Extension to an invertible matrix in convolution algebras of measures supported in $[0, +\infty)$

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Abstract. Let $\mathcal{M}_+$ denote the Banach algebra of all complex Borel measures with support contained in $[0, +\infty)$, with the usual addition and scalar multiplication, and with convolution $*$, and the norm being the total variation of μ. We show that the maximal ideal space $X(\mathcal{M}_+)$ of $\mathcal{M}_+$, equipped with the Gelfand topology, is contractible as a topological space. In particular, it follows that every left invertible matrix with entries from $\mathcal{M}_+$ can be completed to an invertible matrix, that is, the following statements are equivalent for $f \in (\mathcal{M}_+)^{n\times k}$, $k < n$:

1. There exists a matrix $g \in \mathcal{M}_+^{k\times n}$ such that $g * f = I_k$.
2. There exist matrices $F, G \in \mathcal{M}_+^{n\times n}$ such that $G * F = I_n$ and $F_{ij} = f_{ij}$, $1 \le i \le n$, $1 \le j \le k$.

We also show a similar result for all subalgebras of $\mathcal{M}_+$ satisfying a mild condition.

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1. Introduction

The aim of this paper is to show that the maximal ideal space $X(\mathcal{M}_+)$ of the Banach algebra $\mathcal{M}_+$ of all complex Borel measures with support in $[0, +\infty)$ (defined below), is contractible. We also apply this result to the problem of completing a left invertible matrix with entries in $\mathcal{M}_+$ to an invertible matrix over $\mathcal{M}_+$.

Definition 1.1. Let $\mathcal{M}_+$ denote the set of all complex Borel measures with support contained in $[0, +\infty)$. Then $\mathcal{M}_+$ is a complex vector space with addition and scalar multiplication defined as usual, and it becomes a complex algebra if we take convolution of measures as the operation of multiplication. With the norm of μ

taken as the total variation of μ, $\mathcal{M}_+$ is a Banach algebra. Recall that the *total variation* $\|\mu\|$ *of* μ is defined by

$$\|\mu\| = \sup \sum_{n=1}^{\infty} |\mu(E_n)|,$$

the supremum being taken over all *partitions of* $[0, +\infty)$, that is over all countable collections $(E_n)_{n\in\mathbb{N}}$ of Borel subsets of $[0, +\infty)$ such that $E_n \bigcap E_m = \emptyset$ whenever $m \neq n$ and $[0, +\infty) = \bigcup_{n\in\mathbb{N}} E_n$. The identity with respect to convolution in $\mathcal{M}_+$ is the *Dirac measure* δ, given by

$$\delta(E) = \begin{cases} 1 & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E. \end{cases}$$

The above Banach algebra is classical, and we refer the reader to the book [1, §4, p.141-150] for a detailed exposition.

Notation 1.2. Let $X(\mathcal{M}_+)$ denote the maximal ideal space of the Banach algebra $\mathcal{M}_+$, that is the set of all nonzero complex homomorphisms from $\mathcal{M}_+$ to $\mathbb{C}$. We equip $X(\mathcal{M}_+)$ with the Gelfand topology, that is, the weak-$*$ topology induced from the dual space $\mathcal{L}(\mathcal{M}_+;\mathbb{C})$ of the Banach space $\mathcal{M}_+$.

We will show that $X(\mathcal{M}_+)$ is contractible. We recall the notion of contractibility below:

Definition 1.3. A topological space X is said to be *contractible* if there exists a continuous map $H : X \times [0, 1] \to X$ and an $x_0 \in X$ such that for all $x \in X$, $H(x, 0) = x$ and $H(x, 1) = x_0$.

Our main result is the following:

Theorem 1.4. $X(\mathcal{M}_+)$ *is contractible.*

In particular, by a result proved by V. Ya. Lin, the above implies that the ring $\mathcal{M}_+$ is Hermite. Before stating this result, we recall the definition of a Hermite ring:

Definition 1.5. Let R be a ring with an identity element denoted by 1. Let us denote by $I_k \in R^{k\times k}$ the diagonal matrix of size $k \times k$ with all the diagonal entries equal to the identity element 1. A matrix $f \in R^{n\times k}$ is called *left invertible* if there exists a matrix $g \in R^{k\times n}$ such that $gf = I_k$.

The ring R is called a *Hermite ring* if for all $k, n \in \mathbb{N}$ with $k < n$ and all left invertible matrices $f \in R^{n\times k}$, there exist matrices $F, G \in R^{n\times n}$ such that $GF = I_n$ and $F_{ij} = f_{ij}$ for all $1 \le i \le n$ and $1 \le j \le k$.

We now recall Lin's result; [2, Theorem 3, p. 127]:

Proposition 1.6. *Let R be a commutative Banach algebra with identity. If the maximal ideal space $X(R)$ (equipped with the Gelfand topology) of the Banach algebra R is contractible, then R is a Hermite ring.*

Using the above result, our main result given in Theorem 1.4 then implies the following.

Corollary 1.7. *$\mathcal{M}_+$ is a Hermite ring, that is, the following statements are equivalent for $f \in (\mathcal{M}_+)^{n\times k}$, $k < n$:*

1. *There exists a matrix $g \in \mathcal{M}_+^{k\times n}$ such that $g * f = I_k$.*
2. *There exist matrices $F, G \in \mathcal{M}_+^{n\times n}$ such that $G * F = I_n$ and $F_{ij} = f_{ij}$, $1 \le i \le n$, $1 \le j \le k$.*

(In the above, $*$ denotes convolution, and F_{ij}, f_{ij} denote the entries in the ith row and jth column, of the matrices F and f, respectively.)

1.1. Relevance of the Hermiteness of $\mathcal{M}_+$ in control theory

The motivation for proving that $\mathcal{M}_+$ is a Hermite ring arises from control theory, where it plays an important role in the problem of stabilization of linear systems. Let $\widehat{\mathcal{M}_+}$ denote the integral domain of Laplace transforms of elements of $\mathcal{M}_+$. Then $\widehat{\mathcal{M}_+}$ is a class of "stable" transfer functions, in the sense that if the plant transfer function $g = \widehat{\mu}$ belongs to $\widehat{\mathcal{M}_+}$, then nice inputs are mapped to nice outputs in a continuous manner: if the initial state of the system is 0, and the input $u \in L^p(0, +\infty)$, where $1 \le p \le +\infty$, then the corresponding output[1] $y = \mu * u$ is in $L^p(0, +\infty)$ (here μ is the inverse Laplace transform of g). Moreover,

$$\sup_{0 \neq u \in L^p(0,+\infty)} \frac{\|y\|_p}{\|u\|_p} \le \|g\|.$$

In fact one has equality above if $p = 1$ or $p = +\infty$.

The result that $\mathcal{M}_+$ is Hermite implies that if a system with a transfer function G in the field of fractions of $\widehat{\mathcal{M}_+}$ has a right (or left) coprime factorization, then G has a doubly coprime factorization, and the standard Youla parameterization yields all stabilizing controllers for G. For further details on the relevance of the Hermite property in control theory, see [5, Theorem 66, p.347].

Unfortunately, a nice analytic test for checking right invertibility is not available; see [1, Theorem 4.18.5, p.149]. This has been the reason that in control theory, one uses the subalgebra $\mathcal{A}$ of $\mathcal{M}_+$ consisting of those measures from $\mathcal{M}_+$ for which the non-atomic singular part is 0, for which an analytic condition for left invertibility is indeed available [1, Theorem 4.18.6]. The Hermite property of $\mathcal{A}$, which was mentioned as an open problem in Vidyasagar's book [5, p. 360], was proved in [4]. The proof of the Hermite property of $\mathcal{M}_+$ we give here is inspired from the calculation done in [4].

In Section 3, we will give the proof of Theorem 1.4, but before doing that, in Section 2, we first prove a few technical results which will be used in the sequel.

[1] equivalently $\widehat{y}(s) = g(s)\widehat{u}(s)$, for all s in some right half plane in $\mathbb{C}$

2. Preliminaries

In this section, we prove a few auxiliary facts, which will be needed in order to prove our main result.

Definition 2.1. If $\mu \in \mathcal{M}_+$ and $\theta \in [0, 1)$, then we define the complex Borel measure μ_θ as follows:

$$\mu_\theta(E) := \int_E (1-\theta)^t d\mu(t),$$

where E is a Borel subset of $[0, +\infty)$. If $\theta = 1$, then we define

$$\mu_\theta = \mu_1 := \mu(\{0\})\delta.$$

It can be seen that $\mu_\theta \in \mathcal{M}_+$ and that $\|\mu_\theta\| \leq \|\mu\|$. Also $\delta_\theta = \delta$ for all $\theta \in [0, 1]$.

Proposition 2.2. *If $\mu, \nu \in \mathcal{M}_+$, then for all $\theta \in [0, 1]$,*

$$(\mu * \nu)_\theta = \mu_\theta * \nu_\theta.$$

Proof. If E is a Borel subset of $[0, +\infty)$, then

$$(\mu * \nu)_\theta(E) = \int_E (1-\theta)^t d(\mu * \nu)(t) = \iint_{\substack{\sigma+\tau \in E \\ \sigma, \tau \in [0,+\infty)}} (1-\theta)^{\sigma+\tau} d\mu(\sigma) d\nu(\tau).$$

On the other hand,

$$\begin{aligned} (\mu_\theta * \nu_\theta)(E) &= \int_{\tau \in [0,+\infty)} \mu_\theta(E-\tau) d\nu_\theta(\tau) \\ &= \int_{\tau \in [0,+\infty)} \left(\int_{\substack{\sigma \in E-\tau \\ \sigma \in [0,+\infty)}} (1-\theta)^\sigma d\mu(\sigma) \right) d\nu_\theta(\tau) \\ &= \iint_{\substack{\sigma+\tau \in E \\ \sigma, \tau \in [0,+\infty)}} (1-\theta)^{\sigma+\tau} d\mu(\sigma) d\nu(\tau). \end{aligned}$$

This completes the proof. □

The following result says that for a fixed μ, the map $\theta \mapsto \mu_\theta : [0, 1] \to \mathcal{M}_+$ is continuous.

Proposition 2.3. *If $\mu \in \mathcal{M}_+$ and $\theta_0 \in [0, 1]$, then*

$$\lim_{\theta \to \theta_0} \mu_\theta = \mu_{\theta_0}$$

in $\mathcal{M}_+$.

Proof. Consider first the case when $\theta_0 \in [0,1)$. Given an $\epsilon > 0$, first choose an $R > 0$ large enough so that $|\mu|((R,+\infty)) < \epsilon$. Let $\theta \in [0,1)$. There exists a Borel measurable function w such that $d(\mu_\theta - \mu_{\theta_0})(t) = e^{-iw(t)} d|\mu_\theta - \mu_{\theta_0}|(t)$. Thus

$$\begin{aligned}
\|\mu_\theta - \mu_{\theta_0}\| &= |\mu_\theta - \mu_{\theta_0}|([0,+\infty)) = \int_{[0,+\infty)} e^{iw(t)} d(\mu_\theta - \mu_{\theta_0})(t) \\
&= \left| \int_{[0,+\infty)} e^{iw(t)} d(\mu_\theta - \mu_{\theta_0})(t) \right| \\
&= \left| \int_{[0,+\infty)} e^{iw(t)} \Big((1-\theta)^t - (1-\theta_0)^t \Big) d\mu(t) \right|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mu_\theta - \mu_{\theta_0}\| &\leq \left| \int_{[0,R]} e^{iw(t)} \Big((1-\theta)^t - (1-\theta_0)^t \Big) d\mu(t) \right| \\
&\quad + \left| \int_{(R,+\infty)} e^{iw(t)} \Big((1-\theta)^t - (1-\theta_0)^t \Big) d\mu(t) \right| \\
&\leq \max_{t\in[0,R]} \left| (1-\theta)^t - (1-\theta_0)^t \right| |\mu|([0,R]) + 2|\mu|((R,+\infty)) \\
&\leq \max_{t\in[0,R]} \left| (1-\theta)^t - (1-\theta_0)^t \right| |\mu|([0,+\infty)) + 2\epsilon.
\end{aligned}$$

But by the mean value theorem applied to the function $\theta \mapsto (1-\theta)^t$,

$$(1-\theta)^t - (1-\theta_0)^t = (\theta - \theta_0) t (1-c)^{t-1} = (\theta-\theta_0) t \frac{(1-c)^t}{1-c},$$

for some c (depending on t, θ and θ_0) in between θ and θ_0. Since c lies between θ and θ_0, and since both θ and θ_0 lie in $[0,1)$, and $t \in [0,R]$, it follows that $(1-c)^t \leq 1$ and

$$\frac{1}{1-c} \leq \max\left\{ \frac{1}{1-\theta}, \frac{1}{1-\theta_0} \right\}.$$

Thus using the above and the fact that $|t| \leq R$,

$$\begin{aligned}
\max_{t\in[0,R]} \left| (1-\theta)^t - (1-\theta_0)^t \right| &= \max_{t\in[0,R]} |\theta-\theta_0||t||(1-c)^t| \frac{1}{|1-c|} \\
&\leq |\theta - \theta_0| \cdot R \cdot 1 \cdot \max\left\{ \frac{1}{1-\theta}, \frac{1}{1-\theta_0} \right\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \limsup_{\theta\to\theta_0}\left(\max_{t\in[0,R]}\left|(1-\theta)^t-(1-\theta_0)^t\right||\mu|([0,+\infty))\right) \\
\leq\ & \limsup_{\theta\to\theta_0}\left(|\theta-\theta_0|\cdot R\cdot\max\left\{\frac{1}{1-\theta},\frac{1}{1-\theta_0}\right\}\cdot|\mu|([0,+\infty))\right) \\
=\ & 0\cdot R\cdot\frac{1}{1-\theta_0}|\mu|([0,+\infty)) \\
=\ & 0.
\end{aligned}$$

Consequently,

$$\limsup_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|\leq 2\epsilon.$$

But the choice of $\epsilon>0$ was arbitrary, and so

$$\limsup_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|=0.$$

Since $\|\mu_\theta-\mu_{\theta_0}\|\geq 0$, we can conclude that

$$\lim_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|=0.$$

Now let us consider the case when $\theta_0=1$. Let us assume for the moment that $\mu(\{0\})=0$. We will show that

$$\lim_{\theta\to 1}\mu_\theta=0$$

in $\mathcal{M}_+$. Given an $\epsilon>0$, first choose a $r>0$ small enough so that $|\mu|([0,r]))<\epsilon$. (This is possible, since $\mu(\{0\})=0$.) There exists a Borel measurable function w such that $d\mu_\theta(t)=e^{-iw(t)}d|\mu_\theta|(t)$. Thus

$$\begin{aligned}
\|\mu_\theta\| &= |\mu_\theta|([0,+\infty))=\int_{[0,+\infty)}e^{iw(t)}d\mu_\theta(t) \\
&= \int_{[0,+\infty)}e^{iw(t)}(1-\theta)^t d\mu(t)=\left|\int_{[0,+\infty)}e^{iw(t)}(1-\theta)^t d\mu(t)\right| \\
&\leq \left|\int_{[0,r]}e^{iw(t)}(1-\theta)^t d\mu(t)\right|+\left|\int_{(r,+\infty)}e^{iw(t)}(1-\theta)^t d\mu(t)\right| \\
&\leq |\mu|([0,r])+(1-\theta)^r|\mu|((r,+\infty)) \\
&\leq \epsilon+(1-\theta)^r|\mu|([0,+\infty)).
\end{aligned}$$

Consequently,

$$\limsup_{\theta\to 1}\|\mu_\theta-\mu_{\theta_0}\|\leq\epsilon.$$

But the choice of $\epsilon>0$ was arbitrary, and so

$$\limsup_{\theta\to 1}\|\mu_\theta\|=0.$$

Since $\|\mu_\theta\| \geq 0$, we can conclude that

$$\lim_{\theta \to 1} \|\mu_\theta\| = 0.$$

Finally, if $\mu(\{0\}) \neq 0$, then define

$$\nu := \mu - \mu(\{0\})\delta \in \mathcal{M}_+.$$

It is clear that $\nu(\{0\}) = 0$ and $\nu_\theta = \mu_\theta - \mu(\{0\})\delta$. Since

$$\lim_{\theta \to 1} \nu_\theta = 0,$$

we obtain

$$\lim_{\theta \to 1} \mu_\theta = \mu(\{0\})\delta$$

in $\mathcal{M}_+$. $\square$

3. Contractibility of $X(\mathcal{M}_+)$

In this section we will prove our main result.

Proof of Theorem 1.4. Define $\varphi_{+\infty} : \mathcal{M}_+ \to \mathbb{C}$ by $\varphi_{+\infty}(\mu) = \mu(\{0\})$, $\mu \in X(\mathcal{M}_+)$. It can be checked that $\varphi_{+\infty} \in X(\mathcal{M}_+)$; see [1, Theorem 4.18.1, p.147]. We will construct a continuous map $H : X(\mathcal{M}_+) \times [0,1] \to X(\mathcal{M}_+)$ such that

$$\begin{aligned} &\text{for all } \varphi \in X(\mathcal{M}_+),\ H(\varphi, 0) = \varphi, \text{ and} \\ &\text{for all } \varphi \in X(\mathcal{M}_+),\ H(\varphi, 1) = \varphi_{+\infty}. \end{aligned}$$

The map H is defined as follows:

$$(H(\varphi, \theta))(\mu) = \varphi(\mu_\theta), \quad \mu \in \mathcal{M}_+, \quad \theta \in [0,1]. \tag{1}$$

We show that H is well-defined. From the definition, $H(\varphi, 1) = \varphi_{+\infty} \in X(\mathcal{M}_+)$ for all $\varphi \in X(\mathcal{M}_+)$. If $\theta \in [0,1)$, then the linearity of $H(\varphi, \theta) : \mathcal{M}_+ \to \mathbb{C}$ is obvious. Continuity of $H(\varphi, \theta)$ follows from the fact that φ is continuous and $\|\mu_\theta\| \leq \|\mu\|$. That $H(\varphi, \theta)$ is multiplicative is a consequence of Proposition 2.2, and the fact that φ respects multiplication. Finally $(H(\varphi, \theta))(\delta) = \varphi(\delta_\theta) = \varphi(\delta) = 1$.

That $H(\cdot, 0)$ is the identity map and $H(\cdot, 1)$ is a constant map is obvious.

Finally, we show below that H is continuous. Since $X(\mathcal{M}_+)$ is equipped with the Gelfand topology, we just have to prove that for every convergent net $(\varphi_i, \theta_i)_{i \in I}$ with limit (φ, θ) in $X(\mathcal{M}_+) \times [0,1]$, there holds that $(H(\varphi_i, \theta_i))(\mu) \to (H(\varphi, \theta))(\mu)$. We have

$$\begin{aligned} |(H(\varphi_i, \theta_i))(\mu) - (H(\varphi, \theta))(\mu)| &= |\varphi_i(\mu_{\theta_i}) - \varphi_i(\mu_\theta) + \varphi_i(\mu_\theta) - \varphi(\mu_\theta)| \\ &\leq |\varphi_i(\mu_{\theta_i}) - \varphi_i(\mu_\theta)| + |\varphi_i(\mu_\theta) - \varphi(\mu_\theta)| \\ &= |\varphi_i(\mu_{\theta_i} - \mu_\theta)| + |(\varphi_i - \varphi)(\mu_\theta)| \\ &\leq \|\varphi_i\| \cdot \|\mu_{\theta_i} - \mu_\theta\| + |(\varphi_i - \varphi)(\mu_\theta)| \\ &\leq 1 \cdot \|\mu_{\theta_i} - \mu_\theta\| + |(\varphi_i - \varphi)(\mu_\theta)| \to 0. \end{aligned}$$

This completes the proof. $\square$

In [4], we had used the explicit description of the maximal ideal space $X(\mathcal{A})$ of the algebra $\mathcal{A}$ (of those complex Borel measures that do not have a singular non-atomic part) in order to prove that $X(\mathcal{A})$ is contractible. Such an explicit description of the maximal ideal space $X(\mathcal{M}_+)$ of $\mathcal{M}_+$ does not seem to be available explicitly in the literature on the subject.

Our definition of the map H is based on the following consideration, which can be thought of as a generalization of the Riemann-Lebesgue Lemma for functions $f_a \in L^1(0, +\infty)$ (which says that the limit as $s \to +\infty$ of the Laplace transform of f_a is 0):

Theorem 3.1. *If $\mu \in \mathcal{M}_+$, then*

$$\lim_{s\to+\infty} \int_0^{+\infty} e^{-st} d\mu(t) = \mu(\{0\}).$$

The set $X(\mathcal{M}_+)$ contains the half plane

$$\mathbb{C}_{\geq 0} := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$$

in the sense that each $s \in \mathbb{C}_{\geq 0}$, gives rise to the corresponding complex homomorphism $\varphi_s : \mathcal{M}_+ \to \mathbb{C}$, given simply by point evaluation of the Laplace transform of μ at s:

$$\mu \mapsto \varphi_s(\mu) = \int_0^{+\infty} e^{-st} d\mu(t), \quad \mu \in \mathcal{M}_+.$$

If we imagine s tending to $+\infty$ along the real axis we see, in light of the Theorem 3.1 stated above, that φ_s starts looking more and more like $\varphi_{+\infty}$. So we may define

$$H(\varphi_s, \theta) = \varphi_{s-\log(1-\theta)},$$

which would suggest that at least the part $\mathbb{C}_{\geq 0}$ of $X(\mathcal{M}_+)$ is contractible to $\varphi_{+\infty}$. But we see that we can view the action of $H(\varphi_s, \theta)$ defined above as follows:

$$\begin{aligned} (H(\varphi_s, \theta))(\mu) &= \varphi_{s-\log(1-\theta)}(\mu) \\ &= \int_0^{+\infty} e^{-(s-\log(1-\theta))t} d\mu(t) \\ &= \int_0^{+\infty} e^{-st}(1-\theta)^t d\mu(t) \\ &= \varphi_s(\nu), \end{aligned}$$

where ν is the measure such that $d\nu(t) = (1-\theta)^t d\mu(t)$. This motivates the definition of H given in (1).

4. Hermite-ness of some subalgebras of $\mathcal{M}_+$

The proof of Theorem 1.4 shows that in fact it works for all subalgebras R of $\mathcal{M}_+$ which are closed under the operation $\mu \mapsto \mu_\theta$, $\theta \in [0, 1]$.

Theorem 4.1. *Suppose that R is a Banach subalgebra of $\mathcal{M}_+$, such that it has the property:*

$$\text{(P) For all } \mu \in R \text{ and for all } \theta \in [0,1],\ \mu_\theta \in R.$$

Then the maximal ideal space $X(R)$ equipped with the Gelfand topology is contractible. In particular, the ring R is Hermite, that is, the following statements are equivalent for $f \in R^{n\times k}$, $k < n$:

1. *There exists a matrix $g \in R^{k\times n}$ such that $g * f = I_k$.*
2. *There exist matrices $F, G \in R^{n\times n}$ such that $G * F = I_n$ and $F_{ij} = f_{ij}$, $1 \le i \le n$, $1 \le j \le k$.*

As specific examples of R, we consider the following:

(a) Consider the Wiener-Laplace algebra $\mathcal{W}^+$ of the half plane, of all functions defined in the half plane $\mathbb{C}_{\ge 0}$ that differ from the Laplace transform of an $L^1(0,+\infty)$ function by a constant. The Wiener-Laplace algebra $\mathcal{W}^+$ is a Banach algebra with pointwise operations and the norm

$$\|\widehat{f} + \alpha\|_{W^+} = \|\widehat{f}\|_{L^1} + |\alpha|, \quad f \in L^1(0,+\infty),\ \alpha \in \mathbb{C}.$$

Then $\mathcal{W}^+$ is precisely the set of Laplace transforms of elements of the subalgebra of $\mathcal{M}_+$ consisting of all complex Borel measures of the type $\mu_a + \alpha\delta$, where μ_a is absolutely continuous (with respect to the Lebesgue measure) and $\alpha \in \mathbb{C}$. This subalgebra of $\mathcal{M}_+$ has the property (P) demanded in the statement of Theorem 4.1, and so the maximal ideal space $X(\mathcal{W}^+)$ is contractible.

(b) Also we recover the main result in [4], but this time without recourse to the explicit description of the maximal ideal space of $\mathcal{A}$. Indeed, the subalgebra $\mathcal{A}$ of $\mathcal{M}_+$, consisting of all complex Borel measures that do not have a singular non-atomic part, also possesses the property (P).

(c) Finally, we consider the algebra almost-periodic Wiener algebra APW^+, of sums

$$f(s) = \sum_{k=1}^{\infty} f_k e^{-st_k}, \quad s \in \mathbb{C}_{\ge 0}$$

where $t_0 = 0 < t_1, t_2, t_3, \ldots$ and $\displaystyle\sum_{k=0}^{\infty} |f_k| < +\infty$.

This algebra is isometrically isomorphic to the subalgebra of $\mathcal{M}_+$ of atomic measures μ. Since this subalgebra has the property (P), it follows that APW^+ is a Hermite ring.

In each of the above algebras $\mathcal{W}^+$, $\mathcal{A}$ or APW^+, the corona theorem holds, that is, there is an analytic condition which is equivalent to left-invertibility. (The proofs/references of the corona theorems for $\mathcal{W}^+$, $\mathcal{A}$ and APW^+ can be found for example in [3, Theorem 4.3].) Combining the Hermite-ness with the corona theorem, we obtain the following:

Corollary 4.2. *Let R be any one of the algebras $\mathcal{W}^+$, $\mathcal{A}$ or APW^+. Then the following statements are equivalent for $f \in R^{n\times k}$, $k < n$:*

1. *There exists a matrix $g \in R^{k\times n}$ such that $gf = I_k$.*
2. *There exist matrices $F, G \in R^{n\times n}$ such that $GF = I_n$ and $F_{ij} = f_{ij}$ for all $1 \le i \le n$, $1 \le j \le k$.*
3. *There exists a $\delta > 0$ such that for all $s \in \mathbb{C}_{\ge 0}$, $f(s)^* f(s) \ge \delta^2 I$.*

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