Journal of Applied Analysis **16** (2010), 16 pages
DOI 10.1515/JAA.2010.006

Some algebraic properties of the Wiener–Laplace algebra

Raymond Mortini and Amol Sasane

Abstract. We denote by $W^+(\mathbb{C}_+)$ the set of all complex-valued functions defined in the closed right half plane $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$ that differ from the Laplace transform of functions from $L^1(0,\infty)$ by a constant. Equipped with pointwise operations, $W^+(\mathbb{C}_+)$ forms a ring. It is known that $W^+(\mathbb{C}_+)$ is a pre-Bézout ring. The following properties are shown for $W^+(\mathbb{C}_+)$:

$W^+(\mathbb{C}_+)$ is not a GCD domain, that is, there exist functions F_1, F_2 in $W^+(\mathbb{C}_+)$ that do not possess a greatest common divisor in $W^+(\mathbb{C}_+)$.

$W^+(\mathbb{C}_+)$ is not coherent, and in fact, we give an example of two principal ideals whose intersection is not finitely generated.

We will also observe that $W^+(\mathbb{C}_+)$ is a Hermite ring, by showing that the maximal ideal space of $W^+(\mathbb{C}_+)$, equipped with the Gelfand topology, is contractible.

Note 1 Please check the MSC numbers for 2010 (see www.ams.org/msc).

Keywords. GCD domain, Wiener–Laplace algebra, control theory.

2010 Mathematics Subject Classification. Primary 46J15; Secondary 13G05, 93D15.

1 Introduction

The aim of this paper is to study some algebraic properties of the ring $W^+(\mathbb{C}_+)$ (defined below).

We first recall the notion of a GCD domain, a coherent ring and a Hermite ring below.

Definition 1.1. Let R be an integral domain (that is a commutative unital ring having no divisors of zero).

1. An element $d \in R$ is called a *greatest common divisor* (often abbreviated by *gcd*) of $a, b \in R$ if it is a divisor of a and b, and moreover, if k is another divisor, then k divides d.

   R is said to be a *GCD domain* if for all $a, b \in R$, there exists a greatest common divisor d of a, b.

2. R is said to be *pre-Bézout* if for every $a, b \in R$ for which there exists a greatest common divisor d, there exist $x, y \in R$ such that $d = xa + yb$.

3. R is called *coherent* if for any pair (I, J) of finitely generated ideals in R their intersection $I \cap J$ is finitely generated again.
4. A matrix $f \in R^{n\times k}$ is called *left invertible* if there exists a $g \in R^{k\times n}$ such that $gf = I_k$.
5. R is called a *Hermite ring* if for all $k, n \in \mathbb{N}$ with $k < n$ and all left invertible matrices $f \in R^{n\times k}$, there exist $F, G \in R^{n\times n}$ such that $GF = I_n$ and $F_{ij} = f_{ij}$ for all $1 \le i \le n$ and $1 \le j \le k$. We shall also say that in that case R has the *matricial extension property*.

Whether some rings of analytic functions have the above algebraic properties has been investigated in earlier works. For example, von Renteln showed that the Hardy algebra $H^\infty(\mathbb{D})$ (of all bounded and analytic functions in the open unit disc, with pointwise operations) is a GCD domain [17, p. 519], while the disc algebra $A(\mathbb{D})$ (the ring of continuous functions on the closed unit disc $\overline{\mathbb{D}}$, which are analytic in the open unit disc $\mathbb{D}$, with the usual pointwise operations) is not [16, p. 52]. In [10], the first author and von Renteln noted that the Wiener algebra $W^+(\mathbb{D})$ (of all absolutely convergent Taylor series in the open unit disc) is not a GCD domain. In this article, we will show that the ring $W^+(\mathbb{C}_+)$ (defined below) is not a GCD domain.

W. S. McVoy and L. A. Rubel [8] showed that the Hardy algebra $H^\infty(\mathbb{D})$ is coherent, while the disc algebra $A(\mathbb{D})$ is not. In [10], it was also shown that the Wiener algebra $W^+(\mathbb{D})$ (of all absolutely convergent Taylor series in the open unit disc $\mathbb{D}$) is not coherent. In Section 4, we will show that the ring $W^+(\mathbb{C}_+)$ is not coherent, in the same manner as the noncoherence of $W^+(\mathbb{D})$ was shown in [10, Theorem 3, p. 226].

The Hermiteness of $H^\infty(\mathbb{D})$ was first shown by V. Tolokonnikov (see for example [21], [12, §10, p. 293]). A. Quadrat has proved in [15, Corollary 3.30] that $H^\infty(\mathbb{D})$ is a projective free ring, which implies in particular that it is Hermite. The Hermiteness of $A(\mathbb{D})$ and $W^+(\mathbb{D})$ follows from [6] and the fact that their maximal ideal ideal space is homeomorphic to the closed unit disc $\overline{\mathbb{D}}$. Indeed, Lin's theorem (Proposition 2.10 below) says that a Banach algebra whose maximal ideal space is contractible is a Hermite ring. We will show that the maximal ideal space of $W^+(\mathbb{C}_+)$ is homeomorphic to the closed unit disc, and hence, by [6], the ring $W^+(\mathbb{C}_+)$ is Hermite as well.

Throughout the article, we will use the following notation:

$$\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \ge 0\}.$$

Definition 1.2. We denote by $W^+(\mathbb{C}_+)$ the set of all functions $F : \mathbb{C}_+ \to \mathbb{C}$ such that $F(s) = \hat{f_a}(s) + f_0$ ($s \in \mathbb{C}_+$), where $f_a \in L^1(0, \infty)$, $f_0 \in \mathbb{C}$, and $\hat{f_a}$ denotes the Laplace transform of f_a:

$$\hat{f_a}(s) = \int_0^\infty e^{-st} f_a(t)dt, \quad s \in \mathbb{C}_+.$$

Equipped with pointwise operations and the norm $\|F\|_{W^+} = \|f_a\|_{L^1} + |f_0|$, $W^+(\mathbb{C}_+)$ is a Banach algebra.

Remark 1.3. 1. From the application point of view, the above algebra arises as natural classes of transfer functions of stable distributed parameter systems in control theory; see [13].

Following [11], we use the notation $W^+(\mathbb{C}_+)$ in order to highlight the similarity with $W^+(\mathbb{D})$ and call $W^+(\mathbb{C}_+)$ the *Wiener–Laplace algebra*.

2. The algebraic properties of $W^+(\mathbb{C}_+)$ investigated in this article have important consequences in control theory:

The relevance in control theory of the question whether or not $W^+(\mathbb{C}_+)$ is a GCD domain can be found in [13]: Let R denote a ring of stable transfer functions, and $Q(R)$ the field of fractions of R. Then every transfer function $p \in Q(R)$ admits a weak coprime factorisation iff R is a GCD domain.

The importance of the coherence property in control theory can be found in [15, Theorem 3.24, p. 286]. In fact, our Theorem 1.4 answers a question raised in [14, p. 30].

The motivation for proving that $W^+(\mathbb{C}_+)$ is a Hermite ring is that if a transfer function G (with entries from the field of fractions of $W^+(\mathbb{C}_+)$) has a right (or left) coprime factorisation, then G has a doubly coprime factorisation, and then the standard Youla–Kučera parameterisation yields all stabilising controllers for G. For further details, see [24, Theorem 66, p. 347].

Our main results are the following:

Theorem 1.4. *The ring $W^+(\mathbb{C}_+)$ is not a GCD domain.*

Theorem 1.5. *The ring $W^+(\mathbb{C}_+)$ is not coherent.*

Theorem 1.6. *The ring $W^+(\mathbb{C}_+)$ is Hermite.*

The paper is structured as follows. In Section 2, we will first collect a few auxiliary results needed to prove our main theorems. Subsequently in Sections 3, 4 and 5 we will prove Theorem 1.4, 1.5, and 1.6, respectively.

2 Preliminaries

We begin by recalling different (but equivalent) versions of the corona theorem for $W^+(\mathbb{C}_+)$; see [3, p. 112] and [11, Proposition 4.4]. Let $\widehat{L^1(0,\infty)}$ denote the set of Laplace transforms of functions in $L^1(0,\infty)$. Note also that by the non-discrete version of the Riemann–Lebesgue Lemma

$$\lim_{\substack{s\to\infty \\ \mathrm{Re}(s)\geq 0}} F(s) = 0$$

for any $F \in \widehat{L^1(0,\infty)}$. Hence, every function in $W^+(\mathbb{C}_+)$ admits a continuous extension at infinity. In particular, $W^+(\mathbb{C}_+) \subseteq H^\infty(\mathbb{C}_+)$, the set of bounded analytic functions on the open right half plane.

Proposition 2.1. *The following assertions hold:*

1. *The set of maximal ideals in* $W^+(\mathbb{C}_+)$ *is given by*

$$\mathfrak{M}_a = \{f \in W^+(\mathbb{C}_+) \mid f(a) = 0\}, \quad a \in \mathbb{C}_+,$$

and

$$\mathfrak{M}_\infty = \widehat{L^1(0,\infty)} = \{f \in W^+(\mathbb{C}_+) \mid \lim_{\substack{s\to\infty \\ \mathrm{Re}(s)\geq 0}} f(s) = 0\}.$$

2. *The set* $\{\phi_a \mid \mathrm{Re}(a) > 0\}$ *of point evaluations* $\phi_a(f) = f(a)$*, where* $f \in W^+(\mathbb{C}_+)$*, is dense in the maximal ideal space of* $W^+(\mathbb{C}_+)$*.*
3. *For every n-tuple* $(f_1, \dots, f_n)$ *of functions in* $W^+(\mathbb{C}_+)$ *satisfying*

$$\delta := \inf_{\mathrm{Re}(s)>0} \sum_{j=1}^{n} |f_j(s)| > 0,$$

there exists a solution $(g_1, \dots, g_n) \in W^+(\mathbb{C}_+)^n$ *of the Bézout equation* $\sum_{j=1}^n g_j f_j = 1$.

Using a general procedure that allows to pass from n-tuples to matrices (see [24, p. 340]), we obtain the following matricial version of the corona theorem:

Proposition 2.2. *Let* $R = W^+(\mathbb{C}_+)$*. Let* $F \in R^{n\times k}$*. Then the following are equivalent:*

1. *There exists a* $G \in R^{k\times n}$ *such that* $G(s)F(s) = I_k$, $s \in \mathbb{C}_+$.
2. *There exists a* $\delta > 0$ *such that* $F(s)^*F(s) \geq \delta^2 I_k$, $s \in \mathbb{C}_+$.

Here I_k is the identity matrix in $\mathbb{C}^k$ and F^* is the conjugate transpose (or adjoint) of F. We note that conditions 1 or 2 imply that, automatically, $n \geq k$; in particular the rank of every $k \times k$-submatrix of $F(s)$ is k. Condition 2, also, is equivalent to

3. $\|F(s)[x]\|_2 \geq \delta \|x\|_2$ for every $x \in \mathbb{C}^k$, $s \in \mathbb{C}_+$.

In the following, the maximal ideal $\mathfrak{M}_0$, being the kernel of the complex homomorphism $F \mapsto F(0)$, will play an important role. Since every maximal ideal is closed, the set $\mathfrak{M}_0$ is a commutative Banach subalgebra of $W^+(\mathbb{C}_+)$. Obviously the subalgebra $\mathfrak{M}_0$ has no identity element. But there is a substitute, namely the notion of the bounded approximate identity, which will be useful in the sequel.

Definition 2.3. Let R be a commutative Banach algebra (without identity element). Then R has a *bounded approximate identity* if there exists a bounded sequence $(e_n)_n$ of elements e_n in R such that for any $f \in R$, $\lim_{n\to\infty} \|e_n f - f\| = 0$.

It is known that the maximal ideal $\mathfrak{M}_0$ has a bounded approximate identity; see [11, Theorem 4.2.(a), p. 6].

Proposition 2.4. *Let $e_n(s) := \frac{s}{s+\frac{1}{n}}$ ($s \in \mathbb{C}_+$), $n \in \mathbb{N}$. Then $(e_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\mathfrak{M}_0$.*

We will also need the following amazing factorisation theorem:

Theorem 2.5 (Varopoulos, [23]). *Let R be a commutative Banach algebra with a bounded approximate identity. Then for every sequence $(a_n)_{n\geq 1}$ in R there exists a sequence $(b_n)_{n\geq 1}$ in R as well as an element $c \in R$ such that for all $n \geq 1$, $a_n = cb_n$.*

The following was noted in [10, Remark after Theorem 1, p. 224] without proof. A proof is given below. Our proof of the Theorem 1.4 will follow the same method.

Proposition 2.6 (Mortini–von Renteln, [10]). *Let $f_1, f_2 \in W^+(\mathbb{D})$ be defined as follows:*

$$f_1(z) := (1-z)^3 \quad \textit{and} \quad f_2(z) := (1-z)^3 e^{-\frac{1+z}{1-z}} \quad (z \in \mathbb{D}).$$

Then f_1, f_2 do not have a greatest common divisor in $W^+(\mathbb{D})$.

Proof. Suppose that d is a gcd and let $f_1 = dq_1$, $f_2 = dq_2$. Then q_1 is not invertible in $W^+(\mathbb{D})$, otherwise f_1 would divide f_2, which is not the case. Since the only zero of f_1 is at $z = 1$, it follows that $q_1(1) = 0$. Similarly, $q_2(1) = 0$. So

q_1 and q_2 belong to the maximal ideal $m_1 := \{f \in W^+(\mathbb{D}) \mid f(1) = 0\}$, which has a bounded approximate identity (see [2] and [11]). Hence by Theorem 2.5 (applied to $(q_1, q_2, 0, 0, 0, \dots)$), there is a common factor $c \in m_1$ of q_1 and q_2. Thus $k := dc$ divides f_1 and f_2. But d is a gcd of f_1, f_2, and so k must divide d, say $dch = d$ for some $h \in W^+(\mathbb{D})$. Since f_1 is never zero on $\mathbb{D}$, neither is d. So we obtain that on $\mathbb{D}$, $ch = 1$. But $c \in m_1$ and h is bounded and continuous on the closed unit disc. So by passing the limit as $z \to 1$ in $ch = 1$, we obtain the contradiction that $0 = 1$. $\square$

Instead of f_1 and f_2 above, we will use the following in the case of $W^+(\mathbb{C}_+)$:

Lemma 2.7. *For* $\text{Re}(s) > 0$*, let*

$$F_1(s) := \left(1 - \frac{1}{s+1}\right)^3 \quad \text{and} \quad F_2(s) := \left(1 - \frac{1}{s+1}\right)^3 e^{-\frac{s+2}{s}}.$$

Then $F_1, F_2 \in W^+(\mathbb{C}_+)$*. Moreover,* $F_1, F_2 \in \mathfrak{M}_0$*.*

Proof. It is easy to see that $F_1 \in W^+(\mathbb{C}_+)$. It was also noted in [10] that f_2 given by $f_2(z) := (1-z)^3 e^{-\frac{1+z}{1-z}}$ $(z \in \mathbb{D})$ belongs to $W^+(\mathbb{D})$. So if the complex numbers a_n $(n \geq 0)$ are defined via

$$(1-z)^3 e^{-\frac{1+z}{1-z}} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}, \tag{1}$$

then $\sum_{k=0}^{\infty} |a_k| < \infty$. But if $\text{Re}(s) > 0$, then $\frac{1}{s+1} \in \mathbb{D}$, and so

$$\begin{aligned} F_2(s) &= \left(1 - \frac{1}{s+1}\right)^3 e^{-\frac{s+2}{s}} = \left(1 - \frac{1}{s+1}\right)^3 e^{-\frac{1+\frac{1}{s+1}}{1-\frac{1}{s+1}}} \\ &= a_0 + a_1 \frac{1}{s+1} + a_2 \left(\frac{1}{s+1}\right)^2 + a_3 \left(\frac{1}{s+1}\right)^3 + \cdots . \end{aligned} \tag{2}$$

We have $\|\frac{1}{s+1}\|_{W^+} = 1$ and so $\|(\frac{1}{s+1})^n\|_{W^+} \leq \|\frac{1}{s+1}\|_{W^+}^n = 1^n = 1$. Since $\sum_{k=0}^{\infty} |a_k| < \infty$, it follows that the series

$$a_0 + a_1 \frac{1}{s+1} + a_2 \left(\frac{1}{s+1}\right)^2 + a_3 \left(\frac{1}{s+1}\right)^3 + \cdots \tag{3}$$

converges in norm to an element in $W^+(\mathbb{C}_+)$. For $f_a \in L^1(0, \infty)$, we have $|\hat{f}_a(s)| \leq \|f_a\|_{L^1}$ $(s \in \mathbb{C}_+)$, and so for every $F \in W^+(\mathbb{C}_+)$, $|F(s)| \leq \|F\|_{W^+}$ $(s \in \mathbb{C}_+)$. Thus norm-convergence implies pointwise convergence. But by (2), the pointwise limit for $\text{Re}(s) > 0$ is in fact F_2. Thus $F_2 \in W^+(\mathbb{C}_+)$.

That $F_1 \in \mathfrak{M}_0$ is trivial. To see that $F_2 \in \mathfrak{M}_0$, we observe that for $s = x+iy \in \mathbb{C}_+, s \neq 0$, $F_2(s) = (\frac{s}{s+1})^3 e^{-1} e^{-\frac{2}{s}}$ and

$$|s^3 e^{-2/s}| = s^3 e^{-\frac{2x}{x^2+y^2}} \to 0 \quad \text{as } s \to 0. \qquad \square$$

We will need the result below (a Nakayama type lemma) in order to prove that $W^+(\mathbb{C}_+)$ is not coherent. An analytic proof can be given along the lines to the analogous result for $W^+(\mathbb{D})$ [10, Lemma 1]:

Lemma 2.8. *Let $\mathfrak{L} \neq (0)$ be an ideal in $W^+(\mathbb{C}_+)$ contained in the maximal ideal $\mathfrak{M}_0$. If $\mathfrak{L} = \mathfrak{L}\mathfrak{M}_0$, that is, if every function $F \in \mathfrak{L}$ can be factorised in a product $F = HG$ of two functions $H \in \mathfrak{L}$ and $G \in \mathfrak{M}_0$, then $\mathfrak{L}$ cannot be finitely generated.*

We will also need the following technical result, the proof of which is basically a repetition of key steps from Browder's proof of Cohen's factorisation theorem; see [1, Theorem 1.6.5, p. 74]. It uses the fact that $\mathfrak{M}_0$ has a bounded approximate identity. For a detailed exposition, see also [18, Lemma 2.8].

Lemma 2.9. *Let $R_1, R_2 \in \mathfrak{M}_0$ and $\delta > 0$. Let $U(W^+(\mathbb{C}_+))$ denote the set of all invertible elements in $W^+(\mathbb{C}_+)$. Then there exists a sequence $(G_n)_{n\in\mathbb{N}}$ in $W^+(\mathbb{C}_+)$ such that*

1. *for all $n \in \mathbb{N}$, $G_n \in U(W^+(\mathbb{C}_+))$;*
2. *$(G_n)_{n\in\mathbb{N}}$ is convergent in $W^+(\mathbb{C}_+)$ to a limit $G \in \mathfrak{M}_0$;*
3. *for all $n \in \mathbb{N}$, $\|G_n^{-1} R_i - G_{n+1}^{-1} R_i\|_{W^+} \leq \delta/2^n$, $i = 1, 2$.*

We now state Lin's result, which will be used to show that $W^+(\mathbb{C}_+)$ is Hermite; [6, Theorem 3, p. 127]. Recall that a topological space, X, is said to be *contractible* if there exists a continuous map $\kappa : X \times [0,1] \to X$ and $x_0 \in X$ such that $\kappa(x, 0) = x$ for all x and $\kappa(x, 1) = x_0$.

Proposition 2.10. *Let R be a commutative Banach algebra with identity. If the maximal ideal space $X(R)$ of the Banach algebra is contractible, then R is a Hermite ring.*

3 $W^+(\mathbb{C}_+)$ is not a GCD domain

Proof of Theorem 1.4. We claim that F_1, F_2 (defined as in Lemma 2.7) have no gcd. Suppose, on the contrary, that D is a gcd, and let $F_1 = DQ_1$ and $F_2 = DQ_2$ with $Q_1, Q_2 \in W^+(\mathbb{C}_+)$.

Step 1. Q_1 is not invertible in $W^+(\mathbb{C}_+)$. If not, then $D = F_1 Q_1^{-1}$ and so $F_2 = DQ_2 = F_1(Q_1^{-1}Q_2)$. In particular, for $s \neq 0$, we have $e^{-\frac{s+2}{s}} = Q_1^{-1}Q_2 \in W^+(\mathbb{C}_+)$, which is not the case, since $\lim_{\mathbb{R}\ni\omega\to 0} e^{-\frac{i\omega+2}{i\omega}}$ does not exist, while $\lim_{\mathbb{R}\ni\omega\to 0} Q_1^{-1}(i\omega)Q_2(i\omega)$ does (because $Q_1^{-1}Q_2 \in W^+(\mathbb{C}_+)$), a contradiction.

Since we have that $\lim_{\mathbb{C}_+\ni s\to\infty} F_1(s) = 1 \neq 0$, it follows from $F_1 = DQ_1$ and the fact that $W^+(\mathbb{C}_+)$-functions have a continuous extension at infinity, that $\lim_{\mathbb{C}_+\ni s\to\infty} Q_1(s) \neq 0$. So by the Corona Theorem for $W^+(\mathbb{C}_+)$ (Proposition 2.1), we conclude that Q_1 has a zero in $\mathbb{C}_+$. But the only zero of F_1 is 0, and since $F_1 = DQ_1$, we have that this zero of Q_1 must be 0. Consequently, $Q_1(0) = 0$, that is, $Q_1 \in \mathfrak{M}_0$.

Step 2. Also, Q_2 is not invertible in $W^+(\mathbb{C}_+)$. Otherwise $D = F_2 Q_2^{-1}$ and $F_1 = F_2(Q_2^{-1}Q_1)$, so that for $s \neq 0$, we have $1 = e^{-\frac{s+2}{s}}(Q_2^{-1}Q_1)$, and so $e^{\frac{s+2}{s}} = Q_2^{-1}Q_1$. But $\lim_{\mathbb{R}\ni\omega\to 0} e^{\frac{i\omega+2}{i\omega}}$ does not exist, while $\lim_{\mathbb{R}\ni\omega\to 0} Q_2^{-1}(i\omega)Q_1(i\omega)$ does (because $Q_2^{-1}Q_1 \in W^+(\mathbb{C}_+)$), a contradiction.

Since we have that $\lim_{\mathbb{C}_+\ni s\to\infty} F_2(s) = e^{-1} \neq 0$, it follows from $F_2 = DQ_2$ that $\lim_{\mathbb{C}_+\ni s\to\infty} Q_2(s) \neq 0$. Again by the Corona Theorem for $W^+(\mathbb{C}_+)$, we conclude that Q_2 has a zero in $\mathbb{C}_+$. But the only zero of F_2 is 0, and since $F_2 = DQ_2$, we have $Q_2 \in \mathfrak{M}_0$ as well.

Step 3. So Q_1 and Q_2 belong to the maximal ideal $\mathfrak{M}_0$ which has a bounded approximate identity, by Proposition 2.4. From Theorem 2.5 (applied to the sequence $(Q_1, Q_2, 0, 0, 0, \dots)$), it follows that there is a common factor $G \in \mathfrak{M}_0$ of Q_1 and Q_2. Thus $K := DG$ divides F_1 and F_2. But D is a gcd of F_1, F_2, and so K must divide D, say $DGH = D$ for some $H \in W^+(\mathbb{C}_+)$. Since F_1 is never zero for $s \neq 0$, neither is D. So we obtain that for $s \neq 0$, $GH = 1$. But $G \in \mathfrak{M}_0$, and H is bounded and continuous in $\mathbb{C}_+$. Hence by passing the limit as $s \to 0$ in $GH = 1$, we obtain the contradiction that $0 = 1$. □

Remark 3.1. (1) In a similar manner, one can also show that $\mathcal{A}$ is not a GCD domain, where $\mathcal{A}$ denotes the set of all functions $F : \mathbb{C}_+ \to \mathbb{C}$ such that

$$F(s) = \hat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in \mathbb{C}_+),$$

where $f_a \in L^1(0,\infty)$, $(f_k)_{k\geq 0} \in \ell^1$, $t_0 = 0$ and $t_k > 0$ for $k = 1, 2 \dots$.

$\mathcal{A}$ is a Banach algebra when it is equipped with pointwise operations and the norm: $\|F\|_{\mathcal{A}} := \|f_a\|_{L^1} + \|(f_k)_{k\geq 0}\|_{\ell^1}$.

(2) Recall that an integral domain R is said to be *pre-Bézout* if for every $a, b \in R$ for which there exists a greatest common divisor d, there exist $x, y \in R$ such that $d = xa + yb$.

However, as opposed to $W^+(\mathbb{C}_+)$, which is a pre-Bézout domain [11, Theorem 1.5], it turns out that $\mathcal{A}$ is not a pre-Bézout domain. Indeed, consider the elements $U_1, U_2 \in \mathcal{A}$ given by

$$U_1(s) := \frac{1}{s+1} \quad \text{and} \quad U_2(s) := e^{-s}.$$

Note that U_1 is an outer function in H^∞ and U_2 an inner function. Then 1 is a greatest common divisor of U_1 and U_2, but if there exist $G_1, G_2 \in \mathcal{A}$ such that $1 = G_1U_1 + G_2U_2$, then by passing to the limit $s \to +\infty$, $s \in \mathbb{R}$, and using

$$\lim_{(\mathbb{R}\ni)s\to\infty} U_1(s) = 0 = \lim_{(\mathbb{R}\ni)s\to\infty} U_2(s),$$

we obtain the contradiction that $1 = 0$.

4 $W^+(\mathbb{C}_+)$ is not coherent

We use the same approach as the one used to show the noncoherence of $W^+(\mathbb{D})$ in [10, Theorem 3, p. 226]. Nevertheless, we record the details here for the sake of convenience of the reader.

Proof of Theorem 1.5. We present two principal ideals $\mathfrak{I}$ and $\mathfrak{J}$ such that $\mathfrak{I} \cap \mathfrak{J}$ is not finitely generated.

Let

$$P(s) = \left(1 - \frac{1}{1+s}\right)^3 \quad \text{and} \quad S(s) = e^{-\frac{s+2}{s}}$$

for $\mathrm{Re}(s) > 0$. Note that $P = F_1$ and $PS = F_2$, the functions in Lemma 2.7. Also, $S \in H^\infty$, since $|S| \le e^{-1}$.

By Lemma 2.7, $F_1, F_2 \in \mathfrak{M}_0$. We define the ideals $\mathfrak{I} = (P)$ and $\mathfrak{J} = (PS)$. Let

$$\mathfrak{K} := \{PSF \mid F \in W^+(\mathbb{C}_+) \text{ and } SF \in W^+(\mathbb{C}_+)\}.$$

We claim that $\mathfrak{K} = \mathfrak{I} \cap \mathfrak{J}$. Trivially $\mathfrak{K} \subset \mathfrak{I} \cap \mathfrak{J}$. To prove the reverse inclusion, let $G \in \mathfrak{I} \cap \mathfrak{J}$. Then there exist two functions F and H in $W^+(\mathbb{C}_+)$ such that $G = PH = PSF$. Hence $SF = H \in W^+(\mathbb{C}_+)$. So $G \in \mathfrak{K}$.

Let $\mathfrak{L}$ denote the ideal

$$\mathfrak{L} := \{F \in W^+(\mathbb{C}_+) \mid SF \in W^+(\mathbb{C}_+)\}.$$

Then $\mathfrak{K} := PS\mathfrak{L}$.

We first show that $\mathfrak{L} \subset \mathfrak{M}_0$. Let $F \in \mathfrak{L}$. We have

$$\lim_{\omega \searrow 0} F(i\omega)S(i\omega) = \lim_{r \searrow 0} F(r)S(r) = 0,$$

since F is bounded in $\mathbb{C}_+$ and $\lim_{r\searrow 0} S(r) = 0$. Since for $\omega \in \mathbb{R} \setminus \{0\}$ we have

$$S(i\omega) = e^{-1}\, e^{-\frac{2}{i\omega}},$$

it follows that $S(i\omega)$ is invertible and $|[S(i\omega)]^{-1}| = e^{-1}$. Thus

$$\lim_{\omega \searrow 0} F(i\omega) = \lim_{\omega \searrow 0} F(i\omega)S(i\omega)[S(i\omega)]^{-1} = 0,$$

and so $F \in \mathfrak{M}_0$. Consequently, $\mathfrak{L} \subset \mathfrak{M}_0$.

We will show that $\mathfrak{L} = \mathfrak{L}\mathfrak{M}_0$. Let $F \in \mathfrak{L}$. Then $F \in \mathfrak{M}_0$. Also, since $|S|$ is bounded by 1 on $\mathrm{Re}(s) > 0$ and $F(0) = 0$, it follows that $SF \in \mathfrak{M}_0$. We would like to factor $F = HG$ with $H \in \mathfrak{L}$ and $G \in \mathfrak{M}_0$. Applying Lemma 2.9 with $R_1 := F \in \mathfrak{M}_0$ and $R_2 := SF \in \mathfrak{M}_0$, for any $\delta > 0$, there exists a sequence $(G_n)_{n\in\mathbb{N}}$ in $W^+(\mathbb{C}_+)$ such that

1. $G_n \in U(W^+(\mathbb{C}_+))$ $(n \in \mathbb{N})$.
2. $(G_n)_{n\in\mathbb{N}}$ is convergent in $W^+(\mathbb{C}_+)$ to a limit $G \in \mathfrak{M}_0$.
3. $\|G_n^{-1}F - G_{n+1}^{-1}F\|_{W^+} \le \frac{\delta}{2^n}$, $\|G_n^{-1}SF - G_{n+1}^{-1}SF\|_{W^+} \le \frac{\delta}{2^n}$ $(n \in \mathbb{N})$.

Put

$$H_n := G_n^{-1}F \quad \text{and} \quad K_n := G_n^{-1}SF.$$

Then $H_n \in \mathfrak{M}_0$. Also $K_n \in \mathfrak{M}_0$. The estimates above imply that $(H_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $W^+(\mathbb{C}_+)$. Since $\mathfrak{M}_0$ is closed, they converge to elements H and K, respectively, in $\mathfrak{M}_0$: $H_n = G_n^{-1}F \to H$ and $K_n = G_n^{-1}SF = SH_n \to K$.

Let $H^\infty(\mathbb{C}_+)$ denote the Hardy space of all bounded analytic functions in the open right half plane equipped with the norm $\|\varphi\|_\infty := \sup_{\mathrm{Re}(s)>0} |\varphi(s)|$, $\varphi \in H^\infty(\mathbb{C}_+)$. If $f_a \in L^1(0,\infty)$ and $f_0 \in \mathbb{C}$, then we have

$$|\hat{f_a}(s) + f_0| \le |\hat{f_a}(s)| + |f_0| \le \|f_a\|_{L^1} + |f_0| \quad (s \in \mathbb{C}_+),$$

and so it follows that $\|F\|_\infty \leq \|F\|_{W^+}$ for all $F \in W^+(\mathbb{C}_+)$. Hence convergence in $W^+(\mathbb{C}_+)$ implies convergence in $H^\infty(\mathbb{C}_+)$, and so

$$\begin{aligned}
H_n &\xrightarrow{H^\infty(\mathbb{C}_+)} H && (\text{since } H_n \xrightarrow{W^+(\mathbb{C}_+)}\ ) \\
SH_n &\xrightarrow{H^\infty(\mathbb{C}_+)} SH && (\text{since } H_n \xrightarrow{W^+(\mathbb{C}_+)} H \text{ and } S \in H^\infty(\mathbb{C}_+)) \\
SH_n &\xrightarrow{H^\infty(\mathbb{C}_+)} K && (\text{since } K_n \xrightarrow{W^+(\mathbb{C}_+)} K)
\end{aligned}$$

and so $SH = K$. Also, in $W^+(\mathbb{C}_+)$ norm we have

$$F = \lim_{n\to\infty} H_n G_n = HG.$$

Since H and $SH = K$ belong to $\mathfrak{M}_0 \subset W^+(\mathbb{C}_+)$, we see that $H \in \mathfrak{L}$. Moreover, as $G \in \mathfrak{M}_0$, we have got the desired factorisation and $\mathfrak{L} = \mathfrak{L}\mathfrak{M}_0$.

But $\mathfrak{L} \neq (0)$, since $P \in \mathfrak{L}$. By Lemma 2.8, it follows that $\mathfrak{L}$ cannot be finitely generated. Therefore, $PS\mathfrak{L} = \mathfrak{I} \cap \mathfrak{J}$ is not finitely generated. □

Remark 4.1. 1. The ideal $\mathfrak{L}$ in the above proof can be interpreted as an *ideal of denominators*; see [4, p. 396]. Indeed, using the fact that $PS \in W^+(\mathbb{C}_+)$, we have $S \in Q(W^+(\mathbb{C}_+))$, where $Q(W^+(\mathbb{C}_+))$ denotes the field of fractions of $W^+(\mathbb{C}_+)$. The *ideal of denominators* $\mathfrak{L}$ of S, namely

$$\mathfrak{L} = \{d \in W^+(\mathbb{C}_+) \mid dS \in W^+(\mathbb{C}_+)\}$$

is the ideal of $W^+(\mathbb{C}_+)$ consisting of all possible denominators of S, together with 0, when written as a fraction of elements from $W^+(\mathbb{C}_+)$; see the book by Matusumura [7].

2. Following the proof in [10], the second author had proved that $\mathcal{A}$ is also not coherent [18]. The following functions were used there:

$$P(s) = \frac{(1-e^{-s})^3}{s+1} \quad \text{and} \quad S(s) = e^{-\frac{1+e^{-s}}{1-e^{-s}}}.$$

Another way to see that $W^+(\mathbb{C}_+)$ is not a GCD domain, is to use the following observation together with Theorem 1.5 and the fact that $W^+(\mathbb{C}_+)$ is a pre-Bézout domain (see [11]):

Observation 4.2. *Let R be a pre-Bézout domain. Let $f, g \in R$. Suppose that the intersection of the associated principal ideals (f) and (g) is not finitely generated. Then f and g have no greatest common divisor.*

Proof. Assuming the contrary, let d be a greatest common divisor of f and g and write $f = dF, g = dG$. Then $gcd(F, G) = 1$. We claim that $(f) \cap (g) = (dFG)$. In fact, one trivially has that $(dFG) \subseteq (f) \cap (g)$. Now let $h = xdF = ydG$. Then $d(xF - yG) = 0$. Since there are now divisors of zero, and $d \neq 0$, $xF = yG$. Now R has the pre-Bézout property; hence $1 = aF + bG$ for some $a, b \in R$. Thus $y = yaF + b(yG) = yaF + b(xF) = F(ya + bx)$. So $h = y(dG) = F(ya + bx)dG \in (dFG)$. So $(f) \cap (g) \subseteq (dFG)$. □

5 $W^+(\mathbb{C}_+)$ is a Hermite ring

Proof of Theorem 1.6. Consider the algebra

$$A^+ := \left\{ F\left(\frac{1+z}{1-z}\right) \ (z \in \mathbb{D}) \ \Big| \ F \in W^+(\mathbb{C}_+) \right\},$$

with pointwise operations and the same norm as in $W^+(\mathbb{C}_+)$, that is,

$$\left\| z \mapsto F\left(\frac{1+z}{1-z}\right) \right\|_{A^+} := \|F\|_{W^+(\mathbb{C}_+)}, \quad F \in W^+(\mathbb{C}_+).$$

Then this is a Banach algebra that is isometrically isomorphic to $W^+(\mathbb{C}_+)$. Since $\lim_{s \to \infty, s \in \mathbb{C}_+} F(s)$ exists for each element $F \in W^+(\mathbb{C}_+)$, we see that every $f \in A^+$ admits a continuous extension to $z = 1$ and hence $A^+ \subseteq A(\mathbb{D})$. Now we will show that the maximal ideal space of A^+ is homeomorphic to the closed unit disc $\overline{\mathbb{D}}$ in $\mathbb{C}$.

Using the formula

$$1 = -\frac{f - f(a)}{f(a)} + \frac{f}{f(a)}$$

whenever $f \in A^+$ and $f(a) \neq 0, a \in \overline{\mathbb{D}}$, we see that

$$M_a := \{f \in A^+ \mid f(a) = 0\}$$

is a maximal ideal in A^+. But these are all the maximal ideals of A^+. In fact, let M be any maximal ideal in A^+ and suppose that M is not contained in M_a for any $a \in \overline{\mathbb{D}}$. Then by a compactness argument, there exists finitely many functions $f_1, \ldots, f_n \in A^+$ so that $\sum_{j=1}^n |f_j| \geq \delta > 0$ on $\overline{\mathbb{D}}$. Moving back to the algebra $W^+(\mathbb{C}_+)$, we see from the corona theorem 2.1 that the ideal generated by the functions $w \mapsto f(\frac{w-1}{w+1})$, $\mathrm{Re}(w) > 0$ is $W^+(\mathbb{C}_+)$. Hence $(f_1, \ldots, f_n)$ is not proper either. This contradiction shows that $M \subseteq M_a$ for some $a \in \overline{\mathbb{D}}$. The maximality of M_a now implies that $M = M_a$. Finally, since $\overline{\mathbb{D}}$ is compact, it is easy to see that the map $\Psi : \overline{\mathbb{D}} \to X(A^+), a \mapsto \Phi_a$ is a homeomorphism; here $X(A^+)$ is the space of nonzero multiplicative linear functionals on A^+ endowed with the Gelfand topology and Φ_a is the evaluation functional at a.

To sum up, we have shown that the maximal ideal space of A^+ can be identified with $\overline{\mathbb{D}}$.

Since $\overline{\mathbb{D}}$ is contractible, it follows that the maximal ideal space of $W^+(\mathbb{C}_+)$ is contractible. The claim that $W^+(\mathbb{C}_+)$ (and also A^+) are Hermite rings now follows via Lin's result given in Proposition 2.10. □

Remark 5.1. 1. The second author proved that $\mathcal{A}$ is also Hermite [19].

2. It can be shown that a commutative ring R with identity, and having Bass stable rank equal to 1 is a Hermite ring; see [22, p. 3155]. It is known that the Bass stable rank of $W^+(\mathbb{C}_+)$ is 1; see [9, Theorem 1.2]. So this gives another proof of Theorem 1.6.

We conclude this note by showing that A^+ is not contained in $W^+(\mathbb{D})$.

Proposition 5.2. *The following assertions hold:*

1. *If $f \in W^+(\mathbb{D})$, then $f(\frac{1-z}{2}) \in A^+$.*
2. *For $\alpha > 0$,*
$$f_\alpha(z) = (1-z)^\alpha e^{-\frac{1+z}{1-z}} \in W^+(\mathbb{D}) \quad \Longleftrightarrow \quad \alpha > 1/2$$
but $f_\alpha \in A^+$ for all $\alpha > 0$;
3. $f_\alpha\left(\frac{s-1}{s+1}\right) = \frac{2^\alpha}{(1+s)^\alpha} e^{-s} \in \widehat{L^1(0,\infty)}$;
4. $\|z\|_{A^+} = 3$.

Proof. To show 1 and 4, we have to observe that $z = F\left(\frac{z+1}{z-1}\right)$, where $F(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1} = 1 - 2\widehat{e^{-t}}$. Now let $f \in W^+(\mathbb{D})$, say $f(z) = \sum_{n=0}^\infty a_n z^n$ with $\sum_{n=0}^\infty |a_n| < \infty$. As in the proof of Lemma 2.7, we see that the function F given by $F(s) = f\left(\frac{1}{1+s}\right) \in W^+(\mathbb{C}_+)$. Replacing s by $\frac{1+z}{1-z}$, $z \in \mathbb{D}$, we get that $f\left(\frac{1-z}{2}\right) = F\left(\frac{1+z}{1-z}\right) \in A^+$. But it is obvious that $h(z) := \sum a_n \left(\frac{1-z}{2}\right)^n \in W^+(\mathbb{D})$ since $\left\|\frac{1-z}{2}\right\|_{W^+(\mathbb{D})} = 1$, and so the series for h is a Cauchy-sequence in $W^+(\mathbb{D})$.

For 3, we observe that trivially, $f_\alpha\left(\frac{s-1}{s+1}\right) = \frac{2^\alpha}{(1+s)^\alpha} e^{-s} =: G(s)$. But $G(s)$ is the Laplace transform of the $L^1(0,\infty)$-function

$$g(t) = 2^\alpha \frac{e^{-(t-1)} u(t-1)}{\Gamma(1-\alpha)(t-1)^{1-\alpha}},$$

where $u(t)$ is the Heaviside function given by

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Thus $f_\alpha \in A^+$ for each $\alpha > 0$.

The following proof that $f_\alpha \in W^+(\mathbb{D})$ is from Udo Klein [5]. Let

$$g_\beta(z) := (1-z)^{-\beta-1} \exp \frac{\mu z}{z-1}, \quad \beta := -\alpha - 1, \quad \mu = 2.$$

Then $f_\alpha = e^{-1} g_\beta$. Hence it will suffice to show that $g_\beta \in W^+(\mathbb{D})$ if and only $\beta < -\frac{3}{2}$. The Taylor coefficients of g though are (more or less by definition) generalised Laguerre polynomials. Indeed, we have the following expansion (see [20, p. 100]):

$$g_\beta(z) = \sum_{n=0}^{\infty} L_n^{(\beta)}(\mu) z^n.$$

The result now follows since the asymptotic behaviour of the terms $L_n^{(\beta)}(\mu)$ is given by Fejér's formula (see [20, p. 198])

$$L_n^{(\beta)}(\mu) = \frac{1}{\sqrt{\pi}} e^{\frac{\mu}{2}} \mu^{-\frac{\beta}{2}-\frac{1}{4}} n^{\frac{\beta}{2}-\frac{1}{4}} \cos\left(\sqrt{4n\mu} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) + O(n^{\frac{\beta}{2}-\frac{3}{4}}). \quad \square$$

Acknowledgments. The authors thank the anonymous referee for the careful review and for comments that improved the presentation of the article.

Bibliography

[1] A. Browder. *Introduction to Function Algebras*. W.A. Benjamin, Inc., New York, 1969.

[2] V. M. Faĭvyševskiĭ. The structure of the ideals of certain algebras of analytic functions. (Russian) *Dokl. Akad. Nauk SSSR*, 211:537–539, 1973. English translation in *Soviet Math. Dokl.*, 14:1067–1070, 1973.

[3] I. Gelfand, D. Raikov and G. Shilov. *Commutative Normed Rings*. Chelsea Publ. Comp. New York, 1964.

[4] V. P. Havin, S. V. Hruščëv and N. K. Nikol'skiĭ (Editors). *Linear and complex analysis problem book*. Lecture Notes in Mathematics, 1043, Springer-Verlag, Berlin, 1984.

[5] U. Klein. *On the Taylor coefficients of a class of smooth functions*. Unpublished manuscript, 1995.

[6] V. Ya. Lin. Holomorphic fiberings and multivalued functions of elements of a Banach algebra. *Functional Analysis and its Applications*, no. 2, 7:122–128, 1973, English translation.

[7] H. Matsumura. *Commutative ring theory.* Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1989.

[8] W. S. McVoy and L. A. Rubel. Coherence of some rings of functions. *Journal of Functional Analysis*, no. 1, 21:76–87, 1976.

[9] K. M. Mikkola and A. J. Sasane. Bass and topological stable ranks of complex and real algebras of measures, functions and sequences. To appear in *Complex Analysis and Operator Theory*.

[10] R. Mortini and M. von Renteln. Ideals in the Wiener algebra W^+. *Journal of the Australian Mathematical Society (Series A)*, 46: 220–228, 1990.

[11] R. Mortini and A. J. Sasane. On the pre-Bézout property of Wiener algebras on the disc and the half-plane; to appear in New Zealand J. Math.

[12] N. K. Nikol'skiĭ. *Treatise on the Shift Operator. Spectral Function Theory. With an appendix by S. V. Khrushchëv and V. V. Peller.* Translated from the Russian by Jaak Peetre. Grundlehren der Mathematischen Wissenschaften, 273, Springer-Verlag, Berlin, 1986.

[13] A. Quadrat. On a generalization of the Youla-Kučera parametrization. Part I: The fractional ideal approach to SISO systems. *Systems and Control Letters*, 50:135–148, no. 2, 2003.

[14] A. Quadrat. An introduction to internal stabilization of linear infinite dimensional systems. École Internationale d'Automatique de Lille (02–06/09/02): Contrôle de systèmes à paramètres répartis: Théorie et Applications, *Revue e-STA*, vol. 1, no. 1, 2002.

[15] A. Quadrat. The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. Part I: (Weakly) doubly coprime factorizations. *SIAM Journal on Control and Optimization*, 42:266–299, 2003.

[16] M. von Renteln. Divisibility structure and finitely generated ideals in the disc algebra. *Monatshefte für Mathematik*, 82:51–56, 1976.

[17] M. von Renteln. Hauptideale und äussere Funktionen im Ring H^∞. (German) Archiv der Mathematik (Basel), no. 5, 28:519–524, 1977.

[18] A. J. Sasane. Noncoherence of a causal Wiener algebra used in control theory. *Abstract and Applied Analysis*, Volume 2008, Article ID 459310, 13 pages.

[19] A. J. Sasane. The Hermite property of a causal Wiener algebra used in control theory. To appear in *Complex Analysis and Operator Theory*.

Note 2 Please update [19], if possible.

[20] G. Szegő. *Orthogonal Polynomials*. Amererican Mathematical Society Colloquium Publications, 23, New York, 1959.

[21] V. Tolokonnikov. Extension problem to an invertible matrix. *Proc. Amer. Math. Soc.* 117 (1993), 1023–1030.

[22] V. Tolokonnikov. Stable rank of H^∞ in multiply connected domains. *Proc. Amer. Math. Soc.* 123 (1995), no. 10, 3151–3156.

[23] N. Th. Varopoulos. Sur les formes positives d'une algèbre de Banach. (French) *Comptes Rendus Acad. Sci. Paris* 258:2465–2467, Groupe 1, 1964.

[24] M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. MIT Press Series in Signal Processing, Optimization, and Control, 7, MIT Press, Cambridge, MA, 1985.

Received December 9, 2008; revised June 1, 2009.

Author information

Raymond Mortini, Département de Mathématiques et LMAM, Université Paul Verlaine, Ile du Saulcy, 57045 Metz, France.
E-mail: mortini@math.univ-metz.fr

Amol Sasane, Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom.
E-mail: A.J.Sasane@lse.ac.uk