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Inertia theorems for operator Lyapunov inequalities

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Abstract

We study operator Lyapunov inequalities and equations for which the infinitesimal generator is not necessarily stable, but it satisfies the spectrum decomposition assumption and it has at most finitely many unstable eigenvalues. Moreover, the input or output operators are not necessarily bounded, but are admissible. We prove an inertia result: under mild conditions, we show that the number of unstable eigenvalues of the generator is less than or equal to the number of negative eigenvalues of the self-adjoint solution of the operator Lyapunov inequality. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and the main result

The inertia of a square matrix $A \in \mathbb{C}^{n\times n}$ is the triple $(v(A), \zeta(A), \pi(A))$, where

$v(A) =$ number of eigenvalues of A in $\mathbb{C}_-$,

$\zeta(A) =$ number of eigenvalues of A on the imaginary axis,

$\pi(A) =$ number of eigenvalues of A in $\mathbb{C}_+$,

where $\mathbb{C}_- = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Inertia theorems for matrices concern relations between the inertia of Hermitian solutions Q of the Lyapunov equation

$$A^*Q + QA = -C^*C \tag{1}$$

and the matrix A. The fundamental result was by Ostrowski and Schneider [12], and later contributions can be found in [15,2]. We shall generalize the following known theorem (see [7, Theorem 3.3.2, p. 1126]):

Theorem 1.1. *Given the matrices $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{n\times p}$ and a Hermitian solution Q to* (1), *if $\zeta(Q) = 0$, then $\pi(A) \leqslant v(Q)$ and $v(A) \leqslant \pi(Q)$.*

There is little known about such inertia theorems for operator Lyapunov equations, and since the operators may have infinitely many eigenvalues, it is clear that one can only hope for a partial generalization of the matrix results. In [3,4], they consider the case of A a bounded linear operator assuming an exact controllability condition on $\Sigma(A, B, -)$.

We now define the notion of the algebraic multiplicity of an isolated eigenvalue of a closed operator on a Hilbert space.

Let λ_0 be an eigenvalue of a closed linear operator A on a Hilbert space $\mathscr{H}$. Suppose further that this eigenvalue is isolated; that is, there exists an open set $\mathcal{O}$ containing λ_0 such that $\sigma(A) \cap \mathcal{O} = \{\lambda_0\}$. We say

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that λ_0 has order v_0 if for every $x \in \mathscr{H}$,

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{v_0} (\lambda I - A)^{-1} x$$

exists, but there exists a x_0 such that

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{v_0 - 1} (\lambda I - A)^{-1} x_0$$

does not. If for every $v \in \mathbb{N}$ there exists a $x_v \in \mathscr{H}$ such that the limit

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{v} (\lambda I - A)^{-1} x_v$$

does not exist, then the order of λ_0 is infinity. For an isolated eigenvalue λ_0 of finite order v_0, its algebraic multiplicity is defined as $\dim(\ker(\lambda_0 I - A)^{v_0})$.

For example, the eigenvalue 1 of the operator

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathscr{L}(\mathbb{C}^2)$$

has order 2. On the other hand, any isolated eigenvalue of a self-adjoint operator $Q \in \mathscr{L}(\mathscr{H})$ has order 1.

Next, we define $\pi(A)$ for a closed operator A on a Hilbert space $\mathscr{H}$.

Let A be a closed linear operator on a Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ and let $\sigma(A) \cap \mathbb{C}_+$ be a bounded set which is isolated from $\sigma(A) \setminus (\sigma(A) \cap \mathbb{C}_+)$ (by which we mean that it is separated from $\sigma(A) \setminus (\sigma(A) \cap \mathbb{C}_+)$ in such a way that a simple, closed, rectifiable curve Γ can be drawn so as to enclose an open set containing $\sigma(A) \cap \mathbb{C}_+$ in its interior and $\sigma(A) \setminus (\sigma(A) \cap \mathbb{C}_+)$ in its exterior). Let Π denote the spectral projection on $\sigma(A) \cap \mathbb{C}_+$. Then $\mathscr{H} = \mathscr{H}^+ \dotplus \mathscr{H}^-$, where

$$\mathscr{H}^+ := \Pi \mathscr{H}, \qquad \mathscr{H}^- := (I - \Pi)\mathscr{H} \tag{2}$$

and $\dotplus$ denotes the direct sum of the subspaces $\mathscr{H}^+$ and $\mathscr{H}^-$ (see for example, [10, Theorem 6.17, p. 178]). $\dim(\mathscr{H}^+) < \infty$ iff $\sigma(A) \cap \mathbb{C}_+$ consists of a finite system of eigenvalues (see [6, Lemma 2.5.7, pp. 71–72], [10, Problem 6.18, p. 182]). In this case, the total algebraic multiplicity of the eigenvalues in $\mathbb{C}_+$, which we denote by $\pi(A)$, is equal to $\dim(\mathscr{H}^+)$.

We now state our main result about operator Lyapunov inequalities.

Theorem 1.2. *Assume that*

1. *A is a densely defined closed linear operator on a Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$,*
2. *$\sigma(A) \cap \mathbb{C}_+$ is a bounded set which is isolated from $\sigma(A) \setminus (\sigma(A) \cap \mathbb{C}_+)$,*
3. *$\dim(\mathscr{H}^+) < \infty$ (with the notation introduced in (2)),*
4. *Q in $\mathscr{L}(\mathscr{H})$ is a self-adjoint operator such that $0 \notin \sigma_p(Q)$, $\sigma(Q) \cap \mathbb{C}_- = \sigma_p(Q) \cap \mathbb{C}_-$, $v(Q) < \infty$, and Q satisfies the Lyapunov inequality*

$$\langle Qx, Ax \rangle + \langle QAx, x \rangle \leqslant 0 \quad \forall x \in D(A). \tag{3}$$

Then $\pi(A) \leqslant v(Q)$.

The paper is organized as follows. In Section 2, we provide the mathematical background and the proof of our main theorem. In Section 3, we give a few corollaries of our main theorem for operator Lyapunov equations with admissible observation operators.

2. Preliminaries on indefinite inner products and proof of the main result

The proof of our main theorem relies on the fact that any self-adjoint solution of the operator Lyapunov inequality gives rise to a natural indefinite inner product space. So, we will first state a few preliminaries and results about indefinite inner product spaces which will be used in the proof. For more details, see [1].

Let $\mathscr{V}$ be a vector space over $\mathbb{C}$. An indefinite inner product $[\cdot,\cdot]$ on $\mathscr{V}$ is a map $[\cdot,\cdot] : \mathscr{V} \times \mathscr{V} \to \mathbb{C}$ satisfying:

1. $[\alpha x_1 + \beta x_2, y] = \alpha[x_1, y] + \beta[x_2, y]$, $\forall x_1, x_2, y \in \mathscr{V}$ and $\forall \alpha, \beta \in \mathbb{C}$.
2. $[x, y] = \overline{[y, x]}$ for all $x, y \in \mathscr{V}$.

A vector $x \in \mathscr{V}$ is said to be positive, negative or neutral depending on whether $[x,x]$ is > 0, < 0 or $= 0$, respectively. We denote the sets of all positive, negative and neutral vectors of a space by $\mathscr{V}_{++}$, $\mathscr{V}_{--}$ and $\mathscr{V}_0$, respectively, that is

$$\mathscr{V}_{++} = \{x \mid [x,x] > 0\},$$
$$\mathscr{V}_{--} = \{x \mid [x,x] < 0\},$$
$$\mathscr{V}_0 = \{x \mid [x,x] = 0\}.$$

We define

$$\mathscr{V}_+ = \mathscr{V}_{++} \cup \mathscr{V}_0, \quad \mathscr{V}_- = \mathscr{V}_{--} \cup \mathscr{V}_0,$$

the sets of all nonnegative and nonpositive vectors in $\mathscr{V}$, respectively. A subspace $\mathscr{W}$ of $\mathscr{V}$ is said to be nonnegative, nonpositive or neutral if

$$\mathscr{W} \subset \mathscr{V}_+, \quad \mathscr{W} \subset \mathscr{V}_- \quad \text{or} \quad \mathscr{W} \subset \mathscr{V}_0,$$

respectively. A subspace $\mathscr{W}$ of $\mathscr{V}$ is said to be positive (negative) if

$$\mathscr{W} \subset \mathscr{V}_{++} \cup \{0\} \quad (\mathscr{W} \subset \mathscr{V}_{--} \cup \{0\}).$$

The nonnegative and nonpositive subspaces are said to be semidefinite. For a semidefinite subspace $\mathscr{W}$, the following generalization of the Cauchy–Schwarz inequality holds:

$$|[x_1,x_2]|^2 \leqslant [x_1,x_1][x_2,x_2] \quad \forall x_1,x_2 \in \mathscr{W}. \tag{4}$$

The following lemma will be crucial in obtaining the inequality in our main inertia theorem:

Lemma 2.1. *Let $\mathscr{V}$ be a linear space with an indefinite inner product $[\cdot,\cdot]$ which admits a decomposition into a direct sum $\mathscr{V} = \mathscr{V}_+ \dot{+} \mathscr{V}_-$ of a positive subspace $\mathscr{V}_+$ and a negative subspace $\mathscr{V}_-$. Then the dimension of any nonpositive subspace $\mathscr{W}$ of $\mathscr{V}$ does not exceed the dimension of $\mathscr{V}_-$.*

Proof. This follows from [1, Remark 4.4, p. 24]. □

Next, we study indefinite inner products that are induced by a bounded self-adjoint operator.

Let $\mathscr{H}$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$. Let $Q \in \mathscr{L}(\mathscr{H})$ be an arbitrary bounded self-adjoint operator on $\mathscr{H}$. Then $\mathscr{H}$ equipped with the indefinite inner product $[\cdot,\cdot]$ defined by

$$[x,y] = \langle Qx,y\rangle \quad \forall x,y \in \mathscr{H},$$

is called a Q-space and Q is called the Gram operator of the space $(\mathscr{H},[\cdot,\cdot])$. It is clear that

$$|[x,y]| \leqslant \|Q\|\,\|x\|\,\|y\|,$$

where $\|x\| = (\langle x,x\rangle)^{1/2}$, which establishes the continuity of $[\cdot,\cdot] : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$. We denote the sum of two $\langle\cdot,\cdot\rangle$-orthogonal subspaces $\mathscr{W}_1$ and $\mathscr{W}_2$ by

$$\mathscr{W}_1 \langle + \rangle \mathscr{W}_2$$

and the sum of two $[\cdot,\cdot]$-orthogonal subspaces $\mathscr{W}_1$ and $\mathscr{W}_2$ by

$$\mathscr{W}_1 [+] \mathscr{W}_2.$$

We now prove a useful lemma about Q-spaces which will be used in the proof of our main theorem.

Lemma 2.2. *Let $\mathscr{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and let $Q \in \mathscr{L}(\mathscr{H})$ be self-adjoint. Then*

1. *The Q-space $(\mathscr{H},[\cdot,\cdot])$ admits an $\langle\cdot,\cdot\rangle$-orthogonal direct sum decomposition*

$$\mathscr{H} = \mathscr{H}_{Q_-} \langle + \rangle \mathscr{H}_{Q_0} \langle + \rangle \mathscr{H}_{Q_+}, \tag{5}$$

where $\mathscr{H}_{Q_-}$ is a negative subspace, $\mathscr{H}_{Q_0}$ is a neutral subspace and $\mathscr{H}_{Q_+}$ is a positive subspace. Furthermore, $\mathscr{H} = \mathscr{H}_{Q_-}[+]\mathscr{H}_{Q_0}[+]\mathscr{H}_{Q_+}$; that is, the decomposition is also $[\cdot,\cdot]$-orthogonal.

2. *If $\sigma(Q)\cap\mathbb{C}_- = \sigma_p(Q)\cap\mathbb{C}_-$ and $\nu(Q) < \infty$, then* $\dim(\mathscr{H}_{Q_-}) = \nu(Q)$.

3. $\mathscr{H}_{Q_0} = 0$ *iff* $0 \notin \sigma_p(Q)$.

Proof. 1. Let $\mathscr{E} = \{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be the spectral family of spectral projections $E(\lambda) \in \mathscr{L}(\mathscr{H})$, $\lambda \in \mathbb{R}$, corresponding to the self-adjoint operator Q. Define the projections

$$S_- = \int_{-\infty}^{0-} \mathrm{d}E(\lambda) = E(0-),$$

$$S_0 = E(0) - E(0-) \quad \text{and} \quad S_+ := \int_0^{\infty} \mathrm{d}E(\lambda).$$

These projections are pairwise orthogonal and $I = S_- + S_0 + S_+$, and they generate a $\langle\cdot,\cdot\rangle$-orthogonal decomposition of $\mathscr{H}$ into subspaces $\mathscr{H}_{Q_-}$, $\mathscr{H}_{Q_0}$, $\mathscr{H}_{Q_+}$, where

$$\mathscr{H}_{Q_-} = \mathrm{ran}(S_-),$$

$$\mathscr{H}_{Q_0} = \mathrm{ran}(S_0) \quad \text{and} \quad \mathscr{H}_{Q_+} = \mathrm{ran}(S_+).$$

Thus, $\mathscr{H} = \mathscr{H}_{Q_-}\langle+\rangle\mathscr{H}_{Q_0}\langle+\rangle\mathscr{H}_{Q_+}$. We first prove that $[x_-,x_-] = \langle Qx_-,x_-\rangle \leqslant 0$ $\forall x_- \in \mathscr{H}_{Q_-}$. We have

$$\begin{aligned}
[x_-,x_-] = \langle Qx_-,x_-\rangle &= \int_{-\infty}^{\infty} \lambda\,\mathrm{d}\langle E(\lambda)x_-,x_-\rangle \\
&= \int_{-\infty}^{\infty} \lambda\,\mathrm{d}\langle E(\lambda)x_-,S_-x_-\rangle \\
&= \int_{-\infty}^{\infty} \lambda\,\mathrm{d}\langle E(\lambda)x_-,E(0-)x_-\rangle \\
&= \int_{-\infty}^{\infty} \lambda\,\mathrm{d}\langle E(0-)E(\lambda)x_-,x_-\rangle.
\end{aligned}$$

But

$$\langle E(0-)E(\lambda)x_-,x_-\rangle = \begin{cases} \langle E(\lambda)x_-,x_-\rangle, & \lambda < 0 \\ \langle E(0-)x_-,x_-\rangle (= \text{a constant}), & \lambda \geqslant 0 \end{cases}$$

and so

$$\begin{aligned}
\langle Qx_-,x_-\rangle &= \int_{-\infty}^{\infty} \lambda\,\mathrm{d}\langle E(0-)E(\lambda)x_-,x_-\rangle \\
&= \int_{-\infty}^{0-} \lambda\,\mathrm{d}\langle E(\lambda)x_-,x_-\rangle \leqslant 0,
\end{aligned}$$

since $\lambda \mapsto \langle E(\lambda)x_-,x_-\rangle$ is a nondecreasing function and $\lambda < 0$.

Similarly, it can be checked that $[x_+,x_+] = \langle Qx_+,x_+\rangle \geqslant 0$, $\forall x_+ \in \mathscr{H}_{Q_+}$. Since $\mathscr{H}_{Q_-}$, $\mathscr{H}_{Q_0}$

and $\mathscr{H}_{Q_+}$ are Q-invariant, it follows from their $\langle\cdot,\cdot\rangle$-orthogonality that they are $[\cdot,\cdot]$-orthogonal. Finally, to prove that the subspaces $\mathscr{H}_{Q_+}$ and $\mathscr{H}_{Q_-}$ are in fact positive and negative, respectively, we use the generalized Cauchy–Schwarz inequality (4). If $x_+ \in \mathscr{H}_{Q_+}$ and $\langle Qx_+, x_+\rangle = 0$, then

$$
\begin{aligned}
0 &\leqslant |\langle Qx_+, Qx_+\rangle|^2 = |[x_+, Qx_+]|^2 \\
&\leqslant [x_+, x_+][Qx_+, Qx_+] \\
&= \langle Qx_+, x_+\rangle\langle QQx_+, Qx_+\rangle = 0
\end{aligned}
$$

and so $Qx_+ = 0$, that is $x_+ \in \mathscr{H}_{Q_0}$. Consequently, $x_+ \in \mathscr{H}_{Q_+} \cap \mathscr{H}_{Q_0} = \{0\}$.

Similarly, it can be shown that $\mathscr{H}_{Q_-}$ is a negative subspace.

2. If the eigenvalues in $\mathbb{C}_-$ are $\lambda_1, \ldots, \lambda_n$, then

$$
\begin{aligned}
\dim(\mathscr{H}_{Q_-}) &= \dim(E(0-)) \\
&= \sum_{k=1}^{n} \dim(E(\lambda_k) - E(\lambda_k-)) \\
&= \sum_{k=1}^{n} \dim(\ker(\lambda_k I - Q)) = v(Q).
\end{aligned}
$$

3. This follows from the fact that $\mathscr{H}_{Q_0} = \ker(Q)$. □

Remark. We remark that the decomposition in (5) is only one amongst several possible ones. So for example, if $[x,x] < 0$, it does not follow that $x \in \mathscr{H}_{Q_-}$. Indeed, with

$$
Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} (\in \mathscr{L}(\mathbb{C}^2)) \quad \text{and}
$$

$$
x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(\notin \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \mathscr{H}_{Q_-}\right),
$$

we have $[x,x] < 0$.

A linear operator A with an arbitrary domain of definition $D(A)$, operating in a Q-space $\mathscr{H}$, is said to be Q-dissipative if

$$
2\,\mathrm{Re}([Ax,x]) = \langle QAx, x\rangle + \langle Qx, Ax\rangle \leqslant 0
$$

for all $x \in D(A)$.

We quote the following crucial result which is an immediate consequence of [1, Theorem 2.21, p. 98].

Lemma 2.3. *Let $\mathscr{H}$ be a Q-space and A be a closed Q-dissipative operator on $\mathscr{H}$. Furthermore, assume that σ is a bounded subset of $\sigma(A)$ such that $\sigma \subset \mathbb{C}_+$ and σ is isolated from $\sigma(A) \setminus \sigma$. If Π denotes the spectral projection on σ, then $\Pi\mathscr{H}$ is a nonpositive subspace of the Q-space $(\mathscr{H}, [\cdot,\cdot])$.*

We now proceed to give a proof of our main theorem.

Proof of Theorem 1.2. 1. From the Lyapunov inequality, it follows that A is Q-dissipative:

$$
2\,\mathrm{Re}([Ax,x]) = \langle QAx, x\rangle + \langle Qx, Ax\rangle \leqslant 0 \quad \forall x \in D(A).
$$

Using Lemma 2.3, we obtain that $\mathscr{H}^+ = \Pi\mathscr{H}$ (see (2)) is a nonpositive subspace of $(\mathscr{H}, [\cdot,\cdot])$. This is a $\pi(A)$-dimensional nonpositive subspace of $(\mathscr{H}, [\cdot,\cdot])$ (see [11, Problem 6.18, p. 182]).

2. From Lemma 2.2, since $0 \notin \sigma_p(Q)$ it follows that the self-adjoint operator Q induces a $[\cdot,\cdot]$-orthogonal direct sum decomposition

$$
\mathscr{H} = \mathscr{H}_{Q_-}[+]\mathscr{H}_{Q_+},
$$

where $\mathscr{H}_{Q_-}$ and $\mathscr{H}_{Q_+}$ are negative and positive subspaces, respectively, in $(\mathscr{H}, [\cdot,\cdot])$, with $\dim(\mathscr{H}_{Q_-}) = v(Q)$.

3. Finally it follows from Lemma 2.1, that $\dim(\mathscr{H}^+) \leqslant \dim(\mathscr{H}_{Q_-})$, that is, $\pi(A) \leqslant v(Q)$. □

3. Corollaries

In this section, we give a few corollaries of our main theorem applied to Lyapunov equations with possibly unbounded observation operators.

Throughout this section, we assume that X is a Hilbert space and $A : D(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ on X.

Definition 3.1. Let us denote

$$
\sigma_+(A) := \sigma(A) \cap \overline{\mathbb{C}_+},
$$
$$
\sigma_-(A) := \sigma(A) \cap \mathbb{C}_-.
$$

A satisfies the spectrum decomposition assumption if $\sigma_+(A)$ is a bounded set which is separated from $\sigma_-(A)$ in such a way that a rectifiable, simple closed curve, Γ, can be drawn so as to enclose an open set containing $\sigma_+(A)$ in its interior and $\sigma_-(A)$ in its exterior (see Fig. 1).

The decomposition of the spectrum in this way induces a corresponding direct sum decomposition of the state space X:

$$
X = X^+ \dotplus X^-, \quad X^+ = \Pi X, \quad X^- = (I - \Pi)X, \tag{6}
$$

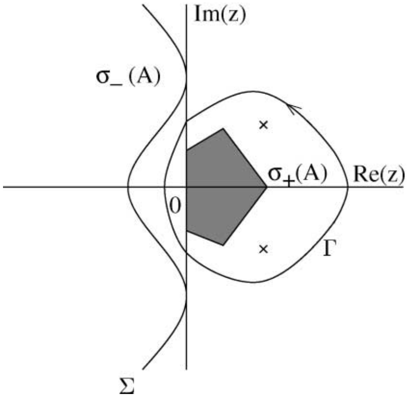

Fig. 1. The spectrum decomposition: Here $\sigma_+(A)$ comprises the shaded region together with the crosses, and $\sigma_-(A)$ is contained in the region to the left of the curve Σ.

where Π is the spectral projection on $\sigma_+(A)$:

$$\Pi x = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda I - A)^{-1} x \, \mathrm{d}\lambda \quad \forall x \in X$$

and Γ is traversed once in the positive direction (counterclockwise).

Here, we allow for unbounded observation operators, for which we require a few preliminaries.

We define the Hilbert space X_1 as $D(A)$ with the norm

$$\|z\|_1 = \|(\beta I - A)z\|,$$

where $\beta \in \rho(A)$ is fixed (this norm is equivalent to the graph norm). Z_{-1} is the completion of X with respect to the norm

$$\|x\|_{-1} = \|(\gamma I - A^*)^{-1} z\|,$$

where $\gamma \in \rho(A^*)$ is fixed. If X is the pivot space (that is, if we identify X with X^*), then it follows that $Z_{-1}^* = X_1$.

We now consider the operator Lyapunov equation

$$A^* Qx + QAx = -C^* Cx \quad \forall x \in D(A), \tag{7}$$

with values in Z_{-1}, where $C \in \mathscr{L}(X_1, Y)$ and Y is a Hilbert space. We say that (7) has a self-adjoint solution $Q = Q^* \in \mathscr{L}(X)$ if (7) holds. For the theory of such Lyapunov equations with a self-adjoint nonnegative definite solutions $Q \in \mathscr{L}(X)$, see [8,9].

Corollary 3.2. *If*
1. *A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geqslant 0}$ on the Hilbert space X,*
2. $C \in \mathscr{L}(X_1, Y)$,
3. *A satisfies the spectrum decomposition assumption,*
4. $\dim(X^+) < \infty$ (*with the notation used in* (6)),
5. *$Q \in \mathscr{L}(X)$ is a self-adjoint solution of* (7) *such that* $0 \notin \sigma_p(Q)$, $\sigma(Q) \cap \mathbb{C}_- = \sigma_p(Q) \cap \mathbb{C}_-$, *and* $\nu(Q) < \infty$,

then $\pi(A) \leqslant \nu(Q)$.

Proof. We observe that

$$\begin{aligned} 2\,\mathrm{Re}([Ax, x]) &= \langle QAx, x\rangle + \langle Qx, Ax\rangle \\ &= -\langle Cx, Cx\rangle \leqslant 0 \quad \forall x \in D(A), \end{aligned}$$

that is, A is Q-dissipative. An application of Theorem 1.2 yields the desired inequality. □

The conditions $\dim(X_+) < \infty$ and on $\sigma(Q)$ in Corollary 3.2 may not be easy to check, and so we replace these by more familiar sufficient conditions on the pair (A, C) in the following corollary:

Corollary 3.3. *If*
1. *$C \in \mathscr{L}(X, Y)$ has finite rank,*
2. *$\Sigma(A, -, C)$ is exponentially detectable, and*
3. *$Q \in \mathscr{L}(X)$ is a self-adjoint solution of* (7),

then A satisfies the spectrum decomposition assumption, A has a pure point spectrum in the closed right half-plane (*that is,* $\sigma(A) \cap \overline{\mathbb{C}_+} = \sigma_p(A) \cap \overline{\mathbb{C}_+}$) *and* $\zeta(A) = 0$.

Furthermore, if $0 \notin \sigma_p(Q)$, $\sigma(Q) \cap \mathbb{C}_- = \sigma_p(Q) \cap \mathbb{C}_-$, $\nu(Q) < \infty$, *then* $\pi(A) \leqslant \nu(P)$.

Proof. From Theorem 5.2.7 [6, p. 235] it follows that A satisfies the spectrum decomposition assumption, and X^+ is finite-dimensional. So from Problem 6.18 [10, p. 182], we conclude that $\sigma_+(A)$ comprises finitely many eigenvalues of finite algebraic multiplicity.

Next, we show that A has no eigenvalues on the imaginary axis. Assume the contrary; that is, suppose that there exists a $\omega_0 \in \mathbb{R}$ and a $x_0\ (\neq 0) \in X$ such that $Ax_0 = \mathrm{i}\omega_0 x_0$. From (8), we obtain that

$$\begin{aligned} -\|Cx_0\|^2 &= -\langle Cx_0, Cx_0\rangle = \langle Ax_0, Qx_0\rangle + \langle x_0, QAx_0\rangle \\ &= \mathrm{i}\omega_0\langle Qx_0, x_0\rangle - \mathrm{i}\omega_0\langle Qx_0, x_0\rangle = 0 \end{aligned}$$

and so $Cx_0 = 0$. Thus, $x_0 \in \ker(C)$. But since $\Sigma(A, -, C)$ is exponentially detectable with a finite-rank C, we have

$$\ker(sI - A) \cap \ker(C) = \{0\} \quad \forall s \in \overline{\mathbb{C}_+}, \tag{8}$$

(see [6, Theorem 5.2.11, pp. 240–241]), and so we arrive at a contradiction.

If $0 \notin \sigma_p(Q)$, $\sigma(Q)\cap\mathbb{C}_-=\sigma_p(Q)\cap\mathbb{C}_-$, $v(Q) < \infty$, then using Corollary 3.2 above, we obtain that $\pi(A) \leqslant v(P)$. □

Remark. The above corollary can be extended to allow for unbounded, but admissible observation operators.

C is said to be an admissible observation operator for $\{T(t)\}_{t\geqslant 0}$ if for every $T > 0$, there exists a $K_T \geqslant 0$ such that

$$\int_0^T \|CT(t)x\|^2 \,\mathrm{d}t \leqslant K_T^2 \|x\|^2 \quad \forall x \in D(A).$$

(See [14].)

Corollary 3.3 still holds for admissible C with finite rank and such that the pair (A, C) is exponential detectable, where by the latter is meant:

There exists a $L \in \mathscr{L}(Y, X_1)$ such that $A^{X_1} + LC$ generates an exponentially stable semigroup on W, where A^{X_1} denotes the restriction of A to

$$D(A^{X_1}) = \{x \in D(A) \mid Ax \in D(A)\}.$$

This is a bounded concept of detectability on the state space X_1 and as in the proof of Corollary 3.3, we conclude that A^{X_1} satisfies the spectrum decomposition assumption on X_1. But the spectra of A and its restriction A^{X_1} are the same (see [5]) and so A satisfies the spectrum decomposition on X and X^+ is finite-dimensional. Similarly, we can argue that (8) holds. Finally, we remark that this concept of exponential detectability is equivalent to the existence of an "admissible control" operator $L \in \mathscr{L}(Y, X)$ such that $A + LC$ generates an exponentially stable semigroup on X (see [5]). The more general concepts of detectability in the literature do not imply that A satisfies the spectrum decomposition assumption, even if C has finite rank (see [13]).

Of course, it is clear from Corollary 3.2 that detectability is not necessary and it is possible to formulate sharper sufficient conditions on (A, C).

Finally, we remark that similar theorems can be proved for control operators $B \in \mathscr{L}(U, X_{-1})$, where the input space U is a Hilbert space (see [9]).

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