (for otherwise, if $s_0 \in \mathbb{C}_{\geq 0}$ is such that it has a neighbourhood *V* in $\mathbb{C}_{\geq 0}$ where there is no point of *E*, then each h_n is identically zero in *V*, and by the identity theorem for holomorphic functions, each h_n is zero; consequently each f_n is zero, and so L = (0), a contradiction). So by continuity, inequality (2.11) holds in $\mathbb{C}_{\geq 0}$. But this contradicts the fact that each g_n vanishes at 0.

Remark 2.4. Lemma 2.3 can be proved purely algebraically using Nakayama's lemma. Indeed, it holds in the following more general algebraic situation: if I is a nonzero ideal of a commutative domain D contained in a maximal ideal M and I = IM, then I cannot be finitely generated. However, we have given an analytic proof in our special case above.

Since every maximal ideal is closed, m_0 is a commutative Banach subalgebra of \mathcal{W}^+ , but obviously without identity element. But there is a substitute, namely the notion of the approximate identity, which turns out to be useful.

Definition 2.5. Let *R* be a commutative Banach algebra (without identity element). One says that *R* has an *approximate identity* if there exists a bounded sequence $(e_n)_n$ of elements e_n in *R* such that for any $f \in R$,

$$\lim_{n \to \infty} \|e_n f - f\| = 0.$$
 (2.13)

We will now prove the following result, which shows that the maximal ideal \mathfrak{m}_0 in \mathcal{W}^+ has an approximate identity.

Theorem 2.6. Let

$$e_n \coloneqq \frac{s}{s+1/n}, \quad n \in \mathbb{N}.$$
(2.14)

Then $(e_n)_{n \in \mathbb{N}}$ *is an approximate identity for* \mathfrak{m}_0 *.*

The existence of an approximate identity for the maximal ideal \mathfrak{m}_0 in \mathcal{W}^+ is not obvious. In order to prove Theorem 2.6, we will need the following lemma.

Lemma 2.7. Suppose $\hat{f} \in \mathfrak{m}_0$. Then, for all $\epsilon > 0$, there exists a $\hat{p} \in \mathfrak{m}_0$ such that p has compact support in $[0, \infty)$, and $\|\hat{f} - \hat{p}\|_{\mathcal{W}^+} < \epsilon$.

Proof. Let $\epsilon > 0$ be given. Suppose that

$$f = f_a + \sum_{k=0}^{\infty} f_k \delta(\cdot - t_k), \qquad (2.15)$$

where $f_a \in L^1(0,\infty)$, $(f_k)_{k\geq 0} \in \ell^1$, and $0 = t_0 < t_1 < t_2 < \cdots$. Since $\int_0^\infty |f_a(t)| dt < \infty$, we can choose an M > 0 large enough such that

$$\int_{M}^{\infty} \left| f_a(t) \right| dt < \frac{\epsilon}{4}. \tag{2.16}$$

With $p_a(t) := f_a(t)$ if $t \in [0, M]$, and 0 otherwise, we have that $p_a \in L^1(0, \infty)$ is compactly supported and

$$\|p_a - f_a\|_{L^1} < \frac{\epsilon}{4}.$$
 (2.17)

Furthermore, select $N \in \mathbb{N}$ such that

$$\sum_{k>N} \left| f_k \right| < \frac{\epsilon}{4}.$$
(2.18)

Now let $T \in (0, \infty)$ be any number satisfying $t_N < T < t_{N+1}$, and define

$$f_T := -\left(\int_0^\infty p_a(t)dt + \sum_{0 \le k \le N} f_k\right).$$
(2.19)

Set

$$p := p_a + \sum_{0 \le k \le N} f_k \delta(\cdot - t_k) + f_T \delta(\cdot - T).$$
(2.20)

Then $\widehat{p} \in \mathcal{W}^+$ and

$$\widehat{p}(0) = \int_0^\infty p(t)dt = \int_0^\infty p_a(t)dt + \sum_{0 \le k \le N} f_k + f_T = 0.$$
(2.21)

So $\hat{p} \in \mathfrak{m}_0$. Clearly *p* has compact support contained in $[0, \infty)$. We have

$$\begin{split} |f_{T}| &= \left| \int_{0}^{\infty} p_{a}(t) dt + \sum_{0 \le k \le N} f_{k} \right| \\ &= \left| \int_{0}^{\infty} f_{a}(t) dt + \sum_{k=0}^{\infty} f_{k} + \int_{0}^{\infty} (p_{a}(t) - f_{a}(t)) dt - \sum_{k>N} f_{k} \right| \\ &\le \left| \int_{0}^{\infty} f(t) dt \right| + \| p_{a} - f_{a} \|_{L^{1}} + \sum_{k>N} |f_{k}| \\ &= |\widehat{f}(0)| + \| p_{a} - f_{a} \|_{L^{1}} + \sum_{k>N} |f_{k}| \\ &< 0 + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{split}$$

$$(2.22)$$

Thus

$$\|\widehat{f} - \widehat{p}\|_{\mathcal{W}^{+}} = \|f_a - p_a\|_{L^1} + \sum_{k>N} |f_k| + |f_T| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$
(2.23)

This completes the proof.

We are now ready to prove the existence of an approximate identity for the maximal ideal \mathfrak{m}_0 in \mathcal{W}^+ .

Proof of Theorem 2.6. We have

$$e_n = \frac{s}{s+1/n} = \frac{s+1/n-1/n}{s+1/n} = 1 - \frac{1}{n} \frac{1}{s+1/n} = 1 + -\frac{1}{n} \frac{1}{e^{-t/n}}.$$
 (2.24)

Thus for an $n \in \mathbb{N}$,

$$\left\| e_n \right\|_{\mathcal{W}^+} = \left\| -\frac{1}{n} e^{-t/n} \right\|_{L^1} + |1| = 1 + 1 = 2.$$
(2.25)

Given $\hat{f} \in \mathcal{W}^+$, and $\epsilon > 0$ arbitrarily small, in view of Lemma 2.7, we can find a $\hat{p} \in \mathfrak{m}_0$ such that p has compact support and $\|\hat{f} - \hat{p}\|_{\mathcal{W}^+} < \epsilon$. Then

$$\|e_{n}\widehat{f} - \widehat{f}\|_{\mathcal{W}^{+}} \leq \|e_{n}\widehat{p} - \widehat{p}\|_{\mathcal{W}^{+}} + \|e_{n}\|_{\mathcal{W}^{+}} \|\widehat{f} - \widehat{p}\|_{\mathcal{W}^{+}} + \|\widehat{f} - \widehat{p}\|_{\mathcal{W}^{+}}.$$
(2.26)

So it is enough to prove that

$$\lim_{n \to \infty} \left\| e_n \hat{p} - \hat{p} \right\|_{\mathcal{W}^+} = 0 \tag{2.27}$$

for all $\hat{p} \in \mathfrak{m}_0$ such that p has compact support in $[0, \infty)$. We do this below.

We have

$$e_n \hat{p} - \hat{p} = \frac{s + 1/n - 1/n}{s + 1/n} \hat{p} - \hat{p} = -\frac{1}{n} \frac{1}{s + 1/n} \hat{p} = -\frac{1}{n} \left(e^{-t/n} * p \right).$$
(2.28)

Let *C* denote the convolution $e^{-t/n} * p$:

$$C(t) := \int_{0}^{t} e^{-(t-\tau)/n} p(\tau) d\tau.$$
 (2.29)

We note that $C \in L^1(0, \infty)$, since $L^1(0, \infty)$ is an ideal in \mathcal{W}^+ . Let T > 0 be such that

$$\operatorname{supp}(p) \subset [0, T]. \tag{2.30}$$

We have

$$\|e_{n}\widehat{p}-\widehat{p}\|_{\mathcal{W}^{+}} = \frac{1}{n}\|C\|_{L^{1}} = \frac{1}{n}\int_{0}^{\infty} |C(t)|dt = \underbrace{\frac{1}{n}\int_{0}^{T}|C(t)|dt}_{(I)} + \underbrace{\frac{1}{n}\int_{T}^{\infty}|C(t)|dt}_{(II)}.$$
(2.31)

We estimate (I) as follows:

$$(I) = \frac{1}{n} \int_{0}^{T} |C(t)| dt = \frac{1}{n} \int_{0}^{T} \left| \int_{0}^{t} e^{-(t-\tau)/n} p(\tau) d\tau \right| dt$$

$$\leq \frac{1}{n} \int_{0}^{T} \int_{0}^{t} e^{-(t-\tau)/n} |p(\tau)| d\tau dt$$

$$\leq \frac{1}{n} \underbrace{\left(\int_{0}^{T} \int_{0}^{t} 1 \cdot |p(\tau)| d\tau dt \right)}_{(III)}.$$
(2.32)

Since the integral (III) does not depend on *n*, we obtain that

$$\lim_{n \to \infty} \frac{1}{n} \int_0^T |C(t)| dt = 0.$$
(2.33)

Furthermore,

$$(II) = \frac{1}{n} \int_{T}^{\infty} |C(t)| dt$$

$$= \frac{1}{n} \int_{T}^{\infty} e^{-t/n} \left| \int_{0}^{t} e^{\tau/n} p(\tau) d\tau \right| dt$$

$$= \frac{1}{n} \int_{T}^{\infty} e^{-t/n} \left| \int_{0}^{\infty} e^{\tau/n} p(\tau) d\tau \right| dt \quad (\text{since supp}(p) \in [0, T])$$

$$= \frac{1}{n} \int_{T}^{\infty} e^{-t/n} \left| \hat{p} \left(-\frac{1}{n} \right) \right| dt.$$
(2.34)

Since *p* has compact support in [0, T], \hat{p} is an entire function by the Payley-Wiener theorem (see, e.g., [8, Theorem 7.2.3, page 122]). Consequently,

$$(II) = \frac{1}{n} \int_{T}^{\infty} e^{-t/n} \left| \hat{p} \left(-\frac{1}{n} \right) \right| dt = e^{-T/n} \left| \hat{p} \left(-\frac{1}{n} \right) \right| \longrightarrow n \longrightarrow \infty 1 \cdot \left| \hat{p}(0) \right| = 1 \cdot 0 = 0.$$
(2.35) completes the proof.

This completes the proof.

We will also need the following lemma, which is basically a repetition of key steps from Browder's proof of Cohen's factorization theorem; see [9, Theorem 1.6.5, page 74]. We will need this version since in our application in the proof of Theorem 1.3, we are not able to use Cohen's factorization theorem directly.

Lemma 2.8. Let $f_1, f_2 \in \mathfrak{m}_0$, and $\delta > 0$. Let $U(\mathcal{W}^+)$ denote the set of all invertible elements in \mathcal{W}^+ . *Then there exists a sequence* $(g_n)_{n \in \mathbb{N}}$ *in* \mathcal{W}^+ *such that*

- (1) for all $n \in \mathbb{N}$, $g_n \in U(\mathcal{W}^+)$;
- (2) $(g_n)_{n\in\mathbb{N}}$ is convergent in \mathcal{W}^+ to a limit $g \in \mathfrak{m}_0$;
- (3) for all $n \in \mathbb{N}$, $\|g_n^{-1}f_i g_{n+1}^{-1}f_i\|_{\mathcal{W}^+} \le \delta/2^n$, i = 1, 2.

Proof. We will first prove two general results in steps (A) and (B), which we will use in the rest of the proof.

(A) Let $e \in \mathfrak{m}_0$ and $||e||_{\mathcal{W}^+} \leq K$, where K > 1. Then $1 - c + ce \in U(\mathcal{W}^+)$, where *c* is a number chosen such that

$$0 < c < \frac{1}{4K} < \frac{1}{4}.$$
(2.36)

Indeed,

$$\left\| \frac{c}{c-1} e \right\|_{\mathcal{W}^+} < \frac{1/(4K)}{3/4} \cdot K = \frac{1}{3} < 1,$$
(2.37)

and so

$$(1 - c + ce)^{-1} = \frac{1}{1 - c} \sum_{k=0}^{\infty} \left(\frac{c}{c - 1}\right)^k e^k.$$
(2.38)

(B) Furthermore, under the same assumptions and notation as in (A) above, we now show that if $||eF - F||_{\mathcal{W}^+}$ is small for some *F*, then so is $||EF - F||_{\mathcal{W}^+}$, where $E := (1 - c + ce)^{-1}$. Since

$$1 = \frac{1}{1-c} \sum_{k=0}^{\infty} \left(\frac{c}{c-1}\right)^k,$$
(2.39)

we have

$$\|EF - F\|_{\mathcal{W}^{+}} = \left\|\frac{1}{1 - c}\sum_{k=0}^{\infty} \left(\frac{c}{c - 1}\right)^{k} \left(e^{k}F - F\right)\right\|_{\mathcal{W}^{+}} \le \frac{1}{1 - c}\sum_{k=0}^{\infty} \left(\frac{c}{1 - c}\right)^{k} \|e^{k}F - F\|_{\mathcal{W}^{+}}.$$
 (2.40)

But

$$\left\|e^{k}F-F\right\|_{\mathcal{W}^{+}} = \left\|\sum_{j=0}^{k-1} \left(e^{j+1}F-e^{j}F\right)\right\|_{\mathcal{W}^{+}} \leq \sum_{j=0}^{k-1} \|e^{j}\|_{\mathcal{W}^{+}} \|eF-F\|_{\mathcal{W}^{+}} \leq \|eF-F\|_{\mathcal{W}^{+}} \sum_{j=0}^{k-1} \|e\|_{\mathcal{W}^{+}}^{j} < \|eF-F\|_{\mathcal{W}^{+}} \frac{K^{k}}{K-1}.$$
(2.41)

Hence

$$\|EF - F\|_{\mathcal{W}^+} < \|eF - F\|_{\mathcal{W}^+} \frac{1}{1 - c} \sum_{k=0}^{\infty} \frac{1}{K - 1} \left(\frac{1}{4(1 - c)}\right)^k < \frac{2}{K - 1} \|eF - F\|_{\mathcal{W}^+}.$$
 (2.42)

This estimate will be used in constructing the sequence of g_n 's.

Let $(e_n)_{n \in \mathbb{N}}$ denote the approximate identity for \mathfrak{m}_0 from Theorem 2.6. Let K > 1 be such that $||e_n||_{\mathcal{W}^+} \leq K$ for all $n \in \mathbb{N}$. Choose *c* such that

$$0 < c < \frac{1}{4K} < \frac{1}{4}.$$
(2.43)

We will inductively define a sequence $(e_{m_k})_{k \in \mathbb{N}}$ with terms from the approximate identity for \mathfrak{m}_0 such that if

$$g_n := c \sum_{k=1}^n (1-c)^{k-1} e_{m_k} + (1-c)^n, \qquad (2.44)$$

then we have $||f_i - g_1^{-1}f_i||_{\mathcal{W}^+} < \delta/2, i = 1, 2, \text{ and}$

- (P1) for all $n \in \mathbb{N}$, $g_n \in U(\mathcal{W}^+)$,
- (P2) for all $n \in \mathbb{N}$, $\|g_n^{-1}f_i g_{n+1}^{-1}f_i\|_{\mathcal{W}^+} < \delta/2^n$, i = 1, 2.

Since $(e_n)_{n \in \mathbb{N}}$ is an approximate identity for \mathfrak{m}_0 , we can choose m_1 such that

$$\|e_{m_1}f_i - f_i\|_{\mathcal{W}^+} < \frac{\delta}{4}(K-1), \quad i = 1, 2.$$
 (2.45)

Define $g_1 = ce_{m_1} + 1 - c$. So by (A), $g_1 \in U(\mathcal{W}^+)$ and using the calculation in (B), we see that

$$\|f_i - g_1^{-1} f_i\|_{\mathcal{W}^+} < \frac{\delta}{2}, \quad i = 1, 2.$$
 (2.46)

Suppose that e_{m_1}, \ldots, e_{m_n} have been constructed, so that g_n defined by (2.44) satisfies (P1) and (P2). We assert that if we choose $e_{m_{n+1}}$ such that

$$\|e_{m_{n+1}}f_i - f_i\|_{\mathcal{W}^+} \quad (i = 1, 2), \qquad \|e_{m_{n+1}}e_{m_k} - e_{m_k}\|_{\mathcal{W}^+} \quad (1 \le k \le n)$$
(2.47)

are sufficiently small, then g_{n+1} defined by (2.44) satisfies (P1) and (P2), completing the induction step.

Indeed, if $E := (1 - c + ce_{m_{n+1}})^{-1}$, we have

$$g_n = E^{-1} c \sum_{k=1}^n (1-c)^{k-1} E e_{m_k} + (1-c)^n,$$

$$g_{n+1} = E^{-1} \left(c \sum_{k=1}^n (1-c)^{k-1} E e_{m_k} + (1-c)^n \right).$$
(2.48)

Let G_n be defined by

$$G_n = c \sum_{k=1}^n (1-c)^{k-1} E e_{m_k} + (1-c)^n.$$
(2.49)

Then we have

$$\|G_{n}-g_{n}\|_{\mathcal{W}^{+}} < c \sum_{k=1}^{n} (1-c)^{k-1} \|Ee_{m_{k}}-e_{m_{k}}\|_{\mathcal{W}^{+}} < \max_{1 \le k \le n} \|Ee_{m_{k}}-e_{m_{k}}\|_{\mathcal{W}^{+}} < \frac{2}{K-1} \max_{1 \le k \le n} \|e_{m_{n+1}}e_{m_{k}}-e_{m_{k}}\|_{\mathcal{W}^{+}}.$$
(2.50)

Hence $G_n \in U(\mathcal{W}^+)$ and moreover $||G_n^{-1} - g_n^{-1}||_{\mathcal{W}^+}$ is small, provided only that $||e_{m_{n+1}}e_{m_k} - e_{m_k}||_{\mathcal{W}^+}$ is small for k = 1, ..., n. (Indeed, this is because $U(\mathcal{W}^+)$ is an open set in \mathcal{W}^+ .)

Since $g_{n+1} = E^{-1}G_n$, we have then $g_{n+1} \in U(\mathcal{W}^+)$, $g_{n+1}^{-1} = G_n^{-1}E$, and so for i = 1, 2,

$$\begin{aligned} \left\|g_{n+1}^{-1}f_{i} - g_{n}^{-1}f_{i}\right\|_{\mathcal{W}^{+}} &= \left\|G_{n}^{-1}Ef_{i} - g_{n}^{-1}f_{i}\right\|_{\mathcal{W}^{+}} \\ &\leq \left\|G_{n}^{-1}Ef_{i} - g_{n}^{-1}Ef_{i}\right\|_{\mathcal{W}^{+}} + \left\|g_{n}^{-1}Ef_{i} - g_{n}^{-1}f_{i}\right\|_{\mathcal{W}^{+}} \\ &\leq \left\|G_{n}^{-1} - g_{n}^{-1}\right\|_{\mathcal{W}^{+}}\left\|Ef_{i}\right\|_{\mathcal{W}^{+}} + \left\|g_{n}^{-1}\right\|_{\mathcal{W}^{+}}\left\|Ef_{i} - f_{i}\right\|_{\mathcal{W}^{+}}. \end{aligned}$$
(2.51)

Moreover, recall that by (B), we know that

$$\left\| Ef_{i} - f_{i} \right\|_{\mathcal{W}^{+}} \leq \frac{2}{K-1} \left\| e_{m_{n+1}} f_{i} - f_{i} \right\|_{\mathcal{W}^{+}}, \quad i = 1, 2.$$
(2.52)

Thus if $||e_{m_{n+1}}f_i - f_i||_{\mathcal{W}^+}$ (i = 1, 2) and $||e_{m_{n+1}}e_{m_k} - e_{m_k}||_{\mathcal{W}^+}$ $(1 \le k \le n)$ are sufficiently small, we will have $\|g_{n+1}^{-1}f_i - g_n^{-1}f_i\|_{\mathcal{W}^+}$ as small as we please. This completes the induction step. Since $\|e_{m_k}\|_{\mathcal{W}^+} \le K$, 0 < 1 - c < 1, and \mathcal{W}^+ is a Banach algebra, it follows that

$$g_n \longrightarrow c \sum_{k=1}^{\infty} (1-c)^{k-1} e_{m_k} =: g \in \mathfrak{m}_0,$$
(2.53)

and the proof is completed.

3. Noncoherence of \mathcal{W}^+

Proof of Theorem 1.3. We will use the characterization that an integral domain is coherent if and only if the intersection of any two finitely generated ideals of the ring is again finitely generated; see [1, Theorem 2.3.2, page 45]. In fact, we present two finitely generated ideals I and *J* such that $I \cap J$ is not finitely generated.

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Let *p*, *S* be given by

$$p = (1 - e^{-s})^3, \qquad S = e^{-(1 + e^{-s})/(1 - e^{-s})}.$$
 (3.1)

Clearly we have $p \in \mathfrak{m}_0$.

It is known (see, e.g., [3, Remark after Theorem 1, page 224]) that

$$(1-z)^{3}e^{-(1+z)/(1-z)} \in W^{+}(\mathbb{D}) := \left\{ f(z) = \sum_{n=0}^{\infty} a_{n} z^{n} \ (z \in \overline{\mathbb{D}}) \left| \sum_{n=0}^{\infty} |a_{n}| < \infty \right\}.$$
(3.2)

Here $\overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \le 1\}$. So if a_n 's are defined via

$$(1-z)^{3}e^{-(1+z)/(1-z)} = a_{0} + a_{1}z + a_{2}z^{2} + a_{3}z^{3} + \cdots, \quad z \in \mathbb{D},$$
(3.3)

then we have

$$\sum_{k=0}^{\infty} |a_k| < \infty. \tag{3.4}$$

If $\operatorname{Re}(s) > 0$, then $e^{-s} \in \mathbb{D}$, and so from (3.3), we have

$$pS = a_0 + a_1 e^{-s} + a_2 e^{-2s} + a_3 e^{-3s} + \cdots, \quad \text{Re}(s) > 0.$$
(3.5)

Since $\sum_{k=0}^{\infty} |a_k| < \infty$, the right-hand side in (3.5) belongs to \mathcal{W}^+ . So $pS \in \mathcal{W}^+$.

We define the ideals I = (p) and J = (pS) of \mathcal{W}^+ .

Let

$$K := \{ pSf \mid f \in \mathcal{W}^+ \text{ and } Sf \in \mathcal{W}^+ \}.$$

$$(3.6)$$

We claim that $K = I \cap J$. Trivially $K \subset I \cap J$. To prove the reverse inclusion, let $g \in I \cap J$. Then there exist two functions f and h in \mathcal{W}^+ such that g = ph = pSf. Since $p \neq 0$ and \mathcal{W}^+ is an integral domain, we obtain that $Sf = h \in \mathcal{W}^+$. So $g \in K$.

Let *L* denote the ideal

$$L := \{ f \in \mathcal{W}^+ \mid Sf \in \mathcal{W}^+ \}.$$

$$(3.7)$$

Then K := pSL. Since *S* has a singularity at s = 0, it follows that $L \subset \mathfrak{m}_0$. We will show that $L = L\mathfrak{m}_0$. Let $f \in L$. We would like to factor f = hg with $h \in L$ and $g \in \mathfrak{m}_0$. Applying Lemma 2.8 with $f_1 := f \in \mathfrak{m}_0$ and $f_2 := Sf \in \mathfrak{m}_0$, for any $\delta > 0$, there exists a sequence $(g_n)_{n \in \mathbb{N}}$ in \mathcal{W}^+ such that

- (1) for all $n \in \mathbb{N}$, $g_n \in U(\mathcal{W}^+)$;
- (2) $(g_n)_{n \in \mathbb{N}}$ is convergent in \mathcal{W}^+ to a limit $g \in \mathfrak{m}_0$;

(3) for all $n \in \mathbb{N}$,

$$\|g_n^{-1}f - g_{n+1}^{-1}f\|_{\mathcal{W}^+} \le \frac{\delta}{2^n}, \qquad \|g_n^{-1}Sf - g_{n+1}^{-1}Sf\|_{\mathcal{W}^+} \le \frac{\delta}{2^n}.$$
(3.8)

Put

$$h_n := g_n^{-1} f, \qquad H_n := g_n^{-1} S f.$$
 (3.9)

Then $h_n \in \mathfrak{m}_0$. Also $H_n \in \mathfrak{m}_0$, since |S| is bounded by 1 on $\operatorname{Re}(s) > 0$ and f(0) = 0. The estimates above imply that $(h_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$ are Cauchy sequences in \mathcal{W}^+ . Since \mathfrak{m}_0 is closed, they converge to elements h and H, respectively, in \mathfrak{m}_0 , that is, $h_n = g_n^{-1}f \to h$ and $H_n = g_n^{-1}Sf =$ $Sh_n \to H$. Since convergence in \mathcal{W}^+ implies convergence in H^{∞} (Lemma 2.2), it follows that

$$h_n \longrightarrow H^{\infty}h \quad (\text{since } h_n \longrightarrow \mathcal{W}^+h),$$

$$Sh_n \longrightarrow H^{\infty}Sh \quad (\text{since } h_n \longrightarrow H^{\infty}h, S \in H^{\infty}),$$

$$Sh_n \longrightarrow H^{\infty}H \quad (\text{since } H_n \longrightarrow \mathcal{W}^+H)$$
(3.10)

and so by the uniqueness of the limit of the sequence $(Sh_n)_{n\in\mathbb{N}}$ in H^{∞} , we have Sh = H. Also, in \mathcal{W}^+ -norm we have

$$f = \lim_{n \to \infty} h_n g_n = hg, \tag{3.11}$$

since multiplication is continuous in the Banach algebra \mathcal{W}^+ . Since *h* and Sh = H belong to $\mathfrak{m}_0 \subset \mathcal{W}^+$, we see that $h \in L$. Moreover, as $g \in \mathfrak{m}_0$, we have got the desired factorization and $L = L\mathfrak{m}_0$.

But $L \neq (0)$, since $p \in L$. By Lemma 2.3, it follows that L cannot be finitely generated. Therefore, $pSL = I \cap J$ is not finitely generated.

Remark 3.1. The ideal *L* in the above proof can be interpreted as an *ideal of denominators*; see [10, page 396]. Using the fact that $pS \in W^+$, we have $S \in Q(W^+)$, where $Q(W^+)$ denotes the field of fractions of W^+ . We can then consider the *fractional ideal* $M := W^+ + W^+S$ of W^+ (see [11, page 19]) and the *ideal of denominators L* of *S*, namely $L = W^+ : M = \{d \in W^+ \mid dS \in W^+\}$.

Based on the results in [12, Theorem 3, Example 3], it follows that $S \in Q(\mathcal{W}^+)$ does not admit a weak coprime factorization, since *L* is not a principal ideal of \mathcal{W}^+ . In particular, *S* does not admit a coprime factorization, that is, there do not exist $d, x, y, n \in \mathcal{W}^+$ such that $d \neq 0$, S = n/d, and dx - ny = 1. Moreover, *S* is not internally stabilizable, since otherwise *L* would be generated by two elements. Finally, the fact that *L* is not finitely generated implies that \mathcal{W}^+ is not a *greatest common divisor domain*: indeed, were it the case that \mathcal{W}^+ is a greatest common divisor domain, then by [12, Corollary 3], every element in $Q(\mathcal{W}^+)$ would admit a weak coprime factorization.

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References

- S. Glaz, Commutative Coherent Rings, vol. 1371 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 1989.
- [2] W. S. McVoy and L. A. Rubel, "Coherence of some rings of functions," *Journal of Functional Analysis*, vol. 21, no. 1, pp. 76–87, 1976.
- [3] R. Mortini and M. von Renteln, "Ideals in the Wiener algebra W⁺," Journal of the Australian Mathematical Society. Series A, vol. 46, no. 2, pp. 220–228, 1989.
- [4] F. M. Callier and C. A. Desoer, "An algebra of transfer functions for distributed linear time-invariant systems," *IEEE Transactions on Circuits and Systems*, vol. 25, no. 9, pp. 651–662, 1978.

- [5] M. Vidyasagar, Control System Synthesis: A Factorization Approach, vol. 7 of MIT Press Series in Signal Processing, Optimization, and Control, MIT Press, Cambridge, Mass, USA, 1985.
- [6] A. Quadrat, "An introduction to internal stabilization of linear infinite dimensional systems," Course Notes, École Internationale d'Automatique de Lille (02-06/09/02): Contrôle de systèmes à paramètres répartis: Théorie et Applications, 2002, http://www-sop.inria.fr/cafe/ Alban.Quadrat/Pubs/Germany2.pdf.
- [7] A. Quadrat, "The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. I. (Weakly) doubly coprime factorizations," *SIAM Journal on Control and Optimization*, vol. 42, no. 1, pp. 266–299, 2003.
- [8] R. S. Strichartz, A Guide to Distribution Theory and Fourier Transforms, World Scientific, River Edge, NJ, USA, 2003.
- [9] A. Browder, Introduction to Function Algebras, W. A. Benjamin, New York, NY, USA, 1969.
- [10] V. P. Havin, S. V. Hruščev, and N. K. Nikol'skiĭ, Eds., *Linear and Complex Analysis Problem Book*, vol. 1043 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1984.
- [11] H. Matsumura, *Commutative Ring Theory*, vol. 8 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1989.
- [12] A. Quadrat, "On a generalization of the Youla-Kučera parametrization. I. The fractional ideal approach to SISO systems," Systems & Control Letters, vol. 50, no. 2, pp. 135–148, 2003.

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