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The Hilbert–Schmidt property of feedback operators

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Abstract

In this article we prove new sufficient conditions under which the feedback operator associated with the Linear Quadratic Regulator control design for distributed parameter systems is nuclear or Hilbert–Schmidt. Examples illustrating the main results are also given.

Keywords: Hilbert–Schmidt; Nuclear; Analytic semigroups; Linear Quadratic Regulator (LQR) control

1. Introduction

The most popular control design for distributed parameter systems (DPS) is the Linear Quadratic Regulator (LQR) control design. These controllers are infinite-dimensional and in practice one approximates these to obtain implementable finite-dimensional controllers. Conditions for the effectiveness of this approach has been the subject of a number of papers (see for example Banks and Kunisch [2], Burns et al. [4], Gibson [10], Ito [12,13], Kappel and Salamon [14], King [16], Opmeer et al. [20]).

Here we focus on two properties that are of importance:

(P1) The gain operator is Hilbert–Schmidt.
(P2) The solution to the Riccati equation is Hilbert–Schmidt or nuclear.

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The aim of this paper is to summarize known sufficient conditions and to give new sufficient conditions for (P1) and (P2) to hold.

The LQR problem we consider is for the abstract linear system

$$\left.\begin{aligned} &\frac{d}{dt}x(t) = Ax(t) + Bu(t) \\ &y(t) = Cx(t) \end{aligned}\right\}, \qquad t \geqslant 0, \qquad x(0) = x_0,$$

where the control input[1] $u \in L_2(\mathbb{R}_+; U)$, the state $x \in C^{\mathrm{loc}}(\mathbb{R}_+; X)$, and output $y \in L_2(\mathbb{R}_+; Y)$, and U, X, Y are Hilbert spaces. We suppose that A generates a C_0-semigroup on X, the operator[2] $C \in \mathcal{B}(X, Y)$, and $(\beta I - A)^{-1}B \in \mathcal{B}(U, X)$ for some $\beta \in \rho(A)$, the resolvent set of the operator A.

The LQR problem is to find $u_{\mathrm{opt}} \in L_2(\mathbb{R}_+; U)$ that minimizes

$$J(x_0, u) = \int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 \, dt.$$

If B is bounded and the system is exponentially stabilizable, then there exists $u_{\mathrm{opt}}(t) = Kx(t)$, where the gain operator K is given by $-B^*Q$, and $Q \in \mathcal{B}(X)$ is the minimal nonnegative solution of the Riccati equation

$$\langle Qx_1, Ax_2\rangle + \langle Ax_1, Qx_2\rangle - \langle B^*Qx_1, B^*Qx_2\rangle + \langle Cx_1, Cx_2\rangle = 0$$

for all $x_1, x_2 \in \mathrm{Dom}(A)$. If, in addition, U and Y are finite-dimensional, then it is easy to show that Q is nuclear and consequently the gain operator is Hilbert–Schmidt (see Section 3). This means that K can be represented as an integral operator (see for instance Weidmann [24, Theorem 6.11, p. 139]), which has advantages for designing practical control laws (see the papers by King on functional gains [1,15,16]).

The nuclear property of Q implies the existence of finite-dimensional approximants $u_{\mathrm{opt}}(t) = K_n x(t)$ that will stabilize the original system (see Curtain [6]). In the robust LQG design presented in [8] it is sufficient that Q be Hilbert–Schmidt.

In many applications the control is implemented on the boundary, in which case B is unbounded. It is important to have conditions for (P1) and (P2) to hold in this case too. It is already known that if A generates an analytic C_0-semigroup and $(\beta I - A)^{-\gamma}$ is Hilbert–Schmidt for some $\gamma \in [0, 1)$, then (P2) holds (see Lasiecka and Triggiani [17, Remark 2.2.2, p. 128]). We recall this result in Theorem 4.1 and Proposition 4.4, and use it to obtain sufficient conditions for (P1) to hold in Theorem 4.6. These results cover the classical parabolic equations with boundary control (as we illustrate in Example 6.1).

There are, however, operators A that generate analytic semigroups, but for which $(\beta I - A)^{-\gamma}$ is not Hilbert–Schmidt. If A has an accumulation point in its spectrum, then $(\beta I - A)^{-\gamma}$ will never be Hilbert–Schmidt. For this class we derive alternative sufficient conditions for (P1) and (P2) to hold in Theorem 4.6. In Example 6.2, these results are applied to show that a controlled flexible beam with boundary control has properties (P1) and (P2). This provides the theoretical justification for the LQG-balancing control design in Opmeer et al. [20].

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[1] Throughout this article, $\mathbb{R}_+$ denotes the set of positive real numbers.

[2] The notation $\mathcal{B}(H_1, H_2)$ is used to denote the space of bounded linear operators from the Hilbert space H_1 to the Hilbert space H_2.

2. Preliminaries

In this section, we recall the notions of Hilbert–Schmidt and nuclear operators, and also list a few properties of these classes of operators that we will use in the sequel. For background information, we refer the reader to Pietsch [22] and to Weidmann [24].

Let H_1 and H_2 be Hilbert spaces. An operator $T \in \mathcal{B}(H_1, H_2)$ is said to be Hilbert–Schmidt if

$$\sum_{i\in I} \|Te_i\|^2 < +\infty$$

for some orthonormal basis $\{e_i\}_{i\in I}$ for H_1. The set of Hilbert–Schmidt operators is denoted by $\mathcal{S}_2(H_1, H_2)$. Hilbert–Schmidt operators are compact, and they form a two-sided ideal:

$$\mathcal{S}_2(H_2, H_3)\mathcal{B}(H_1, H_2) \subset \mathcal{S}_2(H_1, H_3) \quad \text{and} \quad \mathcal{B}(H_2, H_3)\mathcal{S}_2(H_1, H_2) \subset \mathcal{S}_2(H_1, H_3).$$

An operator $T \in \mathcal{B}(H_1, H_2)$ is Hilbert–Schmidt iff its adjoint $T^* \in \mathcal{B}(H_2^*, H_1^*)$ is Hilbert–Schmidt. There are several alternative characterizations of Hilbert–Schmidt operators, and we give one such below.

First we recall the notion of singular values of a bounded linear operator from a Hilbert space H_1 to a Hilbert space H_2. For $n \in \mathbb{N}$, the nth *singular value* of an operator $T \in \mathcal{B}(H_1, H_2)$ (denoted by $\sigma_n(T)$) is defined to be the distance with respect to the norm in $\mathcal{B}(H_1, H_2)$ of T from the set of operators in $\mathcal{B}(H_1, H_2)$ of rank at most $n-1$. Thus $\sigma_1(T) = \|T\|$, and

$$\sigma_1(T) \geqslant \sigma_2(T) \geqslant \sigma_3(T) \geqslant \cdots \geqslant 0.$$

If T is compact, then T^*T is compact and nonnegative, and so the nonzero spectrum of T^*T consists of a pure point spectrum with countably many nonnegative eigenvalues. The square roots of these eigenvalues are then the singular values of T.

An alternative characterization of Hilbert–Schmidt operators is then the following: an operator $T \in \mathcal{B}(H_1, H_2)$ is Hilbert–Schmidt iff

$$\sum_{n\in\mathbb{N}} \big(\sigma_n(T)\big)^2 < +\infty.$$

On the other hand, if the singular values are summable, then the operator is called *nuclear*:

$$\sum_{n\in\mathbb{N}} \sigma_n(T) < +\infty.$$

The set of nuclear operators is denoted by $\mathcal{S}_1(H_1, H_2)$. This space has the following ideal property:

$$\mathcal{S}_1(H_2, H_3)\mathcal{B}(H_1, H_2) \subset \mathcal{S}_1(H_1, H_3) \quad \text{and} \quad \mathcal{B}(H_2, H_3)\mathcal{S}_1(H_1, H_2) \subset \mathcal{S}_1(H_1, H_3).$$

Clearly, every nuclear operator is Hilbert–Schmidt: $\mathcal{S}_1(H_1, H_2) \subset \mathcal{S}_2(H_1, H_2)$. It can also be shown that the product of two Hilbert–Schmidt operators is nuclear, that is,

$$\mathcal{S}_2(H_2, H_3)\mathcal{S}_2(H_1, H_2) \subset \mathcal{S}_1(H_1, H_3).$$

The hierarchy of the classes of operators is shown below, where $\mathcal{K}(H_1, H_2)$ denotes the set of compact operators: $\mathcal{S}_1(H_1, H_2) \subset \mathcal{S}_2(H_1, H_2) \subset \mathcal{K}(H_1, H_2) \subset \mathcal{B}(H_1, H_2)$.

3. Case of bounded, finite rank input and output operators

We recall the following theorem from Curtain and Zwart [8].

Theorem 3.1. *Suppose that U, X, Y are Hilbert spaces, and that A be the infinitesimal generator of a C_0-semigroup $\mathfrak{A}$ on X, $B \in \mathcal{B}(U, X)$, and $C \in \mathcal{B}(X, Y)$. If (A, B) is exponentially stabilizable, then there exists a self-adjoint, nonnegative solution Q in $\mathcal{B}(X)$ such that*:

(1) *$A - BB^*Q$ generates an exponentially stable C_0-semigroup $\widetilde{\mathfrak{A}}$ on X;*
(2) *Q is the minimal solution of the algebraic Riccati equation*

$$0 = \langle Ax_1, Qx_2\rangle_X + \langle Qx_1, Ax_2\rangle_X + \langle Cx_1, Cx_2\rangle_Y - \langle B^*Qx_1, B^*Qx_2\rangle_U \tag{1}$$

for all $x_1, x_2 \in \operatorname{Dom}(A)$;
(3) *$Q = \mathfrak{C}_Q^*\mathfrak{C}_Q$, where $\mathfrak{C}_Q : X \to L_2(\mathbb{R}_+; Y \oplus U)$ is given by*

$$(\mathfrak{C}_Q x)(t) = \begin{bmatrix} \widetilde{\mathfrak{A}}(t)x \\ B^*Q\widetilde{\mathfrak{A}}(t)x \end{bmatrix}, \quad t \geqslant 0,\ x \in X. \tag{2}$$

We quote the following result from Dumortier [9, Proposition 1.0.2] (see also Grabowski [11] and Curtain and Sasane [7, Theorem 4.1]).

Theorem 3.2. *Suppose that A is the infinitesimal generator of an exponentially stable C_0-semigroup on the Hilbert space X, and that $C \in \mathcal{B}(X, \mathbb{C}^p)$. Then the observability operator $\mathfrak{C} : X \to L_2(\mathbb{R}_+; \mathbb{C}^p)$ defined by $(\mathfrak{C}x)(t) = C\mathfrak{A}(t)x$, $t \geqslant 0$, $x \in X$, is Hilbert–Schmidt.*

Using the above two results, we obtain the following easy result (see Curtain [5]):

Theorem 3.3. *Suppose that A is the infinitesimal generator of a C_0-semigroup $\mathfrak{A}$ on the Hilbert space X, $B \in \mathcal{B}(\mathbb{C}^m, X)$, and $C \in \mathcal{B}(X, \mathbb{C}^p)$. If (A, B) is exponentially stabilizable, then the minimal solution of the algebraic Riccati equation* (1) *is nuclear. Furthermore, $B^*Q \in \mathcal{B}(X, \mathbb{C}^m)$ is also nuclear.*

Proof. Since $C \in \mathcal{B}(X, \mathbb{C}^p)$ and $B^*Q \in \mathcal{B}(X, \mathbb{C}^m)$, it follows that $\left[\begin{smallmatrix} C \\ B^*Q \end{smallmatrix}\right] \in \mathcal{B}(X, \mathbb{C}^{p+m})$, which has finite rank. The semigroup generated by $A - BB^*Q$ is exponentially stable, by Theorem 3.1. Applying Theorem 3.2, we obtain that the observability operator $\mathfrak{C}_Q : X \to L_2(\mathbb{R}_+; \mathbb{C}^{p+m})$, defined by (2), is Hilbert–Schmidt. Consequently $Q = \mathfrak{C}_Q^*\mathfrak{C}_Q \in \mathcal{B}(X)$ is nuclear. As $B^* \in \mathcal{B}(X, \mathbb{C}^m)$, it follows that B^*Q is nuclear as well. □

4. Analytic case with unbounded finite rank input and output operators

First we introduce some notation. If A is the generator of a C_0-semigroup $\mathfrak{A}$, then its growth bound is denoted by ω_A, where the growth bound is defined as

$$\omega_A = \inf_{t>0} \frac{1}{t} \log\|\mathfrak{A}(t)\|.$$

Suppose that A is the infinitesimal generator of an analytic C_0-semigroup $\mathfrak{A}$ on the Hilbert space X, and let $\omega > \omega_A$. Then for all $\alpha > 0$, the fractional power of $(\omega I - A)^{-1}$ is defined by setting

$$(\omega I - A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\omega t} \mathfrak{A}(t)\, dt \in \mathcal{B}(X). \tag{3}$$

Furthermore, let $(\omega I - A)^{\alpha} = ((\omega I - A)^{-\alpha})^{-1}$. Let $X_\alpha = \mathrm{Dom}((\omega I - A)^{\alpha}) = \mathrm{Ran}((\omega I - A)^{-\alpha})$ with the norm $\|x\|_{X_\alpha} = \|(\omega I - A)^{\alpha} x\|_X$, $x \in \mathrm{Dom}((\omega I - A)^{\alpha})$. For $\alpha < 0$, let X_α be the completion of X with respect to the norm $\|x\|_{X_\alpha} = \|(\omega I - A)^{\alpha} x\|_X$, $x \in X$. Finally, let X_0 be the Hilbert space X. Hence one obtains a chain of Hilbert spaces X_α, parametrized by $\alpha \in \mathbb{R}$. If $\mathfrak{A}$ is analytic, then $\mathfrak{A}^*$ is also analytic, and for $\alpha \in \mathbb{R}$, we define $X_\alpha^* = (X^*)_\alpha = (X_{-\alpha})^*$. These sets and their topologies (not norms) are independent of ω.

Below we list a few remarks concerning properties of analytic semigroups, which we will use in the sequel.

(R1) If A generates an exponentially stable analytic semigroup, then for each $\alpha \geqslant 0$, there exist constants $M < +\infty$ and $\epsilon > 0$ such that

$$\left\|(-A)^{\alpha} \mathfrak{A}(t)\right\|_{\mathcal{B}(X)} \leqslant M \frac{e^{-\epsilon t}}{t^{\alpha}}, \quad t \in \mathbb{R}_+. \tag{4}$$

(R2) For any $\omega > \omega_A$, there exists $\Theta \in (\frac{\pi}{2}, \pi]$ and there exists $M = M(A, \omega, \Theta) < \infty$ such that $\|(sI - A)^{-1}\| \leqslant \frac{M}{|s-\omega|}$ for all $s \in \Sigma_{\Theta,\omega} := \{s \in \mathbb{C} \mid s \neq \omega, |\arg(s - \omega)| < \Theta\}$. Furthermore,

$$\left\|(\omega I - A)^{\alpha} (sI - A)^{-1}\right\|_{\mathcal{B}(X)} \leqslant \frac{M(1 + |s - \omega_0|^{\alpha})}{|s - \omega_0|}, \\ \left(s \in \Sigma_{\Theta,\omega_0},\ \omega > \omega_0 > \omega_A,\ \alpha \in [0, 1]\right). \tag{5}$$

(R3) For $\alpha \geqslant 0$, the restriction of $\mathfrak{A}$ to X_α is an analytic semigroup, isomorphic to the original one (using $\mathfrak{A}(t)(\omega I - A)^{\alpha} = (\omega I - A)^{\alpha}\mathfrak{A}(t)$), with generator $A|_{X_{\alpha+1}}$. Similarly, for $\alpha \leqslant 0$, $\mathfrak{A}$ has a unique extension to an (isomorphic) analytic semigroup on X_α, and we denote its generator by $A|_{X_{\alpha+1}}$. In particular, $\omega_{A|_{X_\alpha}}$ is the same for each α.

(R4) If we were to start from some X_β, $\beta \in \mathbb{R}$ instead of $X = X_0$, then we would obtain the same spaces, semigroups and generators (in particular, $(X_\beta)_\alpha = X_{\beta+\alpha}$, i.e., $(\omega - A|_{X_{\beta+1}})^{-\alpha} X_\beta = (\omega - A|_{X_1})^{-(\beta+\alpha)} X_0$).

For properties of analytic semigroups and interpolation spaces X_α we refer the reader to Lunardi [18], Pazy [21] or Staffans [23].

We recall the following result (see Lasiecka and Triggiani [17, Theorems 2.2.1 and 2.2.2, pp. 125–127] and also Mikkola [19, §9.5]).

Theorem 4.1. *Suppose that*:

(A1) *A is the infinitesimal generator of an analytic C_0-semigroup $\mathfrak{A}$ on a Hilbert space X.*

(A2) *There exists $\alpha_B \in (-1, 0]$ such that $(\omega I - A)^{\alpha_B} B \in \mathcal{B}(U, X)$, that is, $B \in \mathcal{B}(U, X_{\alpha_B})$, where $\omega > \omega_A$ and U is a Hilbert space.*

(A3) *$C \in \mathcal{B}(X, Y)$, where Y is a Hilbert space.*

(A4) (*Exponential detectability*) *There exists $L \in \mathcal{B}(Y, X)$ such that the analytic C_0-semigroup with generator $A + LC$ is exponentially stable on X.*

(A5) (*Finite cost condition*) *For each $x_0 \in X$, there exists $u \in L_2(\mathbb{R}_+; U)$ such that the mild solution x to $\frac{d}{dt}x(t) = Ax(t) + Bu(t)$, $x(0) = x_0$, satisfies $Cx(\cdot) \in L_2(\mathbb{R}_+; X)$.*

Then there exists a self-adjoint, nonnegative $Q \in \mathcal{B}(X)$ such that:

(1) *The operator $A_Q = A - BB^*Q$ with* $\mathrm{Dom}(A_Q)$ *given by*

$$\begin{aligned}&\bigl\{x \in \mathrm{Dom}\bigl((\omega I - A)^{1+\alpha_B}\bigr) \bigm| \\ &(\omega I - A)^{1+\alpha_B}x - (\omega I - A)^{\alpha_B} BB^*Qx \in \mathrm{Dom}\bigl((\omega I - A)^{-\alpha_B}\bigr)\bigr\}\end{aligned} \tag{6}$$

is the infinitesimal generator of an exponentially stable, analytic C_0-semigroup $\widetilde{\mathfrak{A}}$ on X.

(2) *Q is the unique self-adjoint nonnegative solution of the following algebraic Riccati equation*

$$0 = \langle Ax_1, Qx_2\rangle_X + \langle Qx_1, Ax_2\rangle_X + \langle Cx_1, Cx_2\rangle_X - \bigl\langle B^*Qx_1, B^*Qx_2\bigr\rangle_U \tag{7}$$

for all $x_1, x_2 \in \mathrm{Dom}((\omega I - A)^\epsilon)$, and any $\epsilon > 0$.

(3) *$Q = \mathfrak{C}_Q^*\mathfrak{C}_Q$, where $\mathfrak{C}_Q \in \mathcal{B}(X, L_2(\mathbb{R}_+; Y \oplus U))$ is given by*

$$(\mathfrak{C}_Q x)(t) = \begin{bmatrix} C\widetilde{\mathfrak{A}}(t)x \\ B^*Q\widetilde{\mathfrak{A}}(t)x \end{bmatrix}, \quad t \geqslant 0,\ x \in X. \tag{8}$$

(4) *$B^*Q \in \mathcal{B}(X, U)$.*

The result in Theorem 4.3 below is a consequence of the following standard result from Weidmann [24, Theorem 6.12, p. 140].

Theorem 4.2. *Let K be a bounded linear operator from a Hilbert space H into $L_2(\mathbb{R}_+; \mathbb{C})$. If there exists a function $k \in L_2(\mathbb{R}_+; \mathbb{C})$ such that $|(Kx)(t)| \leqslant k(t)\|x\|$ for almost all $t \in \mathbb{R}_+$ and all $x \in H$, then K is a Hilbert–Schmidt operator.*

Theorem 4.3. *Suppose that A is the infinitesimal generator of an exponentially stable, analytic C_0-semigroup $\mathfrak{A}$ on the Hilbert space X, and let $C \in \mathcal{B}(X_{\alpha_C}, \mathbb{C}^p)$. If $\gamma > \alpha_C - \frac{1}{2}$, then the observability operator $\mathfrak{C} : X_\gamma \to L_2(\mathbb{R}_+; \mathbb{C}^p)$ defined by $(\mathfrak{C}x)(t) = C\mathfrak{A}(t)x$, $t \geqslant 0$, $x \in X_\gamma$, is Hilbert–Schmidt.*

Proof. The result follows from an immediate application of Theorem 4.2 above; for details, see for instance the proof of part (2) of Theorem 6 on page 1266 of [7]. □

We recall the following result from Lasiecka and Triggiani [17, Remark 2.2.2, p. 128].

Proposition 4.4. *Let assumptions* (A1) *to* (A5) *hold. Furthermore, if the operator $(\omega I - A)^{-\alpha}$ is Hilbert–Schmidt on X for some $\alpha \in (0, 1)$, then the unique nonnegative solution of the algebraic Riccati equation* (7) *is Hilbert–Schmidt on X.*

However, as mentioned in the introduction, $(\omega I - A)^{-\alpha}$ being Hilbert–Schmidt is an assumption which may not always be satisfied. For instance, this is never satisfied if the spectrum of A has an accumulation point as in Example 6.2. This motivates the result in Theorem 4.6 below, which gives alternative conditions that guarantee the Hilbert–Schmidt/nuclearity properties of the solution to the Riccati equation and the gain operator. This new result is obtained by a simple application of Theorems 4.1 and 4.3 and the following technical result in Proposition 4.5.

Proposition 4.5. *Let A be the generator of an analytic semigroup on the Banach space X. Let $\Delta \in \mathcal{B}(X_\alpha, X_\beta)$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha - \beta < 1$. Let $\gamma \in [\alpha - 1, \beta + 1]$. Then the following hold*:

(S1) *The operator*[3] $A + \Delta$ *generates an analytic semigroup* $\widetilde{\mathfrak{A}}$ *on* $X_{\alpha-1}$.
(S2) *The space* X_γ *is invariant under* $\widetilde{\mathfrak{A}}$, *and the restriction* $\widetilde{\mathfrak{A}}|_{X_\gamma}$ *to* X_γ *is an analytic semigroup on* X_γ.
(S3) *The generator of* $\widetilde{\mathfrak{A}}|_{X_\gamma}$ *is the part of* $A + \Delta$ *in* X_γ; *it equals* $(A + \Delta)|_{X_{\gamma+1}}$ *if* $\gamma \in [\alpha - 1, \beta]$.
(S4) *We have* $(\omega I - (A + \Delta))^{-(\gamma-(\alpha-1))} X_{\alpha-1} = X_\gamma$ $(\omega > \omega_{A+\Delta})$.
(S5) *If* $0 \in [\alpha - 1, \beta + 1]$ (*so that* $\widetilde{\mathfrak{A}}|_X$ *is an analytic semigroup on* X) *and if we let* $\widetilde{X}_\delta$ $(\delta \in \mathbb{R})$ *be the analogues of the spaces* X_δ *with* A *replaced by the part of* $A + \Delta$ *in* X, *then* $\widetilde{X}_\delta = X_\delta$ *for all* $\delta \in [\alpha - 1, \beta + 1]$.

(The part of $A + \Delta$ in X_γ has the domain $\{x \in X_\alpha \cap X_\gamma \mid (A + \Delta)x \in X_\gamma\}$, by definition. By $\widetilde{X}_\gamma = X_\gamma$ we mean that the vector spaces and topologies coincide (so that the norms are equivalent). Naturally, in (S4) we refer to (3) with $\widetilde{\mathfrak{A}}$ in place of $\mathfrak{A}$ and $X_{\alpha-1}$ in place of X.)

The proof of Proposition 4.5 is given in Section 5.

Theorem 4.6. *Under the assumptions* (A1) *to* (A5) *from Theorem* 4.1, *with* $U = \mathbb{C}^m$ *and* $Y = \mathbb{C}^p$, *the following hold*:

(1) *The self-adjoint nonnegative solution of the algebraic Riccati equation* (7) *is a nuclear operator from* X_γ *to* $(X_\gamma)^*$ *for all* $\gamma > -\frac{1}{2}$.
(2) *If* $\alpha_B > -\frac{1}{2}$, *then* B^*Q *is a Hilbert–Schmidt operator from* X_γ *to* $\mathbb{C}^m$ *for all* $\gamma > -\frac{1}{2}$.

Proof. From Theorem 4.1, we know that the semigroup $\widetilde{\mathfrak{A}}$ is analytic and exponentially stable. Furthermore, $\left[\begin{smallmatrix} C \\ B^*Q \end{smallmatrix}\right] \in \mathcal{B}(X, \mathbb{C}^{p+m})$. Since $BB^*Q \in \mathcal{L}(X, X_{\alpha_B})$ with $-\alpha_B < 1$, using Proposition 4.5 above, it follows that the interpolation spaces corresponding to the semigroup $\widetilde{\mathfrak{A}}$ are the same topological spaces as the ones corresponding to $\mathfrak{A}$. By Theorem 4.3 and (8) above, it follows that $\mathfrak{C}_Q \in \mathcal{B}(X_\gamma, L_2(\mathbb{R}_+; \mathbb{C}^{p+m}))$ is Hilbert–Schmidt for all $\gamma > -\frac{1}{2}$. Consequently, $Q = \mathfrak{C}_Q^* \mathfrak{C}_Q \in \mathcal{B}(X_\gamma, (X_\gamma)^*)$ is nuclear for all $\gamma > -\frac{1}{2}$.

We have $\mathfrak{C}_Q \in \mathcal{B}(X_\gamma, L_2(\mathbb{R}_+; \mathbb{C}^{p+m}))$ is Hilbert–Schmidt for all $\gamma > -\frac{1}{2}$. So $B^*Q = B^*\mathfrak{C}_Q^*\mathfrak{C}_Q$ will also be Hilbert–Schmidt from X_γ to $\mathbb{C}^m$ if $B^*\mathfrak{C}_Q^* \in \mathcal{B}(L_2(\mathbb{R}_+; \mathbb{C}^{p+m}), \mathbb{C}^m)$, that is, if $\mathfrak{C}_Q B \in \mathcal{B}(\mathbb{C}^m, L_2(\mathbb{R}_+; \mathbb{C}^{p+m}))$. Now

$$\mathfrak{C}_Q B = \begin{bmatrix} C \\ B^*Q \end{bmatrix} \widetilde{\mathfrak{A}}(t) B$$

and

$$\int_0^\infty \|\mathfrak{C}_Q B\|^2 \, dt = \int_0^\infty \left\| \begin{bmatrix} C \\ B^*Q \end{bmatrix} \widetilde{\mathfrak{A}}(t) B \right\|^2 dt$$

[3] By $A + \Delta$ we refer to $A|_{X_\alpha} + \Delta$.

$$\leqslant \left\| \begin{bmatrix} C \\ B^*Q \end{bmatrix} \right\|^2 \int_0^\infty \left\| (-A + BB^*Q)^{-\alpha_B} \widetilde{\mathfrak{A}}(t) \right\|^2 \left\| (-A + BB^*Q)^{\alpha_B} B \right\|^2 dt$$

$$\leqslant (\text{constant}) \cdot \int_0^\infty \frac{e^{-\epsilon t}}{t^{-2\alpha_B}} \, dt, \quad \big(\text{using (4)}\big)$$

where C, B^*Q and $(\omega I - A)^{\alpha_B} B$ are all bounded, and $\widetilde{\mathfrak{A}}$ is an exponentially stable analytic semigroup. So it follows that $\mathfrak{C}_Q B$ is bounded provided that $\alpha_B > -\frac{1}{2}$. □

In other words, under the assumptions (A1) to (A5), the solution to the Riccati equation is always nuclear for all $\gamma > -\frac{1}{2}$ ((P2) holds). If in addition, the input operator B is sufficiently smooth ($\alpha_B > -\frac{1}{2}$), then the gain operator B^*Q is Hilbert–Schmidt ((P1) holds).

5. Proof of Proposition 4.5

In this section we prove Proposition 4.5. We shall use freely the fact that (R1)–(R4) and that the facts above them (except the sentence on adjoints) hold also when X is a Banach space.

Proposition 4.5 and its proof were sketched in Lemma 9.4.2 of Mikkola [19] and in somewhat more detail in Theorem 3.10.11 of Staffans [23]. Because the result appears to be new, and it is the key to our new results on properties (P1) and (P2), we include an expanded proof below. We start with some auxiliary results.

Lemma 5.1. *Let $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$ be C_0-semigroups on Banach spaces X and $\widetilde{X}$ with generators A and $\widetilde{A}$, respectively, and let $X \subset \widetilde{X}$ continuously.*

(1) *If $A \subset \widetilde{A}$ (that is, if $\widetilde{A}|_{\mathrm{Dom}(A)} = A$), then $\mathfrak{A} = \widetilde{\mathfrak{A}}|_X$.*
(2) *If $\mathfrak{A} = \widetilde{\mathfrak{A}}|_X$, then A is the part of $\widetilde{A}$ in X, that is,* $\mathrm{Dom}(A) = \{x \in \mathrm{Dom}(\widetilde{A}) \cap X \mid Ax \in X\}$ *and $A \subset \widetilde{A}$.*

Proof. (1) Take $\alpha > \omega_A, \omega_{\widetilde{A}}$, and observe that $(\alpha I - A)^{-1} = (\alpha I - \widetilde{A})^{-1}|_X$. Then with

$$A_\alpha := \alpha^2(\alpha I - A)^{-1} - \alpha I \quad \text{and} \quad \widetilde{A}_\alpha := \alpha^2(\alpha I - \widetilde{A})^{-1} - \alpha I,$$

we have $A_\alpha = \widetilde{A}_\alpha|_X$, and $e^{tA_\alpha} = (e^{t\widetilde{A}_\alpha})|_X$, and so $\mathfrak{A}(t) = \lim_{\alpha \to +\infty} e^{tA_\alpha} = \widetilde{\mathfrak{A}}(t)|_X$ (see [23, Theorem 3.7.3]).

(2) This follows from Theorem 3.14.14 of [23]. □

Lemma 5.2. *Let X, Y and Z be Banach spaces. If $X \subset Z$ continuously, $Y \subset Z$ continuously and $X \subset Y$, then $X \subset Y$ continuously.*

Proof. This is a simple consequence of the closed-graph theorem. □

The following is well known (see, for example, [19, Lemma A.4.4]):

Proposition 5.3. *If A generates a strongly continuous semigroup on a Banach space X and $\omega > \omega_A$, then there exists $M < \infty$ such that for all real $\lambda \geqslant \omega$,*

$$\left\| \lambda(\lambda I - A)^{-1} \right\|_{\mathcal{B}(X)} \leqslant M \quad \textit{and} \quad \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{B}(X, \mathrm{Dom}(A))} \leqslant M. \tag{9}$$

Now we are ready to prove Proposition 4.5; we start from a special case:

Lemma 5.4. *Proposition* 4.5 *holds under the additional restrictions* $\alpha = 1$, $\beta > 0$, $\gamma, \delta \in [0, 1]$.

Proof of Lemma 5.4. By the assumption, $\Delta \in \mathcal{B}(X_1, X_\beta)$. Without loss of generality, we assume that $\beta \leqslant 1$: indeed, if $\Delta \in \mathcal{B}(X_1, X_\beta)$ with $\beta > 1$, then $\Delta \in \mathcal{B}(X_1, X_1)$.

$\underline{1}^\circ$ In this step we prove (S1) and make a few additional remarks.
By Propositions 2.4.1(ii) (and 2.1.4(i)) and 2.2.15 of Lunardi [18], $A + \Delta$ with domain $\mathrm{Dom}(A + \Delta) := \mathrm{Dom}(A)$ generates an analytic semigroup $\widetilde{\mathfrak{A}}$ on X. Hence $\widetilde{X}_1 = X_1$, and by Lemma 5.2, the norms are equivalent. Obviously, $A + \Delta$ equals the part of $A + \Delta$ in X_0.
We also have $X_\delta \subset \widetilde{X}_\gamma$ when $1 \geqslant \delta > \gamma \geqslant 0$, and this can be seen as follows. For $\delta = 1$ or $\gamma = 0$ this holds because $X_1 = \widetilde{X}_1 \subset \widetilde{X}_\gamma$ and $X_\delta \subset X = \widetilde{X}_0$. If $1 > \delta > \gamma > 0$, then this follows from Propositions 2.2.15 and 1.2.3 of [18] ($X_\delta \subset (X, X_1)_{\delta,\infty} \subset (X, X_1)_{\gamma,1} \subset \widetilde{X}_\gamma$).

$\underline{2}^\circ$ In this step we show that $(\omega I - A - \Delta)^{-\gamma} \in \mathcal{B}(X, X_\gamma)$ for $\gamma \in [0, 1]$.
The case $\gamma = 0$ is trivial and the case $\gamma = 1$ follows from 1°, and so we assume that $\gamma \in (0, 1)$. Fix some $\omega > \max\{\omega_A, \omega_{A+\Delta}\}$. Then

$$(sI - A)^{-1}\big(I + \Delta(sI - A - \Delta)^{-1}\big)$$
$$= (sI - A - \Delta)^{-1}, \quad s \in \rho(A) \cap \rho(A + \Delta), \tag{10}$$

and so from (6.4) of [21] (or Lemma 3.9.9 of Staffans [23]), it follows that

$$(\omega I - A - \Delta)^{-\gamma} - (\omega I - A)^{-\gamma}$$
$$= \frac{\sin(\pi\gamma)}{\pi} \int_0^\infty \underbrace{s^{-\gamma}\big((\omega + s)I - A\big)^{-1}\Delta\big((\omega + s)I - A - \Delta\big)^{-1}}_{F(s):=} \, ds. \tag{11}$$

But for $s \geqslant 0$ we have $\|\Delta((\omega + s)I - A - \Delta)^{-1}\|_{\mathcal{B}(X, X_\beta)} \leqslant M_\omega$ (apply Proposition 5.3 to $A + \Delta$ and recall that $\widetilde{X}_1 = X_1$ and $\Delta \in \mathcal{B}(X_1, X_\beta)$), and so

$$\big\|\big((\omega + s)I - A\big)^{-1}\Delta\big((\omega + s)I - A - \Delta\big)^{-1}\big\|_{\mathcal{B}(X, X_{\beta+1})} \leqslant M'_\omega.$$

Thus, the $\int_0^1 F(s)\, ds$ part of the integral in (11) converges in $\mathcal{B}(X, X_{\beta+1})$, hence in $\mathcal{B}(X, X_\gamma)$. Therefore, we only need to show that $\int_1^\infty \|(\omega I - A)^\gamma F(s)\|_{\mathcal{B}(X)}\, ds < \infty$ in order to establish that $\int_1^\infty F(s)\, ds \in \mathcal{B}(X, X_\gamma)$ and hence $(\omega I - A - \Delta)^{-\gamma} \in \mathcal{B}(X, X_\gamma)$. But this follows by using the estimate (5), choosing any $\omega_0 \in (\omega_A, \omega)$:

$$\int_1^\infty s^{-\gamma} \frac{M(1 + |s + \omega - \omega_0|^{\gamma-\beta})}{|s + \omega - \omega_0|}\, ds < \infty.$$

Here we have used the fact that

$$(\omega I - A)^\gamma\big((\omega + s)I - A\big)^{-1}(\omega I - A)^{-\beta} = (\omega I - A)^{\gamma-\beta}\big((\omega + s)I - A\big)^{-1},$$

by Lemma 3.9.7(ii) of Staffans [23].

$\underline{3}^\circ$ In this step we prove (S4) and (S5), i.e., we show that $X_\gamma = \widetilde{X}_\gamma$ for $\gamma \in [0, 1]$.
By 2° we already know that $\widetilde{X}_\gamma = (\omega I - A - \Delta)^{-\gamma}\widetilde{X} = (\omega I - A - \Delta)^{-\gamma}X \subset X_\gamma$.
By 1°, $\widetilde{A} := (A + \Delta)|_{X_1}$ generates an analytic semigroup on X. From 1° it also follows that $\widetilde{X}_1 = X_1$ and $X_\beta \subset \widetilde{X}_{\beta/2}$, and since all the embeddings here are continuous by Lemma 5.2,

we conclude that $-\Delta \in \mathcal{B}(\widetilde{X}_1, \widetilde{X}_{\beta/2})$. Now apply 2° to $\widetilde{A}$ and $-\Delta$ to observe that $(\omega I - A)^{-\gamma} \in \mathcal{B}(\widetilde{X}, \widetilde{X}_\gamma)$, and so $X_\gamma = (\omega I - A)^{-\gamma} X \subset \widetilde{X}_\gamma$.

By Lemma 5.2, the topologies also coincide.

4° By 3° and (R3), (S2) holds.

5° By (S2) and Lemma 5.1.5.1, the generator $R_\gamma = (A+\Delta)|_{\widetilde{X}_{\gamma+1}}$ of $\widetilde{\mathfrak{A}}|_{X_\gamma}$ equals the part of $(A+\Delta)|_{X_1}$ in X_γ. If $\gamma \leqslant \beta$, then $R_\gamma = (A+\Delta)|_{X_{\gamma+1}}$: indeed, then we have

$$\begin{aligned}\operatorname{Dom}(R_\gamma) &= \bigl\{x \in X_1 \cap X_\gamma \mid (A+\Delta)x \in X_\gamma\bigr\} = \{x \in X_1 \mid Ax \in X_\gamma\} \\ &= \bigl\{x \in X_1 \mid (\lambda I - A)x \in X_\gamma\bigr\} \quad \bigl(\lambda \in \rho(A)\bigr) \\ &= X_1 \cap X_{\gamma+1} = X_{\gamma+1}. \end{aligned} \tag{12}$$

This completes the proof. □

Proof of Proposition 4.5. We only need to prove (S1), (S2) and (S4), because then (S3) then follows from (S2) (and (S1)) as in the proof of Lemma 5.4, and (S4) implies (S5), by the property (R4).

We have divided the proof into several steps below.

1° In this step we show that (S1), (S2) and (S4) hold under the restriction that $\gamma \in [\alpha-1, \alpha]$. We shall apply Lemma 5.4 to the space $X_{\alpha-1}$. Indeed, set $Z := X_{\alpha-1}$, $\beta' := \beta - (\alpha-1)$. Naturally, we define the interpolation spaces Z_δ ($\delta \in \mathbb{R}$), with respect to $\mathfrak{A} = \mathfrak{A}|_Z$. By (R4), we have $Z_t = X_{\alpha-1+t}$ ($t \in \mathbb{R}$) and hence $\Delta \in \mathcal{B}(Z_1, Z_{\beta'})$.
Therefore, Lemma 5.4 applied to Z implies that (S1) holds and that the spaces Z_t ($t \in [0,1]$) (that is, X_δ ($\delta \in [\alpha-1,\alpha]$)) are invariant under $\widetilde{\mathfrak{A}}|_Z = \widetilde{\mathfrak{A}}$ etc., hence (S2) and (S4) hold.
(Note that, since (S1) is independent of γ, in the other steps below we only need to establish (S2) and (S4).)

2° Let $k \in \{0,1,2,\ldots\}$ and $\alpha + k < \beta$ (skip this step if no such k exists). In this step we assume that (S1), (S2) and (S4) hold under the restriction that $\gamma \in [\alpha-1, \alpha+k]$ and show that then they hold under the restriction that $\gamma \in [\alpha+k, \alpha+k+1]$.
By the assumption (mainly (S4)) and (R2), $\omega_{R_{\alpha+k}} = \omega_{R_{\alpha-1}} = \omega_{A+\Delta}$, where R_δ stands for the generator of $\widetilde{\mathfrak{A}}|_{X_{\alpha+k}}$.
Set $Z := X_{\alpha+k}$, $\beta' := \beta - (\alpha+k)$ to have $Z_t = X_{\alpha+k+t}$ ($t \in \mathbb{R}$) and $\Delta \in \mathcal{B}(Z_{-k}, Z_{\beta'})$, hence $\Delta \in \mathcal{B}(Z_1, Z_{\beta'})$. Now Lemma 5.4 applied to Z implies that (S2) holds (because $\widetilde{\mathfrak{A}}|_{X_\gamma} = (\widetilde{\mathfrak{A}}|_{X_{\alpha+k}})|_{X_\gamma}$) and that $(\omega - R_{\alpha+k})^{-(\gamma-(\alpha+k))} X_{\alpha+k} = (\omega - R_{\alpha+k})^{-(\gamma-(\alpha+k))} Z = Z_{\gamma-(\alpha+k)} = X_\gamma$. Combine this with (R4) (with $X_{\alpha-1}$ in place of X and $\widetilde{\mathfrak{A}}$ in place of $\mathfrak{A}$) and with the assumption that (S4) holds with $\alpha+k$ in place of γ, to conclude that (S4) holds (for the current γ).

3° If (S1), (S2) and (S4) hold when $\gamma \in [\alpha-1, \beta]$, then they hold also when $\gamma \in [\beta, \beta+1)$. Indeed, this can be shown as in 2° (with r in place of $\alpha+k$) by setting $Z := X_r$, $\beta' := \beta - r$, where $r := \max\{\gamma-1, \alpha-1\}$, because $r \in [\alpha-1, \beta)$, $\Delta \in \mathcal{B}(Z_1, Z_{\beta'})$ and $\beta' > 0$.

4° By induction, from the above we conclude that (S1)–(S5) hold for $\gamma, \delta \in [\alpha-1, \beta+1)$ and that it only remains to establish (S2) and (S4) in the case that $\gamma = \beta+1$. For this purpose, we first need an auxiliary result, 4.1°.

4.1° In this auxiliary step we only assume that $\Delta \in \mathcal{B}(X_\alpha, X)$, $\alpha < 1$, $\beta = 0$, $\gamma = 1$ and we only prove certain results needed in 4.2° to complete the proof.

By Proposition 2.2.15 and 2.4.1(i) of Lunardi [18], $(A + \Delta)|_{X_1}$ is sectorial and generates an analytic semigroup on X (if $\alpha < 0$, use the fact that then $\Delta \in \mathcal{B}(X_0, X)$). By Lemma 5.1.1, that semigroup equals the restriction $\widetilde{\mathfrak{A}}|_X$ of $\widetilde{\mathfrak{A}} = \widetilde{\mathfrak{A}}|_{X_{\alpha-1}}$ to X. (By 1°, $\widetilde{\mathfrak{A}}$ exists.)
Therefore, X_1 is invariant under $\widetilde{\mathfrak{A}}|_X$, hence also under $\widetilde{\mathfrak{A}}$, and $\widetilde{\mathfrak{A}}|_{X_1}$ is analytic. Moreover, $(\omega - S)^{-1}X = X_1$ ($\omega > \omega_S$), where $R := (A + \Delta)|_{X_1}$.

4.2° Set $Z := X_\beta$, so that $Z_t = X_{\beta+t}$ ($t \in \mathbb{R}$) and apply 4.1° with $\Delta \in \mathcal{B}(Z_{\alpha-\beta}, Z)$ (note that $Z_{\alpha-\beta-1} = X_{\alpha-1}$). It follows that (S2) holds and that $(\omega - (A + \Delta)|_{X_{\beta+1}})^{-1} X_\beta = X_{\beta+1}$ ($\omega > \omega_S$), where $S := (A + \Delta)|_{X_{\beta+1}}$. Combine this with (R4) and the already proved case of (S4) to obtain (S4) for $\gamma = \beta + 1$, as in 2° (for $\omega > \max\{\omega_{A+\Delta}, \omega_S\}$, but then $\omega_{A+\Delta} = \omega_S$, by (R3)).

This completes the proof. □

6. Examples

Example 1 *(Classical parabolic equations on $X = L_2(\Omega)$).* In Lasiecka and Triggiani [17], the following example was considered. Given a smooth bounded domain $\Omega \in \mathbb{R}^N$, let A be the realization in $L_2(\Omega)$ of an elliptic operator of order $2d$, subject to appropriate boundary conditions (see [17, Chapter 3, Appendix 3A]). A generates a strongly continuous analytic semigroup on $L_2(\Omega)$, and also $\mathrm{Dom}(A^*) \subset H^{2m}(\Omega)$ (Sobolev space). The following result was obtained in [17] (see pp. 128–129): Q is Hilbert–Schmidt on X if $4d > N$.

On the other hand, when $U = \mathbb{C}^m$, applying Theorem 4.6 we obtain that the operator Q is nuclear, and hence it is also Hilbert–Schmidt (for all N and d).

Using the result mentioned above from [17], it can also be shown that if $(\omega I - A)^{-\alpha}$ is Hilbert–Schmidt on X for some $\alpha \in (0, 1)$ the gain operator B^*Q is Hilbert–Schmidt from X to $\mathbb{C}^m$ if $1 + \alpha_B \geqslant \alpha > \frac{N}{4d}$.

On the other hand, in the case when $U = \mathbb{C}^m$, the result in Theorem 4.6 says that the gain operator B^*Q is Hilbert–Schmidt from X_γ to $\mathbb{C}^m$ for all $\gamma > -\frac{1}{2}$ provided that $\alpha_B > -\frac{1}{2}$.

Example 2 *(Model of a damped flexible beam).* In Bontsema [3], several models of flexible beams were considered. They were based on one-dimensional Euler–Bernoulli beam equations with free ends and various types of damping structures. Here we consider the so-called viscous damping model, with an external free force and an external moment $M(t)$ acting at the center of the beam. (See Fig. 1.) The equations describing the displacement w are given by:

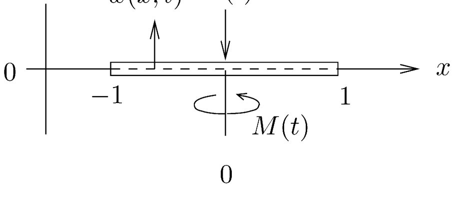

Fig. 1. Flexible beam.

$$\rho a\frac{\partial^2 w}{\partial t^2}(x,t)+E^*I\frac{\partial^5 w}{\partial t\partial x^4}(x,t)+EI\frac{\partial^4 w}{\partial x^4}(x,t)=0,\quad x\in[-1,1]\setminus\{0\},$$
$$EI\frac{\partial^3 w}{\partial x^3}(0+,t)-EI\frac{\partial^3 w}{\partial x^3}(0-,t)+E^*I\frac{\partial^4 w}{\partial t\partial x^3}(0+,t)-E^*I\frac{\partial^4 w}{\partial t\partial x^3}(0-,t)=F(t),$$
$$-EI\frac{\partial^2 w}{\partial x^2}(0+,t)+EI\frac{\partial^2 w}{\partial x^2}(0-,t)-E^*I\frac{\partial^3 w}{\partial t\partial x^2}(0+,t)+E^*I\frac{\partial^3 w}{\partial t\partial x^2}(0-,t)=M(t),$$
$$\frac{\partial^2 w}{\partial x^2}(-1,t)=0,\qquad \frac{\partial^2 w}{\partial x^2}(1,t)=0,\qquad \frac{\partial^3 w}{\partial x^3}(-1,t)=0,\qquad \frac{\partial^3 w}{\partial x^3}(1,t)=0,$$

where

$w(x,t)$	the vertical displacement of the beam at position x and at time t,
a	the cross sectional area of the beam,
p	the mass density of the beam,
E	the Young's modulus of elasticity,
E^*	a constant reflecting the stress-strain relation in the beam,
I	the moment of inertia of the beam per cross section,
F	the external force acting at the center of the beam,
M	the external moment acting at the center of the beam.

The external force F and the moment M are taken as the two inputs: $u_1(t)=F(t)$, $u_2(t)=M(t)$, and the measurements of the displacement and the angle of rotation at the center of the beam are taken as the two outputs: $y_1(t)=w(0,t)$, $y_2(t)=\frac{\partial w}{\partial x}(0,t)$. This model can be thought of as an idealization of a very large flexible space structure with a central hub at $x=0$, where the actuators and sensors are located. Let $\alpha_1:=\frac{E^*I}{\rho a}$ and $\alpha_2:=\frac{EI}{\rho a}$, and introduce the operator A_0, defined as follows: $A_0=\frac{d^4}{dx^4}$, with

$$\operatorname{Dom}(A_0)=\left\{f\in L_2(-1,1)\;\middle|\;\begin{array}{l}\frac{df}{dx},\frac{d^2f}{dx^2},\frac{d^3f}{dx^3}\text{ are absolutely continuous,}\\ \frac{d^4f}{dx^4}\in L_2(-1,1),\text{ and}\\ \frac{d^2f}{dx^2}(-1)=0,\ \frac{d^2f}{dx^2}(1)=0,\ \frac{d^3f}{dx^3}(-1)=0,\ \frac{d^3f}{dx^3}(1)=0\end{array}\right\}.$$

As shown in [3], A_0 is densely defined, self-adjoint and positive. We introduce the Hilbert space $X=\operatorname{Dom}(A_0^{\frac12})\oplus L_2(-1,1)$, with the inner product defined as follows:

$$\left\langle\begin{bmatrix}x_1\\x_2\end{bmatrix},\begin{bmatrix}y_1\\y_2\end{bmatrix}\right\rangle_X=\langle x_1,y_1\rangle_{L_2}+\alpha_2\big\langle A_0^{\frac12}x_1,A_0^{\frac12}y_1\big\rangle_{L_2}+\langle x_2,y_2\rangle_{L_2}.$$

By introducing the state vector $x(t)=\begin{bmatrix}w(\cdot,t)\\\frac{\partial w}{\partial t}(\cdot,t)\end{bmatrix}$, the uncontrolled beam equation can be formulated as an abstract differential equation on X: $\frac{d}{dt}x(t)=Ax(t)$, where

$$A\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_2\\-\alpha_2A_0(x_1+\frac{\alpha_1}{\alpha_2}x_2)\end{bmatrix},\quad \operatorname{Dom}(A)=\left\{\begin{bmatrix}x_1\\x_2\end{bmatrix}\in X\;\middle|\;\begin{array}{l}x_2\in\operatorname{Dom}\big(A_0^{\frac12}\big),\\ x_1+\frac{\alpha_1}{\alpha_2}x_2\in\operatorname{Dom}(A_0)\end{array}\right\}.$$

A is a Riesz spectral operator and it generates a strongly continuous semigroup on X (see [3]).

Spectrum of A. The spectrum of the operator A is $\sigma(A)=\sigma_c(A)\cup\sigma_p(A)$, where

$$\sigma_c(A) = \left\{ -\frac{\alpha_2}{\alpha_1} \right\},$$

$$\sigma_p(A) = \{\lambda_m \mid m \in \mathbb{Z} \setminus \{0\}\}, \quad \lambda_{\pm 1, \pm 2} = 0,\ \lambda_{\pm n} = \frac{-\alpha_1 \mu_n^4 \pm \sqrt{\alpha_1^2 \mu_n^2 - 4\alpha_2 \mu_n^4}}{2}, n \geqslant 3,$$

where μ_ns are the real, positive solutions of $(\sinh \mu_n)(\cos \mu_n) + (-1)^n (\cosh \mu_n)(\sin \mu_n) = 0$. The point spectrum of the operator A lies on a circle with center $(-\frac{\alpha_2}{\alpha_1}, 0)$ and radius $\frac{\alpha_2}{\alpha_1}$, or it lies on the real line, with limit points $-\infty$ and $-\frac{\alpha_2}{\alpha_1}$.

Eigenvectors. The eigenvalue 0 of the operator A has algebraic multiplicity 4, with two eigenvectors and two generalized eigenvectors given respectively by

$$\varphi_1 = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \qquad \varphi_{-1} = \begin{bmatrix} v_2 \\ 0 \end{bmatrix}, \qquad \varphi_2 = \begin{bmatrix} 0 \\ v_1 \end{bmatrix}, \qquad \varphi_{-2} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix},$$

where $v_1(x) = \frac{1}{\sqrt{2}}$ and $v_2(x) = \sqrt{\frac{3}{2}} x$. The eigenvectors of A corresponding to the eigenvalues λ_n, λ_{-n} for $n \geqslant 3$ are given as follows:

$$\varphi_n = \eta_n \begin{bmatrix} v_n \\ \lambda_n v_n \end{bmatrix}, \qquad \varphi_{-n} = \eta_{-n} \begin{bmatrix} v_n \\ \lambda_{-n} v_n \end{bmatrix},$$

where

$$\begin{aligned} v_n(x) &= \frac{\cosh \mu_n + \cos \mu_n}{\sqrt{(\cosh \mu_n)^2 + (\cos \mu_n)^2}} \\ &\quad \times \left(\cos(\mu_n x) + \cos \mu_n \frac{\cosh(\mu_n x) - \cos(\mu_n x)}{\cosh \mu_n + \cos \mu_n x} \right) \quad \text{for odd } n, \\ v_n(x) &= \frac{\sinh \mu_n + \sin \mu_n}{\sqrt{(\sinh \mu_n)^2 - (\sin \mu_n)^2}} \\ &\quad \times \left(\sin(\mu_n x) + \sin \mu_n \frac{\sinh(\mu_n x) - \sin(\mu_n x)}{\sinh \mu_n + \sin \mu_n x} \right) \quad \text{for even } n, \end{aligned}$$

and $\eta_{\pm n} = \frac{1}{\sqrt{1 + \alpha_2 \mu_n^4 + |\lambda_{\pm n}|^2}}$.

The eigenvalue 0 of the operator A^* has eigenvectors $f_1 = \varphi_1$, $f_{-1} = \varphi_{-1}$, and two generalized eigenvectors $f_2 = \varphi_2$, $f_{-2} = \varphi_{-2}$. The eigenvectors f_n of A^* corresponding to the eigenvalue $\bar{\lambda}_n$ for $|n| \geqslant 3$ are given as follows:

$$f_n = \begin{bmatrix} v_n \\ -\frac{1 + \alpha_2 \mu_n^4}{\alpha_2 \mu_n^4} \bar{\lambda}_n v_n \end{bmatrix}, \qquad f_{-n} = \begin{bmatrix} v_n \\ -\frac{1 + \alpha_2 \mu_n^4}{\alpha_2 \mu_n^4} \bar{\lambda}_{-n} v_n \end{bmatrix}.$$

ψ_n, ψ_{-n} are defined by $\psi_n = \frac{1}{\langle f_n, \varphi_n \rangle_X} f_n$, $|n| \geqslant 1$. For[4] $m, n \in \mathbb{Z} \setminus \{0\}$, $\langle \psi_m, \varphi_n \rangle = \delta_{mn}$, and so $(\psi_m)_{m \in \mathbb{Z} \setminus \{0\}}$ is a biorthogonal sequence to $(\varphi_n)_{n \in \mathbb{Z} \setminus \{0\}}$.

Spectral decomposition. A is a Riesz spectral operator with the spectral decomposition

$$Ax = \langle x, \psi_2 \rangle \varphi_1 + \langle x, \psi_{-2} \rangle \varphi_{-1} + \sum_{n=3}^{\infty} \langle x, \psi_n \rangle \varphi_n + \sum_{n=3}^{\infty} \langle x, \psi_{-n} \rangle \varphi_{-n},$$

[4] If $m, n \in \mathbb{Z} \setminus$, then $\delta_{mn} = 0$ if $m \neq n$, and 1 if $m = n$.

for x in $\mathrm{Dom}(A) = \{x \in X \mid \sum_{n\in\mathbb{Z}\setminus\{0\}} |\lambda_n|^2 |\langle x, \psi_n\rangle|^2 < +\infty\}$. The resolvent of A is given by

$$\begin{aligned}(sI - A)^{-1}x = {}& \frac{1}{s}\langle x, \psi_1\rangle\varphi_1 + \frac{1}{s^2}\langle x, \psi_2\rangle\varphi_1 + \frac{1}{s}\langle x, \psi_2\rangle\varphi_2 \\ &+ \frac{1}{s}\langle x, \psi_{-1}\rangle\varphi_{-1} + \frac{1}{s^2}\langle x, \psi_{-2}\rangle\varphi_{-1} + \frac{1}{s}\langle x, \psi_{-2}\rangle\varphi_{-2} \\ &+ \sum_{n=3}^{\infty} \frac{1}{s-\lambda_n}\langle x, \psi_n\rangle\varphi_n + \sum_{n=3}^{\infty} \frac{1}{s-\lambda_{-n}}\langle x, \psi_{-n}\rangle\varphi_{-n},\end{aligned} \tag{13}$$

for all $s \in \rho(A)$. The open right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ is contained in $\rho(A)$ and $\omega_A = 0$. From (13), we obtain that there exists $M < +\infty$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, $\|(sI - A)^{-1}\| \leqslant M \sup_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{|s-\lambda_n|}$. If $\Theta \in [0, \frac{\pi}{2})$ is such that $\sin\Theta = \frac{\alpha_2/\alpha_1}{1+\alpha_2/\alpha_1}$, then we have $\|(sI - A)^{-1}\| \leqslant \frac{\frac{M}{\cos\Theta}}{|s-1|}$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. So A generates an analytic semigroup on X. The semigroup is given as follows:

$$\begin{aligned}\mathfrak{A}(t)x = {}& \langle x, \psi_1\rangle\varphi_1 + t\langle x, \psi_2\rangle\varphi_1 + \langle x, \psi_2\rangle\varphi_2 \\ &+ \langle x, \psi_{-1}\rangle\varphi_{-1} + t\langle x, \psi_{-2}\rangle\varphi_{-1} + \langle x, \psi_{-2}\rangle\varphi_{-2} \\ &+ \sum_{n=3}^{\infty} e^{\lambda_n t}\langle x, \psi_n\rangle\varphi_n + \sum_{n=3}^{\infty} e^{\lambda_{-n} t}\langle x, \psi_n\rangle\varphi_n.\end{aligned}$$

In Proposition 4.4 (the result quoted from [17]), the following condition is given for Q to be a Hilbert–Schmidt operator on X: $(\omega I - A)^{-\alpha}$ is Hilbert–Schmidt on X, $\alpha \in (0, 1)$, $\omega > \omega_A$.

We show that for our present example, this condition is never met. By using properties of analytic semigroups (for instance, Lemma 9.4.2.1 of Mikkola [19]), it follows that for $\omega > \omega_A$ and $\alpha \in (0, 1)$,

$$\begin{aligned}(\omega I - A)^{-\alpha}x = {}& \frac{1}{\omega^\alpha}\langle x, \psi_1\rangle\varphi_1 + \frac{\alpha}{\omega^{\alpha+1}}\langle x, \psi_2\rangle\varphi_1 + \frac{1}{\omega^\alpha}\langle x, \psi_2\rangle\varphi_2 \\ &+ \frac{1}{\omega^\alpha}\langle x, \psi_{-1}\rangle\varphi_{-1} + \frac{1}{\omega^{\alpha+1}}\langle x, \psi_{-2}\rangle\varphi_{-1} + \frac{1}{\omega^\alpha}\langle x, \psi_{-2}\rangle\varphi_{-2} \\ &+ \sum_{n=3}^{\infty} \frac{1}{(\omega-\lambda_n)^\alpha}\langle x, \psi_n\rangle\varphi_n + \sum_{n=3}^{\infty} \frac{1}{(\omega-\lambda_{-n})^\alpha}\langle x, \psi_{-n}\rangle\varphi_{-n}.\end{aligned}$$

If $(\omega I - A)^{-\alpha}$ is Hilbert–Schmidt, then since $(\varphi_n)_{n\geqslant 3}$ is an orthonormal sequence, it follows that $\sum_{n=3}^{\infty} \|(\omega I - A)^{-\alpha}\varphi_n\|^2 < +\infty$, and in particular,

$$\lim_{n\to\infty} \left\|(\omega I - A)^{-\alpha}\varphi_n\right\| = 0, \quad \text{that is,} \quad \lim_{n\to\infty} \left|(\omega - \lambda_n)^\alpha\right| = +\infty. \tag{14}$$

But for positive n, $\mu_n = O(n)$, and so $|\lambda_n| = O(1)$. Consequently, $|(\omega - \lambda_n)^\alpha| = O(1)$, and so (14) does not hold. So $(\omega I - A)^{-\alpha}$ is not Hilbert–Schmidt.

Finally, we show that our main Theorem 4.6 does apply in this case. First we introduce the input and output operators B and C below.

***B* and *C*.** The output operator $C \in \mathcal{B}(X, \mathbb{C}^2)$ is defined as follows: $C\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1(0) \\ x_1'(0) \end{smallmatrix}\right]$. Formally, we can think of the input operator as the following distribution operator:

$$B\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{\rho a}\begin{bmatrix} 0 & 0 \\ \delta_0 & -\delta_0' \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where δ_0 denotes the Dirac distribution with support in 0, and δ_0' is its derivative. The dual of B is given by

$$B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\rho a} \begin{bmatrix} x_1(0) \\ x_1'(0) \end{bmatrix}.$$

To see that $B \in \mathcal{B}(\mathbb{C}^2, X_{\alpha_B})$ with appropriate α_B, we use $Bu = \sum_{n\in\mathbb{Z}\setminus\{0\}} \langle u, b_n\rangle_{\mathbb{C}^2} \varphi_n$, with $b_n = B^*\psi_n$, where

$$b_{-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad b_{-2} = \frac{1}{\rho a} \begin{bmatrix} v_2(0) \\ v_2'(0) \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad b_2 = \frac{1}{\rho a} \begin{bmatrix} v_1(0) \\ v_1'(0) \end{bmatrix},$$

$$b_n = \frac{-\bar{\lambda}_n}{\rho a \langle f_n, \varphi_n\rangle_X} \left(\frac{1 + \alpha_2 \mu_n^4}{\alpha_2 \mu_n^4} \right) \begin{bmatrix} v_n(0) \\ v_n'(0) \end{bmatrix}, \quad n \geqslant 3,$$

$$b_{-n} = \frac{-\bar{\lambda}_{-n}}{\rho a \langle f_{-n}, \varphi_{-n}\rangle_X} \left(\frac{1 + \alpha_2 \mu_n^4}{\alpha_2 \mu_n^4} \right) \begin{bmatrix} v_n(0) \\ v_n'(0) \end{bmatrix}, \quad n \geqslant 3.$$

We have the following estimates for large positive values of n: $|\lambda_n| = O(1)$ and $|\lambda_{-n}| = O(n^4)$, and this yields the following estimates for b_n:

$$\|b_n\|_{\mathbb{C}^2} = \begin{cases} O(\frac{1}{n^2}), & n > 0,\ n \text{ odd}, \\ O(\frac{1}{n}), & n > 0,\ n \text{ even}, \\ O(1), & n < 0,\ n \text{ odd}, \\ O(n), & n < 0,\ n \text{ even}. \end{cases}$$

Using the above, it can be seen that $(I - A)^{\alpha_B} B \in \mathcal{B}(\mathbb{C}^2, X)$ for $\alpha_B < -\frac{3}{8}$. Hence assumptions (A1)–(A3) from Theorem 4.6 hold. In Bontsema [3], it was shown that the pair (A, C) is exponentially detectable if $\frac{\alpha_2}{\alpha_1} > 0$, and so it follows that (A4) is also satisfied.

Hence under the finite cost condition, Theorem 4.6 applies, and we obtain that $Q \in \mathcal{S}_1(X_\gamma, (X_\gamma)^*)$ for all $\gamma > -\frac{1}{2}$. In particular, with $\gamma = 0$, we obtain that $Q \in \mathcal{B}(X)$ is nuclear. Furthermore, B^*Q is a Hilbert–Schmidt operator from X_γ to $\mathbb{C}^m$ for all $\gamma > -\frac{1}{2}$. This provides the theoretical justification for the LQG control design in [20].

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