IDEAL STRUCTURE AND STABLE RANK OF $\mathbb{C}e + \ell^2(I)$ WITH THE HADAMARD PRODUCT

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Abstract. Let I be any index set. We consider the Banach algebra $\mathbb{C}e + \ell^2(I)$ with the Hadamard product, and prove that its Bass and topological stable ranks are both equal to 1. We also characterize divisors, maximal ideals, closed ideals and closed principal ideals. For $I = \mathbb{N}$ we also characterize all prime z-ideals in this Banach algebra.

1. Introduction

The *Hadamard product* $f \odot g$ of power series f and g is defined by

$$(f \odot g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n, \quad \text{where } f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Algebras of power series with the Hadamard product, and the structure of their ideals have been studied in several works; see for example Brooks [1], Caveny [3], Brück and Müller [2], Render and Sauer [8], and Render [7].

In this article, we study the algebra of square summable sequences, indexed by an arbitrary index set I with respect to the Hadamard product. We use the idea of summability of a series where the terms depend on any set I of indices whatsoever and where consequently the terms of the series are not ordered; see for example Laurent Schwartz [10, I.I.I]. This definition is recalled below:

Definition 1.1. Let I be any set of indices and $(u_i)_{i\in I}$ be a family of complex numbers parameterized by the set of indices I. Then the series $\sum_{i\in I} u_i$ is said to be *summable with sum* S and is written $\sum_{i\in I} u_i = S$ if for every $\epsilon > 0$, there is a finite subset of indices $J \subset I$ such that for any finite subset of indices K with $J \subset K$, we have $|S - S_K| \leq \epsilon$, where $S_K := \sum_{i\in I} u_i$.

It can be shown that if $\sum_{i\in I} u_i$ is summable with sum S, then all the terms are zero except for a at most countable subset $C \subset I$, that is, $u_i = 0$ for all $i \notin C$.

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Definition 1.2. Define $\ell^2(I)$ to be the set of all families of complex numbers $a = (a_i)_{i\in I}$ parameterized by I such that $\sum_{i\in I} |a_i|^2$ is summable.

With addition and scalar multiplication defined term-wise, this is a complex vector space, and it becomes a complex algebra with the Hadamard product, defined by:

$$(a \odot b)_i = a_i b_i \ \ (i \in I), \quad a = (a_i)_{i\in I}, b = (b_i)_{i\in I} \in \ell^2(I).$$

Moreover, it is a Banach algebra with the norm

$$\|a\|_2 := \sqrt{\sum_{i\in I} |a_i|^2}.$$

However it does not have an identity element. We unitize the Banach algebra by attaching the identity element e, to obtain the Banach algebra $\mathbb{C}e+\ell^2(I)$, consisting of all expressions of the type $\alpha e + a$, where $\alpha \in \mathbb{C}$ and $a \in \ell^2(I)$. The multiplication $\odot$ is extended from $\ell^2(I)$ to $\mathbb{C}e + \ell^2(I)$ as follows: if $\alpha, \beta \in \mathbb{C}$ and $a, b \in \ell^2(I)$, then

$$(\alpha e + a) \odot (\beta e + b) = \alpha\beta e + \beta a + \alpha b + a \odot b.$$

The norm on $\mathbb{C}e + \ell^2(I)$ is given by $\|\alpha e + a\| = |\alpha| + \|a\|_2$, for $\alpha \in \mathbb{C}$ and $a \in \ell^2(I)$. We will denote the Banach algebra $\mathbb{C}e + \ell^2(I)$ by $\mathcal{A}$. For a given element $x = \alpha e + a \in \mathcal{A}$, where $\alpha \in \mathbb{C}$ and $a = (a_i)_{i\in I} \in \ell^2(I)$, we call the a_i's the *Fourier coefficients* of a (and of x).

A standard model of $\mathcal{A}$ in case $I = \mathbb{N}$ by holomorphic functions in the disk $\mathbb{D} = \{z \mid |z| < 1\}$ can be constructed as follows. The elements of $\mathcal{A}$ can be viewed as functions f of the form

$$f(z) = \alpha \frac{1}{1-z} + \sum_{n=1}^{\infty} a_n z^{n-1} = \sum_{n=1}^{\infty} (\alpha + a_n) z^{n-1} \quad (z \in \mathbb{D}),$$

where $\alpha \in \mathbb{C}$, and

$$\sum_{n=1}^{\infty} a_n z^{n-1} \in H^2(\mathbb{D}).$$

Here $H^2(\mathbb{D})$ denotes the Hardy space of the unit disk. The multiplication $\odot$ in $\ell^2(\mathbb{N})$ now corresponds to convolution in $H^2(\mathbb{D})$. The unit element in this model for $\mathcal{A}$ is the function e, given by

$$e(z) = \frac{1}{1-z} \quad (z \in \mathbb{D}).$$

Our main results are the following:

(1) In Section 2, we describe the maximal ideal space of $\mathcal{A}$ as a topological space (when it is equipped with the Gelfand topology). We show that the maximal ideal space is homeomorphic with the Alexandroff compactification I_∞ of the index set I, where I is given the discrete

topology. Moreover, we show that the covering dimension of the maximal ideal space of $\mathcal{A}$ is 0.

(2) We show in Section 3 that the Bass and topological stable rank of $\mathcal{A}$ are both equal to 1. Moreover, $\mathcal{A}$ has unit 1-stable range.

(3) In Section 4 a necessary and sufficient condition for $x \in \mathcal{A}$ to be a divisor of a given $z \in \mathcal{A}$ is given.

(4) In Section 5, we characterize all closed ideals of $\mathcal{A}$. Moreover, we describe closed principal ideals.

(5) In Section 6, we study prime ideals of $\mathcal{A}$. In particular, we investigate which z-ideals are prime, and prove that every prime ideal is contained in $\ker \varphi_\infty = \ell^2(I)$. Finally, when I is countable, we establish a correspondence between free ultrafilters on I and all nonmaximal, prime z-ideals of $\mathcal{A}$ contained in $\ker \varphi_\infty = \ell^2(I)$.

2. The maximal ideal space of $\mathcal{A}$

In this section we will describe the maximal ideal space of $\mathcal{A}$. In particular, we will show that the maximal ideal space of $\mathcal{A}$ equipped with the Gelfand topology can be identified with the one point Alexandroff compactification I_∞ of I, where I is given the discrete topology.

Theorem 2.1. *The maximal ideal space of $\mathcal{A}$ is given by*

$$\Delta = \{\varphi_i | i \in I\} \cup \{\varphi_\infty\},$$

where for $\alpha \in \mathbb{C}$ and $a = (a_i)_{i \in I} \in \ell^2(I)$,

$$\begin{aligned} \varphi_i(\alpha e + a) &:= \alpha + a_i \quad (i \in I), \\ \varphi_\infty(\alpha e + a) &:= \alpha. \end{aligned}$$

Proof. One can check that φ_i $(i \in I)$ as well as φ_∞ are complex homomorphisms.

Suppose on the other hand, that φ is a complex homomorphism. For $j \in I$, define $e_j \in \ell^2(I)$ by

$$e_j(i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1}$$

We note that

$$e_i \odot e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus

$$\varphi(e_i)\varphi(e_j) = \varphi(e_i \odot e_j) = \begin{cases} \varphi(e_i) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence it follows that $\varphi(e_i) \in \{0, 1\}$ for all $i \in I$. We have the following two cases:

$\underline{1}^\circ$ There exists an index $i_* \in I$ such that $\varphi(e_{i_*}) = 1$. For $j \in I$, we then have

$$\varphi(e_j) = \varphi(e_j)\cdot 1 = \varphi(e_j)\cdot\varphi(e_{i_*}) = \left\{\begin{array}{ll} 1 & \text{if } j = i_*, \\ 0 & \text{if } j \neq i_*. \end{array}\right.$$

Because $\varphi(e) = 1$, we conclude $\varphi(\alpha e + a) = \alpha + \varphi(a) = \alpha + a_{i_*} = \varphi_{i_*}(\alpha e + a)$, for all $\alpha \in \mathbb{C}$ and $a \in \ell^2(I)$. Consequently, $\varphi = \varphi_{i_*}$.

$\underline{2}^\circ$ For all indices $i \in I$, we have $\varphi(e_i) = 0$. By using the continuity of φ we then have

$$\varphi(\alpha e + a) = \alpha + \sum_{i\in I} a_i \varphi(e_i) = \alpha = \varphi_\infty(\alpha e + a),$$

for all $\alpha \in \mathbb{C}$ and $a \in \ell^2(I)$. Consequently, $\varphi = \varphi_\infty$.

This completes the proof. $\square$

Analogous to [7, Theorem 4], from the functional calculus for Banach algebras, we have the following consequence of the above Theorem 2.1:

Corollary 2.2. *Let* $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ *and* $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. *Suppose that* U *is an open neighbourhood of* $0 \in \mathbb{C}$ *and* $\varphi : U \to \mathbb{C}$ *is holomorphic with* $\varphi(0) = 0$. *Let*

$$f = \sum_{\underline{k}\in\mathbb{N}_0^n} a_{\underline{k}} \underline{z}^{\underline{k}} \in H^2(\mathbb{D}^n)$$

(here $\underline{z}^{\underline{k}} := z_1^{k_1}\dots z_n^{k_n}$ *for* $\underline{k} = (k_1,\dots,k_n) \in \mathbb{N}_0^n$*), be such that* $a_{\underline{k}} \in U$ *for all* $\underline{k} \in \mathbb{N}_0^n$. *Then*

$$F := \sum_{\underline{k}\in\mathbb{N}_0^n} \varphi(a_{\underline{k}}) \underline{z}^{\underline{k}} \in H^2(\mathbb{D}^n).$$

(Here $H^2(\mathbb{D}^n)$ denotes the Hardy space of the polydisk $\mathbb{D}^n$.)

Proof. This follows from the fact that the spectrum of f (when f is considered as an element in the Banach algebra $\mathcal{A}$ with $I = \mathbb{N}_0^n$), is contained in U. $\square$

Endowed with the weak-$*$ topology τ_*, (Δ, τ_*) is a compact Hausdorff space.

Lemma 2.3. *All the one point sets* $\{\varphi_i\}$, $i \in I$ *are clopen sets in* (Δ, τ_*). *If* I *is an infinite set, then* $\{\varphi_\infty\}$ *is closed, but not open in* (Δ, τ_*).

Proof. By the very definition of the weak-$*$ topology, given an $a \in \ell^2(I)$, its Gelfand transform, $\widehat{a}$ is a continuous function on Δ. Let us now take a equal to $e_i \in \ell^2(I)$ for a fixed i (where we use the notation from (1)). We have

$$\widehat{a}(\varphi) = \varphi(a) = \varphi(e_i) = \left\{\begin{array}{ll} 1 & \text{if } \varphi = \varphi_i, \\ 0 & \text{if } \varphi = \varphi_j \text{ and } j \neq i, \\ 0 & \text{if } \varphi = \varphi_\infty. \end{array}\right.$$

Thus if $D\left(1,\frac{1}{2}\right)$ denotes the open disk in $\mathbb{C}$ around 1 with radius $\frac{1}{2}$, then

$$\{\varphi_i\}=\widehat{e_i}^{-1}\left(D\left(1,\frac{1}{2}\right)\right)$$

is open. Also, if $D\left(0,\frac{1}{2}\right)$ denotes the open disk in $\mathbb{C}$ around 0 of radius $\frac{1}{2}$, then

$$\Delta\setminus\{\varphi_i\}=\widehat{e_i}^{-1}\left(D\left(0,\frac{1}{2}\right)\right)$$

is open as well. So $\{\varphi_i\}$, $i\in I$ are clopen sets in (Δ,τ_*).

Since $\{\varphi_i\}$, $i\in I$ are all open, so is their union. Hence

$$\varphi_\infty=\Delta\setminus\left(\bigcup_{i\in I}\{\varphi_i\}\right)$$

is closed. If $\{\varphi_\infty\}$ is also open, then the weak-$*$ topology would be the discrete topology (that is, when every subset is open). In the discrete topology only sets with finitely many points are compact. So if I is infinite, $\{\varphi_\infty\}$ cannot be open, since we know that Δ is compact, and it has an infinite number of elements. □

We now recall the construction of the Alexandroff compactification. From now on, we will assume that I is an infinite index set. Equipped with the discrete topology, I is obviously not compact. Hence we take a new element, say ∞, and define the *Alexandroff topology* τ_c on $I_\infty:=I\cup\{\infty\}$ as follows:

(1) All open sets in I are open sets in I_∞.
(2) All sets of the form $\{\infty\}\cup U$ are open, where U is open in I and $I\setminus U$ is compact.

The restriction of τ_c to I is the discrete topology, and so the only compact subsets of I are finite sets. All open sets V in τ_c with $\infty\in V$ have the form $V=\{\infty\}\cup J$, where J is all of I except for a finite number of points.

Theorem 2.4. *The topological spaces* (Δ,τ_*) *and* (I_∞,τ_c) *are homeomorphic.*

Proof. We consider the embedding $\iota:(\Delta,\tau_*)\to(I_\infty,\tau_c)$ given by $\iota(\varphi_i)=i$ $(i\in I)$, and $\iota(\varphi_\infty)=\infty$. It is bijective.

We prove that ι is continuous by showing that pre-images of open sets are open. This is trivial for open subsets of I. Now let V be an open set in (I_∞,τ_c) such that $\infty\in V$. Then $V=\{\infty\}\cup J$, where $J=I\setminus K$ for some finite subset K of I. We have

$$\iota^{-1}(V)=\{\varphi_\infty\}\cup\{\varphi_i\mid i\in I\setminus K\}=\Delta\setminus\underbrace{\left(\bigcup_{i\in K}\{\varphi_i\}\right)}_{=:F}.$$

Since each $\{\varphi_i\}$ $(i\in K)$ is closed, and since K is finite, the set F is closed as well. Hence $\iota^{-1}(V)=\Delta\setminus F$ is open in Δ. Consequently, ι is continuous.

By the topological result that a one-to-one continuous function between compact Hausdorff spaces has a continuous inverse, we conclude that ι must be a homeomorphism. □

Theorem 2.5. *The covering dimension of* (Δ, τ_*) *is zero.*

Proof. Since the topological spaces (Δ, τ_*) and (I_∞, τ_c) are homeomorphic, we simply prove that the covering dimension of (I_∞, τ_c) is zero. We know from [6, Corollary, p. 192] that a normal space has covering dimension zero if and only if for all open sets U and all closed sets F such that $F \subset U$, there exists an open set V with empty boundary such that $F \subset V \subset U$. So assume that F is closed and U is open in (I_∞, τ_c) and $F \subset U$. Then we have the following two cases:

$\underline{1}^\circ$ $\infty \notin F$. But F is closed subset of a compact Hausdorff space. Hence F is compact in I_∞, and so in I. Thus F is a finite set, since in I we have the discrete topology. But then $V := F$ is clopen in I, and so it has empty boundary.

$\underline{2}^\circ$ $\infty \in F$. Then ∞ belongs to the open set U, and so $U = \{\infty\} \cup J$, where J is all of I except for a finite set K. But the boundary of U is the boundary of its complement. Thus the boundary of U is the boundary of the finite set K, which is clopen, and therefore it is void. Consequently, $V := U$ does the job.

This completes the proof. □

3. The Bass and topological stable ranks of $\mathcal{A}$

In this section, we prove that the topological stable rank of $\mathcal{A}$ is 1. It follows that the Bass stable rank of $\mathcal{A}$ is then equal to 1 as well. We recall the pertinent definitions below:

Definition 3.1. Let R be a commutative ring with an identity element, denoted by 1. Let $n \in \mathbb{N}$. An element $a = (a_1, \dots, a_n) \in R^n$ is called *unimodular* if there exists a $b = (b_1, \dots, b_n) \in R^n$ such that

$$\sum_{k=1}^{n} b_k a_k = 1.$$

We denote by $U_n(R)$ the set of unimodular elements of R^n.

We say that $a = (a_1, \dots, a_n) \in U_n(R)$ is *reducible*, if there exist elements $h_1, \dots, h_{n-1} \in R$ such that

$$(a_1 + h_1 a_n, \dots, a_{n-1} + h_{n-1} a_n) \in U_{n-1}(R).$$

The *Bass stable rank of* R, is the least $n \in \mathbb{N}$ such that every $a \in U_{n+1}(R)$ is reducible, and it is infinite if no such integer n exists.

Now let R denote a commutative unital Banach algebra. The *topological stable rank of* R, is the minimum $n \in \mathbb{N}$ such that $U_n(R)$ is dense in R^n, and it is infinite if no such integer exists.

Theorem 3.2. *The topological stable rank of $\mathcal{A}$ is* 1.

Proof. We prove that the invertible elements of the Banach algebra $\mathcal{A}$ are dense. Suppose that $\alpha \in \mathbb{C}$ and $a \in \ell^2(I)$ and let $\epsilon > 0$.

First choose a nonzero $\beta \in \mathbb{C}$ such that $|\alpha - \beta| < \frac{\epsilon}{2}$. Next choose a finite subset $J \subset I$ such that for all $i \in I \setminus J$, $|a_i| < \frac{\beta}{2}$. (Such a choice of J is possible; see [10, Note, p. 18].)

Next choose the finitely many complex numbers b_j ($j \in J$) such that

$$|\beta + b_j| > 0 \text{ and } \sum_{j \in J} |a_j - b_j|^2 < \frac{\epsilon^2}{4}.$$

Finally, for $i \in I \setminus J$, define the complex numbers b_i by $b_i = a_i$. Then $b := (b_i)_{i \in I} \in \ell^2(I)$.

We claim that $\beta e + b$ is invertible in $\mathcal{A}$. Indeed for $i \in I \setminus J$, we have

$$|\varphi_i(\beta e + b)| = |\beta + b_i| = |\beta + a_i| \geq |\beta| - |a_i| \geq |\beta| - \frac{|\beta|}{2} = \frac{|\beta|}{2} > 0.$$

Furthermore, for the finitely many $j \in J$, we have $|\varphi_j(\beta e + b)| = |\beta + b_j| > 0$. Finally, $|\varphi_\infty(\beta e + b)| = |\beta| > 0$. Hence

$$\inf_{\varphi \in \Delta} |\varphi(\beta e + b)| > 0,$$

and so $\beta e + b$ is invertible in $\mathcal{A}$. Moreover,

$$\|(\alpha e + a) - (\beta e + b)\| = |\alpha - \beta| + \|a - b\|_2 < \frac{\epsilon}{2} + \sqrt{\sum_{j \in J} |a_j - b_j|^2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

Recall that a commutative ring R with an identity element is said to have *unit* 1-*stable range* if whenever $a, b \in R$ satisfy $aR + bR = 1$, there exist units $u, v \in R$ such that $au + bv = 1$. By Theorem 3.2, we can also show that in fact the ring $\mathcal{A}$ has unit 1-stable range:

Corollary 3.3. *$\mathcal{A}$ has unit* 1-*stable range.*

Proof. Suppose that the pair $(a, b) \in \mathcal{A}^2$ is unimodular, that is, there exist $x, y \in \mathcal{A}$ such that $x \odot a + y \odot b = e$. Since the topological stable rank of $\mathcal{A}$ is 1, it follows that we can approximate x, y by units u, v, respectively, to any degree of accuracy. We have

$$u \odot a + y \odot b = e - \underbrace{((x - u) \odot a + (y - v) \odot b)}_{=:h \in \mathcal{A}}.$$

By choosing u, v sufficiently close to x, y, respectively, we can ensure that $\|h\| < 1$, and so $(e - h)^{-1} \in \mathcal{A}$. Hence $U \odot a + V \odot b = e$, where $U := u \odot (e - h)^{-1}$ and $V := v \odot (1 - h)^{-1}$ are units. Consequently, $\mathcal{A}$ has unit 1-stable range. □

In fact for any real or complex Banach algebra R with topological stable rank 1, a similar proof as that of Corollary 3.3 shows that R also has unit 1-stable range.

Corollary 3.3 is in sharp contrast to a result of the first author and Raymond Mortini [9, Corollary 3], where it was shown that if $H(G)$ denotes the ring of holomorphic functions in a planar domain $G \subset \mathbb{C}$, equipped with pointwise addition and *pointwise* multiplication, then every subring R such that

$$\mathbb{C} \subsetneq R \subset H(G)$$

does not have unit 1-stable range!

We also note that Corollary 3.3 implies in particular that the Bass stable rank of $\mathcal{A}$ is 1.

Corollary 3.4. *The Bass stable rank of $\mathcal{A}$ is* 1.

4. Divisors in $\mathcal{A}$

In this section we give a necessary and sufficient condition for $x \in \mathcal{A}$ to be a divisor of a given $z \in \mathcal{A}$. Define the *zero set* of the Gelfand transform as

$$Z(\widehat{x}) := \{\varphi \in I_\infty \mid \widehat{x}(\varphi) = 0\}.$$

For a divisor x of z, there exists a $y \in \mathcal{A}$ such that $x \odot y = z$, which implies

$$Z(\widehat{z}) = Z(\widehat{xy}) = Z(\widehat{x}) \cup Z(\widehat{y})$$

and so $Z(\widehat{x}) \subset Z(\widehat{z})$.

Theorem 4.1. *Let the index set I be infinite, $x = \alpha e + a, z = \gamma e + c \in \mathcal{A}$, $Z(\widehat{x}) \subset Z(\widehat{z})$, and assume that $\alpha \neq 0$. Then x divides z, that is, there exists $y = \beta e + b \in \mathcal{A}$ such that $x \odot y = z$.*

Proof. The equation $x \odot y = z$ is equivalent to

$$(\alpha + a_i)(\beta + b_i) = (\gamma + c_i) \ \ (i \in I) \quad \text{ and } \quad \alpha\beta = \gamma,$$

where $b \in \ell^2(I)$. By assumption $\alpha \neq 0$. In light of the last condition, we define $\beta \in \mathbb{C}$ by

$$\beta := \frac{\gamma}{\alpha}.$$

We now construct the sequence $b \in \ell^2(I)$. Let J denote the (at most countable) set of indices such that $a_j \neq 0$ or $c_j \neq 0$. In order to define b_i $(i \in I)$, we consider the following two cases separately:

$\underline{1}^\circ$ $i \in I \setminus J$. In this case we have $a_i = 0$ and $c_i = 0$. Consequently, $b_i := 0$ will do.

$\underline{2}^\circ$ $i \in J$. The equation $\alpha + a_i = 0$ can be satisfied for only finitely many indices $i \in \{j_1, \cdots, j_N\} \subset J$ (since $\alpha \neq 0$ and $a_i \to 0$, where the latter means that for any $\epsilon > 0$, there exists a finite subset K of I such that for each $i \notin K$, $|a_i| \leq \epsilon$).

Suppose that $i \in \{j_1, \cdots, j_N\} \subset J$. In this case we have $\alpha + a_i = 0$. It follows that $\gamma + c_j = 0$ since $Z(\widehat{x}) \subset Z(\widehat{z})$. Again, $b_i := 0$ does the job.

Now suppose, on the other hand, that $i \in J \setminus \{j_1, \cdots, j_N\}$. Then we must solve

$$\beta + b_i = \frac{\gamma + c_i}{\alpha + a_i},$$

that is,

$$b_i = \frac{\gamma + c_i}{\alpha + a_i} - \frac{\gamma}{\alpha} = \frac{\alpha c_i - \gamma a_i}{\alpha(\alpha + a_i)}.$$

We note that since $(|\alpha + a_i|)_{i \in J \setminus \{j_1, \cdots, j_N\}}$ is bounded below away from 0 (because $\alpha \neq 0$ and $a_i \to 0$), we have $\sum_{i \in J \setminus \{j_1, \cdots, j_N\}} |b_i|^2 < \infty$.

Thus, in all cases we defined b_i, and we have $b \in \ell^2(I)$. □

The next result characterizes divisors in the remaining case when $\alpha = 0$ (note that the condition $Z(\widehat{x}) \subset Z(\widehat{z})$ implies that $\gamma = 0$ as well).

Theorem 4.2. *Let the index set I be infinite, $x = \alpha e + a, z = \gamma e + c \in \mathcal{A}$, $Z(\widehat{x}) \subset Z(\widehat{z})$, and assume that $\alpha = \gamma = 0$. Define $J := \{j \in I \mid a_j \neq 0\}$. Then x divides z, that is, there exists $y = \beta e + b \in \mathcal{A}$ such that $x \odot y = z$ if and only if*

$$\beta := \lim_{j \in J} \frac{c_j}{a_j} \text{ exists and } \sum_{j \in J} \left| \frac{c_j}{a_j} - \beta \right|^2 < +\infty \ .$$

(If J is finite, then $\beta := 0$, $b_j := \frac{c_j}{a_j}$ ($j \in J$), $b_i := 0$ ($i \in I \setminus J$) will do.)

Proof. The equation $x \odot y = z$ is equivalent to $a_i(\beta + b_i) = c_i$ ($i \in I$), where $b \in \ell^2(I)$.

Assume that $x \odot y = z$ has a solution $y \in \mathcal{A}$. Then

$$\beta = \lim_{j \in J} (\beta + b_j) = \lim_{j \in J} \frac{c_j}{a_j}$$

exists and is finite. Moreover,

$$\sum_{j \in J} \left| \frac{c_j}{a_j} - \beta \right|^2 = \sum_{j \in J} |b_j|^2 < +\infty.$$

Now assume that

$$\beta := \lim_{j \in J} \frac{c_j}{a_j} \text{ exists and } \sum_{j \in J} \left| \frac{c_j}{a_j} - \beta \right|^2 < +\infty.$$

Then we define $b_i := 0$ for all indices $i \notin J$, and define

$$b_j = \frac{c_j}{a_j} - \beta, \quad j \in J.$$

Then $b = (b_i)_{i \in I} \in \ell^2(I)$, and since $a_i(\beta + b_i) = c_i$ for all $i \in I$, we have $x \odot y = z$, where $y := \beta e + b \in \mathcal{A}$. □

Theorem 4.2 shows that x divides z if, for example, J is finite or $c_j \neq 0$ only for finitely many indices j.

5. Closed ideals in $\mathcal{A}$

In this section we characterize all closed ideals of $\mathcal{A}$. For a given **closed** subset $B \subset I_\infty$ we define

$$\mathfrak{i}_B := \{x \in \mathcal{A} \mid \widehat{x}(\varphi) = 0 \text{ for all } \varphi \in B\}$$

It is an easy exercise to show that $\mathfrak{i}_B$ is a closed ideal. For closed sets this defining set B is unique, that is:

Theorem 5.1. *If B_1, B_2 are closed subsets of I_∞ such that $\mathfrak{i}_{B_1} = \mathfrak{i}_{B_2}$, then $B_1 = B_2$.*

Proof. Assume that $\mathfrak{i}_{B_1} = \mathfrak{i}_{B_2}$, but $B_1 \neq B_2$. Without loss of generality, we may assume that there exists $j_1 \in B_1 \setminus B_2$. Two cases are possible for j_1:

$\underline{1}^\circ$ $j_1 = \infty$. Since B_2 is a closed subset of the compact Hausdorff space I_∞, it follows that B_2 is compact. Since $\infty \notin B_2$, we must have that $B_2 \subset I$, and so B_2 is compact in I. Because I has the discrete topology, we know that the only compact subsets of I are ones which are finite. Therefore B_2 is finite. Defining $x := e - \sum_{j \in B_2} e_i$, we see that $x \in \mathfrak{i}_{B_2}$, while $x \notin \mathfrak{i}_{B_1}$ (since $\infty \in B_1$ and $\widehat{x}(\infty) = 1 \neq 0$).

$\underline{2}^\circ$ $j_1 \neq \infty$. But then $e_{j_1} \notin \mathfrak{i}_{B_1}$ (since $j_1 \in B_1$), while $e_{j_1} \in \mathfrak{i}_{B_2}$ (since $j_1 \notin B_2$).

This completes the proof. □

Following the ideas in [2], we prove the following characterization of closed ideals.

Theorem 5.2. *Let the index set I be infinite. An ideal $\mathfrak{c}$ is closed in $\mathcal{A}$ if and only if $\mathfrak{c} = \mathfrak{i}_B$ for some closed $B \subset I_\infty$. In fact, $B = \bigcap_{x \in \mathfrak{c}} Z(\widehat{x})$.*

Proof. If $B \subset I_\infty$ is closed, then $\mathfrak{c} := \mathfrak{i}_B$ is always a closed ideal. So we assume that $\mathfrak{c}$ is an arbitrary closed ideal. Define the closed set

$$B := \bigcap_{x \in \mathfrak{c}} Z(\widehat{x}) = \{\varphi \in \Delta \mid \widehat{x}(\varphi) = 0 \text{ for all } x \in \mathfrak{c}\}.$$

Obviously we have the inclusion $\mathfrak{c} \subset \mathfrak{i}_B$. We now prove the reverse implication. To this end, we distinguish two cases: there exists $w = \delta e + d \in \mathfrak{c}$ with $\delta \neq 0$ or we have $\mathfrak{c} \subset \ker \phi_\infty$:

$\underline{1}^\circ$ Suppose that there exists $w = \delta e + d \in \mathfrak{c}$ with $\delta \neq 0$. We have $B \subset Z(\widehat{w})$. Since $Z(\widehat{w}) \subset I$ is compact, and I has dicrete topology, we can conclude that $Z(\widehat{w})$, and hence also B, must be finite.

Claim: There exists $x \in \mathfrak{c}$ such that $Z(\widehat{x}) = B$.

Let us denote by J the set of at most countable many indices with nonzero Fourier coefficients of $d = w - \delta e$. For all $j \in J \setminus B$, we must have $\delta + d_j \neq 0$, while for all $j \in B$ we have $d_j = -\delta$. Since B is finite we may write

$$w = \delta e + \sum_{j\in J} d_j e_j = \delta e + \sum_{j\in B} d_j e_j + \sum_{j\in J\setminus B} d_j e_j,$$

that is,

$$w = \delta \left(e - \sum_{j\in B} e_j \right) + \sum_{j\in J\setminus B} d_j e_j.$$

Since for all $j \in J \setminus B$ we have $\delta + d_j \neq 0$, it follows that

$$e_j = w \odot \left(\frac{e_j}{\delta + d_j} \right) \in \mathfrak{c}, \quad j \in J \setminus B.$$

Thus $w \in \mathfrak{c}$ now implies that $e_j \in \mathfrak{c}$ for all $j \in J \setminus B$. Hence $\sum_{j\in J\setminus B} d_j e_j \in \mathfrak{c}$ as well, because $\mathfrak{c}$ is closed. Therefore

$$x := \underbrace{w}_{\in\mathfrak{c}} - \underbrace{\sum_{j\in J\setminus B} d_j e_j}_{\in\mathfrak{c}} = \delta \left(e - \sum_{j\in B} e_j \right)$$

belongs to $\mathfrak{c}$. This proves the claim.

Given $z \in \mathfrak{i}_B$, we have $B \subset Z(\widehat{z})$, that is, $Z(\widehat{x}) \subset Z(\widehat{z})$. By Theorem 4.1, there exists a $y \in \mathcal{A}$ such that $x \odot y = z$, and so $z \in \mathfrak{c}$ (because $x \in \mathfrak{c}$). Consequently, $\mathfrak{i}_B \subset \mathfrak{c}$ in the first case.

$\underline{2}^\circ$ Now suppose that $\mathfrak{c} \subset \ker \varphi_\infty$. It follows that $\infty \in B$. Let $z \in \mathfrak{i}_B$ be given, and let J be the index set of nonzero Fourier coefficients c_j of z. We must prove that $z \in \mathfrak{c}$. For $j \in J \setminus B$ there exist $x_j \in \mathfrak{c}$ such that $\widehat{x_j}(j) = a_j \neq 0$. Hence for $j \in J \setminus B$, we have

$$e_j = x_j \odot \left(\frac{e_j}{a_j} \right) \in \mathfrak{c},$$

because $x_j \in \mathfrak{c}$. Since $B \subset Z(\widehat{z})$, we have

$$z = \sum_{j\in J} c_j e_j = \sum_{j\in J\setminus B} c_j e_j.$$

Let $J \setminus B = \{j_1, j_2, j_3, \dots\}$. Hence starting with the partial sums of z, namely,

$$z_n = \sum_{k=1}^{n} c_{j_k} e_{j_k} \ ,$$

we conclude that all partial sums z_n belong to $\mathfrak{c}$. Thus also $z \in \mathfrak{c}$, proving the reverse implication $\mathfrak{i}_B \subset \mathfrak{c}$ in this second case when $\mathfrak{c} \subset \ker \varphi_\infty$.

This completes the proof. $\square$

The question now arises: when is a closed ideal principal? We answer this in Theorem 5.4, but first we make the following observation.

Lemma 5.3. *Let the index set be infinite, and let $\mathfrak{c}$ be a closed ideal in $\mathcal{A}$, and let $B := \bigcap_{x\in\mathfrak{c}} Z(\widehat{x})$. If $\mathfrak{c}$ is principal and $\mathfrak{c} = (x_0)$ for some $x_0 \in \mathcal{A}$, then we have $B = Z(\widehat{x_0})$.*

Proof. We know that $\mathfrak{c} = \mathfrak{i}_B$.

Since $x_0 \in \mathfrak{c}$, we have $B = \bigcap_{x\in\mathfrak{c}} Z(\widehat{x}) \subset Z(\widehat{x_0})$.

If $x \in \mathfrak{c} = (x_0)$, then $x = x_0 \odot y$ for some $y \in \mathcal{A}$, and so $\widehat{x} = \widehat{x_0}\widehat{y}$. Thus it follows that $Z(\widehat{x_0}) \subset Z(\widehat{x})$. Since this happens with each $x \in \mathfrak{c}$, it follows that $Z(\widehat{x_0}) \subset \bigcap_{x\in\mathfrak{c}} Z(x) = B$. Consequently, $B = Z(\widehat{x_0})$. □

Theorem 5.4. *Let the index set be infinite, and let $\mathfrak{c}$ be a closed ideal in $\mathcal{A}$, and let $B := \bigcap_{x\in\mathfrak{c}} Z(\widehat{x})$.*

(1) *If $\mathfrak{c}$ is principal and $\mathfrak{c} = (x_0)$, where $x_0 = \alpha e + a$, $a \in \ell^2(I)$ and $0 \neq \alpha \in \mathbb{C}$, then B is finite.*
(2) *If B is finite, then*

$$x_0 := e - \sum_{j\in B} e_j$$

is a generator for $\mathfrak{c}$.
(3) *If $\mathfrak{c}$ is principal and $\mathfrak{c} = (x_0)$, where $x_0 \in \ell^2(I)$, then $I \setminus B$ is finite.*
(4) *If $I \setminus B$ is finite, then $\mathfrak{c}$ is principal, and a generator is given by*

$$x_0 := \sum_{i\in I\setminus B} e_i.$$

Proof. We know that $\mathfrak{c} = \mathfrak{i}_B$.

(1) Assume that $\mathfrak{c}$ is principal and $\mathfrak{c} = (x_0)$, where $x_0 = \alpha e + a$, $a \in \ell^2(I)$ and $0 \neq \alpha \in \mathbb{C}$. By Lemma 5.3, $B = Z(\widehat{x_0})$. Now we will show that $Z(\widehat{x_0})$ is finite. Since $\alpha \neq 0$, we must have $\alpha + a_j = 0$ for all $j \in Z(\widehat{x_0})$. But $a_i \to 0$ (that is, for every $\epsilon > 0$, there exists a finite subset K of I such that for each $i \in I \setminus K$, $|a_i| \leq \epsilon$), and so there are only finitely many indices $i \in I$ such that $\alpha + a_i = 0$. Hence $B = Z(\widehat{x})$ is finite.
(2) For the reverse assertion we now assume that B is finite. Define

$$x_0 := e - \sum_{j\in B} e_j.$$

Then we have

$$Z(\widehat{x_0}) = B = \bigcap_{x\in\mathfrak{c}} Z(\widehat{x}).$$

So for a given $z \in \mathfrak{c} = \mathfrak{i}_B$, there holds that $Z(\widehat{x_0}) \subset Z(\widehat{z})$. From Theorem 4.1, there exists $y \in \mathcal{A}$ such that $x_0 \odot y = z$. Consequently, $\mathfrak{i}_B \subset (x_0)$. Since $Z(\widehat{x_0}) = B$, it is clear that $x_0 \in \mathfrak{i}_B$, and so we have $(x_0) \subset \mathfrak{i}_B$. Hence $(x_0) = \mathfrak{i}_B = \mathfrak{c}$.

(3) Now assume that $\mathfrak{c}$ is principal, $\mathfrak{c} = (x_0)$, where $x_0 \in \ell^2(I)$. By Lemma 5.3, $B = Z(\widehat{x_0})$. Since $x_0 \in \ell^2(I)$, $I \setminus Z(\widehat{x_0})$ is at most countable, and so also $I \setminus B$ must be at most countable. We must prove that it is in fact finite. Suppose, on the contrary, that $I \setminus B$ is infinite, say $I \setminus B = \{j_1, j_2, j_3, \cdots\}$. We have

$$x_0 = \sum_{i \in I} a_i e_i = \sum_{j \in I \setminus B} a_j e_j = \sum_{k \in \mathbb{N}} a_{j_k} e_{j_k}.$$

We now define

$$z := \sum_{k \in \mathbb{N}} (-1)^k a_{j_k} e_{j_k}.$$

Of course we have $Z(\widehat{x_0}) = Z(\widehat{z}) = I \setminus B$, but the limit (see the notation of Theorem 4.2)

$$\beta := \lim_{j \in I \setminus B} \frac{c_j}{a_j} = \lim_{k \in \mathbb{N}} (-1)^k$$

doesn't exist. By Theorem 4.2, it follows that $z \notin (x_0)$, which is a contradiction, since we know that $z \in \mathfrak{i}_B = \mathfrak{c}$. Consequently, $I \setminus B$ must be finite.

(4) Assuming that $I \setminus B$ is finite, define

$$x_0 := \sum_{k \in I \setminus B} e_k.$$

Clearly, $Z(\widehat{x_0}) = B$, and so $x_0 \in \mathfrak{i}_B = \mathfrak{c}$. Thus $(x_0) \subset \mathfrak{c}$. For the reverse inclusion, we use Theorem 4.2. If $z \in \mathfrak{c}$, then we have $B = \bigcap_{x \in \mathfrak{c}} Z(\widehat{x}) \subset Z(\widehat{z})$, and so $Z(\widehat{x_0}) \subset Z(\widehat{z})$. But the set of indices J of nonzero Fourier coefficients of x_0 is $I \setminus B$, which is finite. So by Theorem 4.2, we obtain that z is divisible by x_0, that is, $z \in (x_0)$. Consequently, $\mathfrak{c} \subset (x_0)$.

This completes the proof. □

6. Prime ideals in $\mathcal{A}$

The results in this section follow closely some of the results on prime ideals from Gillman and Jerison [4, 2.9, 2.11, 14.G.3], but there the results were proved for the ring of real-valued continuous functions on a topological space.

6.1. **Which z-ideals are prime?**

Definition 6.1.

(1) If $x \in \mathcal{A}$, then the *zero set* $Z(x)$ *of* $x \in \mathcal{A}$ is

$$Z(x) = Z(\widehat{x}) \ \ (\subset I_\infty).$$

(2) If $\mathfrak{i}$ is an ideal in $\mathcal{A}$, then the *zero set* $Z[\mathfrak{i}]$ *of the ideal* $\mathfrak{i}$ is

$$Z[\mathfrak{i}] = \{Z(x) \mid x \in \mathfrak{i}\}.$$

(3) An ideal $\mathfrak{i}$ in $\mathcal{A}$ is called a *z-ideal* if whenever $x \in \mathcal{A}$ is such that $Z(x) \in Z[\mathfrak{i}]$, then we have $x \in \mathfrak{i}$.
(4) A nonempty subfamily $\mathcal{F}$ of $Z[\mathcal{A}]$ is called a *z-filter in* $Z[\mathcal{A}]$ if
  (a) $\emptyset \notin \mathcal{F}$,
  (b) if $Z_1, Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$,
  (c) if $Z_1 \in \mathcal{F}$, $Z_2 \in Z[\mathcal{A}]$, and $Z_1 \subset Z_2$, then $Z_2 \in \mathcal{F}$.

Lemma 6.2. *Let $x \in \mathcal{A}$. Then $\widehat{x}^*$ and $|\widehat{x}|$ are the Gelfand transforms of some elements from $\mathcal{A}$, where $\cdot^*$ denotes complex conjugation. Moreover, if $\widehat{x}$ is real-valued, then* $\max\{\widehat{x}, 0\}$, $\min\{\widehat{x}, 0\}$ *are also the Gelfand transforms of some elements from $\mathcal{A}$.*

Proof. Let $x = \alpha e + a$, $\alpha \in \mathbb{C}$, $a = (a_i)_{i\in I} \in \ell^2(I)$. Define $z = |\alpha| e + \zeta$, where $\zeta = (|\alpha + a_i| - |\alpha|)_{i\in I}$. We have

$$\|\zeta\|_2^2 = \sum_{i\in I} \big||\alpha + a_i| - |\alpha|\big|^2 \leq \sum_{i\in I} |\alpha + a_i - \alpha|^2 = \sum_{i\in I} |a_i|^2 = \|a\|_2^2 < \infty,$$

and so $\zeta \in \ell^2(I)$. Thus $z \in \mathcal{A}$. Moreover,

$$\begin{aligned} \widehat{z}(\varphi_\infty) &= |\alpha| = |\widehat{x}(\varphi_\infty)|, \\ \widehat{z}(\varphi_i) &= |\alpha| + |\alpha + a_i| - |\alpha| = |\alpha + a_i| = |\widehat{x}(\varphi_i)|, \end{aligned}$$

and so $\widehat{z} = |\widehat{x}|$.

Similarly, with $y := \alpha^* e + (a_i^*)_{i\in I}$, we have that $y \in \mathcal{A}$ and $\widehat{y} = \widehat{x}^*$.

Observing that

$$\max\{\widehat{x}, 0\} = \frac{\widehat{x} + |\widehat{x}|}{2} \quad \text{and} \quad \min\{\widehat{x}, 0\} = \frac{\widehat{x} - |\widehat{x}|}{2},$$

the remaining claims are also proved. □

Lemma 6.3. *If $\mathfrak{i}$ is a proper ideal in $\mathcal{A}$, then $Z[\mathfrak{i}]$ is a z-filter in $Z[\mathcal{A}]$.*

Proof. Since $\mathfrak{i}$ does not contain a unit, $\emptyset \notin Z[\mathfrak{i}]$.

Let $Z_1 = Z(x_1), Z_2 = Z(x_2) \in Z[\mathfrak{i}]$, where $x_1, x_2 \in \mathfrak{i}$. By Lemma 6.2, there exist $z_1, z_2 \in \mathcal{A}$ such that $\widehat{z_1} = \widehat{x_1}^*$ and $\widehat{z_2} = \widehat{x_2}^*$. Since $\mathfrak{i}$ is an ideal, it follows that $z_1 \odot x_1 + z_2 \odot x_2 \in \mathfrak{i}$. Hence

$$Z(\widehat{x_1}) \cap Z(\widehat{x_2}) = Z(|\widehat{x_1}|^2 + |\widehat{x_2}|^2) = Z(\widehat{z_1}\widehat{x_1} + \widehat{z_2}\widehat{x_2}) = Z(z_1 \odot x_1 + z_2 \odot x_2) \in Z[\mathfrak{i}],$$

and so $Z_1 \cap Z_2 \in Z[\mathfrak{i}]$.

Finally, let $Z_1 = Z(x_1)$, where $x_1 \in \mathfrak{i}$, and let $Z_2 = Z(x_2) \supset Z_1$, where $x_2 \in \mathcal{A}$. Since $\mathfrak{i}$ is an ideal, it follows that $x_2 \odot x_1 \in \mathfrak{i}$. Thus

$$Z_2 = Z_1 \cup Z_2 = Z(x_2 \odot x_1) \in Z[\mathfrak{i}].$$

So $Z[\mathfrak{i}]$ is a z-filter in $Z[\mathcal{A}]$. □

Theorem 6.4. *For any z-ideal $\mathfrak{i}$ in $\mathcal{A}$, the following are equivalent:*
(1) *$\mathfrak{i}$ is prime.*
(2) *$\mathfrak{i}$ contains a prime ideal.*
(3) *For all $x, y \in \mathcal{A}$, $\widehat{x}\widehat{y} = 0$, then $x \in \mathfrak{i}$ or $y \in \mathfrak{i}$.*

(4) *For every $x \in \mathcal{A}$ such that $\widehat{x}$ is real-valued, there is a zero set in $Z[\mathfrak{i}]$ on which $\widehat{x}$ does not change sign.*

Proof. (1) implies (2): Trivial.

(2) implies (3): Suppose that $\mathfrak{i}$ contains a prime ideal $\mathfrak{p}$. If $x, y \in \mathcal{A}$ satisfy $\widehat{x}\widehat{y} = 0$, then $\widehat{x \odot y} = 0$, and so $x \odot y = 0 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Consequently, $x \in \mathfrak{i}$ or $y \in \mathfrak{i}$.

(3) implies (4): Suppose that $x \in \mathcal{A}$ is such that $\widehat{x}$ is real-valued. Consider

$$\varphi := \max\{\widehat{x}, 0\} \quad \text{and} \quad \psi := \min\{\widehat{x}, 0\}.$$

From Lemma 6.2, there exist $u, v \in \mathcal{A}$ such that $\widehat{u} = \varphi$ and $\widehat{v} = \psi$. We have $\widehat{u}\widehat{v} = \varphi\psi = 0$, and so by the hypothesis, $u \in \mathfrak{i}$ or $v \in \mathfrak{i}$. But if $u \in \mathfrak{i}$, then for all $i \in Z(\widehat{u}) \in Z[\mathfrak{i}]$, we have $\widehat{x} \leq 0$. And if $v \in \mathfrak{i}$, then for all $i \in Z(\widehat{v}) \in Z[\mathfrak{i}]$, we have $\widehat{x} \geq 0$.

(4) implies (1): Let $u, v \in \mathcal{A}$ be such that $u \odot v \in \mathfrak{i}$. Consider the function

$$\varphi := |\widehat{u}| - |\widehat{v}|.$$

By Lemma 6.2, there exists an $x \in \mathcal{A}$ such that $\widehat{x} = \varphi$. By the hypothesis, there is a zero set $J \in Z[\mathfrak{i}]$ on which φ is nonnegative, say. Then for all $i \in J$, $|\widehat{u}| \geq |\widehat{v}|$. But this implies that every zero of $\widehat{u}$ in J is also a zero of $\widehat{v}$. Hence

$$Z(v) \supset J \cap Z(u) = J \cap Z(u \odot v) \in Z[\mathfrak{i}].$$

Since $\mathfrak{i}$ is a z-ideal, it follows that $v \in \mathfrak{i}$. The proof in the case when φ is nonpositive is similar. Consequently, $\mathfrak{i}$ is prime. □

6.2. Every prime ideal is contained in $\ker \varphi_\infty$.

Theorem 6.5. *If a nonzero prime ideal in $\mathcal{A}$ is contained in a maximal ideal and is not equal to it, then the maximal ideal is $\ker \varphi_\infty$.*

Proof. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{A}$ and suppose that $\mathfrak{p} \subsetneq \ker \varphi_i$ for some $i \in I_\infty$. Thus there exists an $x \in (\ker \varphi_i) \setminus \mathfrak{p}$. But $x \in \ker \varphi_i$ implies that

$$x \odot e_i = \varphi_i(x \odot e_i)e_i = 0\ .$$

Since $\mathfrak{p}$ is prime, $0 = x \odot e_i \in \mathfrak{p}$, and x is not in $\mathfrak{p}$, we must have $e_i \in \mathfrak{p}$, contradicting $\mathfrak{p} \subset \ker \varphi_i$. □

6.3. Description of all prime z-ideals when $I = \mathbb{N}$.

In this section we establish a one-to-one correspondence between free ultrafilters and nonmaximal prime z-ideals when the index set is countable. First we recall the definition of an ultrafilter.

Definition 6.6. A *filter* on a set X is a collection $\mathcal{F}$ of subsets of X satisfying

(F1) $X \in \mathcal{F}$, but $\emptyset \notin \mathcal{F}$.
(F2) If $U \in \mathcal{F}$ and $U \subset V \subset X$, then $V \in \mathcal{F}$.
(F3) A finite intersection of sets in $\mathcal{F}$ is in $\mathcal{F}$.

An *ultrafilter* on a set X is a filter $\mathcal{F}$ on X which is maximal with respect to inclusion, that is, it is a filter $\mathcal{F}$ for which any other filter $\mathcal{F}'$ on X satisfying $\mathcal{F} \subset \mathcal{F}'$ actually satisfies $\mathcal{F}' = \mathcal{F}$.

An ultrafilter $\mathcal{F}$ on a set X is called a *free ultrafilter* if $\bigcap_{U\in\mathcal{F}} U = \emptyset$.

In order to prove Theorem 6.8, we will need the following characterization of ultrafilters; see for example [5, p. 83, L.(e)].

Proposition 6.7. *A filter $\mathcal{F}$ on a set X is an ultrafilter if and only if for every set $U \subset X$, either $U \in \mathcal{F}$ or $X \setminus U \in \mathcal{F}$.*

Theorem 6.8. *The mapping*

$$\mathcal{F} \mapsto \mathfrak{p} := \{x \in \mathcal{A} \mid Z(x) \setminus \{\infty\} \in \mathcal{F}\}$$

is one-to-one from the family of all free ultrafilters on $\mathbb{N}$ *onto the family of all nonmaximal, prime z-ideals of $\mathcal{A}$ contained in* $\ker \varphi_\infty$.

Proof. Let $\mathcal{F}$ denote a free ultrafilter on $\mathbb{N}$ and define

$$\mathfrak{p} := \{x \in \mathcal{A} \mid Z(x) \setminus \{\infty\} \in \mathcal{F}\}.$$

First we show that $\mathfrak{p}$ is a nonmaximal, prime z-ideal of $\mathcal{A}$ contained in $\ker \varphi_\infty$.

(1) $\mathfrak{p}$ *is an ideal.*

$0 \in \mathfrak{p}$, since $Z(0) \setminus \{\infty\} = \mathbb{N} \in \mathcal{F}$.

Given $x, y \in \mathfrak{p}$, we have $Z(x) \setminus \{\infty\} \in \mathcal{F}$ and $Z(y) \setminus \{\infty\} \in \mathcal{F}$. Hence $Z(x+y) \setminus \{\infty\} \supset (Z(x) \setminus \{\infty\}) \cap (Z(y) \setminus \{\infty\}) \in \mathcal{F}$. So $Z(x+y) \setminus \{\infty\} \in \mathcal{F}$, as it is a superset of a set in the filter $\mathcal{F}$.

Finally, let $x \in \mathfrak{p}$ and $y \in \mathcal{A}$. Then $Z(x \odot y) = Z(x) \cup Z(y) \supset Z(x)$. So $Z(x \odot y) \setminus \{\infty\} \in \mathcal{F}$, since it contains $Z(x) \setminus \{\infty\} \in \mathcal{F}$.

(2) $\mathfrak{p}$ *is not trivial and not all of $\mathcal{A}$.*

Since $\mathcal{F}$ is an ultrafilter, given subset $U \subset \mathbb{N}$, either $U \in \mathcal{F}$ or $\mathbb{N} \setminus U \in \mathcal{F}$. Taking $U = \{1\}$, we have either $e - e_1 \in \mathfrak{p}$ or $e_1 \in \mathfrak{p}$. Since neither $e - e_1$ nor e_1 is 0, we conclude that $\mathfrak{p}$ is different from the zero ideal.

Since $\emptyset \notin \mathcal{F}$, it follows that $\mathfrak{p}$ cannot contain a unit of $\mathcal{A}$.

(3) $\mathfrak{p}$ *is a prime ideal.*

Let $x, y \in \mathcal{A}$ be such that $x \odot y \in \mathfrak{p}$. But then we have that $U := Z(x \odot y) \setminus \{\infty\} = (Z(x) \setminus \{\infty\}) \cup (Z(y) \setminus \{\infty\}) \in \mathcal{F}$. If neither $Z(x) \setminus \{\infty\}$ nor $Z(y) \setminus \{\infty\}$ belong to $\mathcal{F}$, then their complements do belong to $\mathcal{F}$, and so does the intersection of these complements, which is equal to $\mathbb{N} \setminus (Z(x \odot y) \setminus \{\infty\}) = \mathbb{N} \setminus U$. But this furthermore implies that $\emptyset = U \cap (\mathbb{N} \setminus U) \in \mathcal{F}$, a contradiction. Thus either x or y belongs to $\mathfrak{p}$. Consequently, $\mathfrak{p}$ is prime.

(4) $\mathfrak{p}$ *is a z-ideal.*

Given $x \in \mathcal{A}$ such that $Z(x) = Z(y)$ for a $y \in \mathfrak{p}$, we have that $Z(x) \setminus \{\infty\} = Z(y) \setminus \{\infty\} \in \mathcal{F}$, and so $x \in \mathfrak{p}$.

(5) $\mathfrak{p}$ *is contained in* $\ker \varphi_\infty$.
Since $\mathcal{F}$ is a free ultrafilter on $\mathbb{N}$, we have $\bigcap_{U \in \mathcal{F}} U = \emptyset$. So it follows that $\bigcap_{x \in \mathfrak{p}} Z(x) \subset \{\infty\}$. Hence $\mathfrak{p}$ is not contained in any of the maximal ideals $\ker \varphi_n$, $n \in \mathbb{N}$. By Theorem 6.5, it follows that $\mathfrak{p}$ is contained in $\ker \varphi_\infty$.

(6) $\mathfrak{p}$ *is not maximal.*
Suppose that $\mathfrak{p} = \ker \varphi_\infty$. Defining $x = \sum_{n=1}^{\infty} \frac{1}{n} e_n$, we see that $x \in \ker \varphi_\infty = \mathfrak{p}$. Since $Z(x) = \{\infty\}$, we have $\emptyset = Z(x) \setminus \{\infty\} \in \mathcal{F}$, a contradiction.

Now $\mathfrak{p}$ denote a nonmaximal prime z-ideal in $\ker \varphi_\infty$. We now show that

$$\mathcal{F} := \{Z(x) \setminus \{\infty\} \mid x \in \mathfrak{p}\}$$

is a free ultrafilter on $\mathbb{N}$.

(1) $\mathcal{F}$ *is a filter.*

(F1) $\mathbb{N}$ belongs to $\mathcal{F}$ since $0 \in \mathfrak{p}$. Also, $\emptyset \notin \mathcal{F}$ since otherwise there exists an element $x \in \mathfrak{p}$ such that $Z(x) = \{\infty\}$, and we prove now that this implies that $\mathfrak{p} = \ker \varphi_\infty$. Let $y \in \ker \varphi_\infty \setminus \mathfrak{p}$. By Lemma 6.2, $\widehat{x}^* = \widehat{x_0}$ and $\widehat{y}^* = \widehat{y_0}$ for some $x_0, y_0 \in \mathcal{A}$. So $Z(x \odot x_0 + y \odot y_0) = Z(x) \cap Z(y) = \{\infty\} = Z(x)$. Since $\mathfrak{p}$ is a z-ideal, we conclude that $x \odot x_0 + y \odot y_0 \in \mathfrak{p}$. Because $x \in \mathfrak{p}$, it follows that $y \odot y_0 \in \mathfrak{p}$. By assumption we have $y \notin \mathfrak{p}$, and so $y_0 \in \mathfrak{p}$. But this leads to a contradiction since $Z(y) = Z(y_0)$ and $\mathfrak{p}$ being a z-ideal, gives $y \in \mathfrak{p}$. So no such function $y \in \ker \varphi_\infty \setminus \mathfrak{p}$ can exist.

(F2) Next we show that that if $U \in \mathcal{F}$ and $U \subset V$, then $V \in \mathcal{F}$. Take $x \in \mathfrak{p}$ such that $Z(x) \setminus \{\infty\} = U$. Since $V \subset \mathbb{N}$, we can find a $y \in \mathcal{A}$ such that $Z(y) \setminus \{\infty\} = V$ (for example, we can take $y = \sum_{n \notin V} \frac{1}{n} e_n$). But then $x \odot y \in \mathfrak{p}$ and furthermore $Z(x \odot y) \setminus \{\infty\} = (Z(x) \setminus \{\infty\}) \cup (Z(y) \setminus \{\infty\}) = U \cup V = V$. Consequently, $V \in \mathcal{F}$.

(F3) If $U, V \in \mathcal{F}$, the we now show that $U \cap V \in \mathcal{F}$. Take $x, y \in \mathfrak{p}$ such that $Z(x) \setminus \{\infty\} = U$ and $Z(y) \setminus \{\infty\} = V$. Let $x_0, y_0 \in \mathcal{A}$ be such that $\widehat{x_0} = \widehat{x}^*$ and $\widehat{y_0} = \widehat{y}^*$. Then the intersection $U \cap V = Z(x \odot x_0 + y \odot y_0) \setminus \{\infty\} \in \mathcal{F}$ because $x \odot x_0 + y \odot y_0 \in \mathfrak{p}$. Thus a finite intersection of sets in $\mathcal{F}$ is in $\mathcal{F}$.

Hence $\mathcal{F}$ is a filter.

(2) $\mathcal{F}$ *is an ultrafilter.*
We will use the characterization of ultrafilters given in Proposition 6.7. Let $U \subset \mathbb{N}$ be given. Take functions $x, y \in \ker \varphi_\infty$ such that $Z(x) \setminus \{\infty\} = U$ and $Z(y) \setminus \{\infty\} = \mathbb{N} \setminus U$ (for example, $x := \sum_{n \notin U} \frac{1}{n} e_n$ and $y := \sum_{n \in U} \frac{1}{n} e_n$ do the job). But then $Z(x \odot y) = \mathbb{N} \cup \{\infty\} = \mathbb{N}_\infty$, that is, $x \odot y = 0 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, proving that either $U \in \mathcal{F}$ or $\mathbb{N} \setminus U \in \mathcal{F}$. Consequently, $\mathcal{F}$ is an ultrafilter.

(3) $\mathcal{F}$ *is free.*

Suppose on the contrary that $\mathcal{F}$ is not free, that is, there exists a $n \in \mathbb{N}$ such that $n \in \bigcap_{U \in \mathcal{F}} U$. Since $\mathcal{F}$ is an ultrafilter, it follows that either $\{n\} \in \mathcal{F}$ or $\mathbb{N} \setminus \{n\} \in \mathcal{F}$. We consider the two cases separately:

$\underline{1}^\circ$ $\mathbb{N} \setminus \{n\} \in \mathcal{F}$. This is impossible because $n \in \bigcap_{U \in \mathcal{F}} U$.

$\underline{2}^\circ$ $\{n\} \in \mathcal{F}$. Then every superset of $\{n\}$ also belongs to $\mathcal{F}$, that is, $\mathcal{F} = \{U \subset \mathbb{N} \mid n \in U\}$. This means that every Gelfand transform vanishing at n belongs to $\mathfrak{p}$ since $\mathfrak{p}$ is a z-ideal. Indeed, if $y \in \mathcal{A}$ is such that $n \in Z(y)$, then $Z(y) \in \mathcal{F}$, and so there exists a $x \in \mathfrak{p}$ such that $Z(y) = Z(x) \setminus \{\infty\}$. Thus either we have that $Z(y) = Z(x) \in Z[\mathfrak{p}]$ (which implies by the z-ideal property of $\mathfrak{p}$ that $y \in \mathfrak{p}$) or $Z(y) \cup \{\infty\} = Z(x)$. But the latter implies $Z(y \odot \sum_{n \in \mathbb{N}} \frac{1}{n} e_n) = Z(y) \cup \{\infty\} = Z(x) \in Z[\mathfrak{p}]$, which implies, by the z-ideal property of $\mathfrak{p}$, that $y \odot \sum_{n \in \mathbb{N}} \frac{1}{n} e_n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, we have $y \in \mathfrak{p}$ because $\sum_{n \in \mathbb{N}} \frac{1}{n} e_n \notin \mathfrak{p}$ (otherwise $Z(\sum_{n \in \mathbb{N}} \frac{1}{n} e_n) \setminus \{\infty\} = \emptyset \in \mathcal{F}$, which is absurd. So we have shown that every function vanishing at n belongs to $\mathfrak{p}$. But this implies that $\mathfrak{p} = \ker \varphi_n$, showing $\ker \varphi_n = \mathfrak{p} \subset \ker \varphi_\infty$, a contradiction. Thus the case $\{n\} \in \mathcal{F}$ is impossible as well.

Consequently, the ultrafilter $\mathcal{F}$ is free.

With the above defined ultrafilter $\mathcal{F}$ corresponding to the given prime ideal $\mathfrak{p}$, we can check that

$$\mathfrak{p} = \{x \in \mathcal{A} \mid Z(x) \setminus \{\infty\} \in \mathcal{F}\}.$$

Hence the mapping in the statement of the theorem is onto.

Finally we show that the mapping is one-to-one. Suppose the two distinct ultrafilters $\mathcal{F}$ and $\mathcal{F}'$ give rise to the same prime ideal $\mathfrak{p}$, that is,

$$\mathfrak{p} = \{x \in \mathcal{A} \mid Z(x) \setminus \{\infty\} \in \mathcal{F}\} \quad \text{and} \quad \mathfrak{p} = \{x \in \mathcal{A} \mid Z(x) \setminus \{\infty\} \in \mathcal{F}'\}.$$

We can assume without loss of generality that $U \in \mathcal{F} \setminus \mathcal{F}'$. If $x := \sum_{n \notin U} \frac{1}{n} e_n$, then $Z(x) = U$, and so $x \in \mathfrak{p}$. But then $U = Z(x) \setminus \{\infty\} \in \mathcal{F}'$, a contradiction. □

Remark 6.9. The set of all free ultrafilters on $\mathbb{N}$ is very large. Indeed, it can be indentified with the set $\beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ denotes the Stone-Čech-compactification of $\mathbb{N}$. From [4], Theorem 9.2, we known that the cardinality of $\beta\mathbb{N}$ is

$$|\beta\mathbb{N}| = 2^c \text{ and } |\beta\mathbb{N} \setminus \mathbb{N}| = 2^c ,$$

where c denotes the cardinality of $\mathbb{R}$, that is $2^{\aleph_0}$. Hence there are many nonmaximal prime z-ideals in Theorem 6.8.

An interesting open question that now arises is what the analogue of Theorem 6.8 would be when I is uncountably infinite.

References

[1] R.M. Brooks. A ring of analytic functions. *Studia Mathematica*, 24:191-210, 1964.

[2] R. Brück, J. Müller. Closed ideals in a convolution algebra of holomorphic functions. *Canadian Journal of Mathematics*, no. 5, 47:915-928, 1995.

[3] J. Caveny. Bounded Hadamard products of H^p functions. *Duke Mathematical Journal*, 33:389-394, 1966.

[4] L. Gillman and M. Jerison. *Rings of continuous functions.* Reprint of the 1960 edition. Graduate Texts in Mathematics, No. 43, Springer-Verlag, New York-Heidelberg, 1976.

[5] J.L. Kelley. *General topology.* D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[6] J. Nagata. *Modern Dimension Theory.* Bibliotheca Mathematica, Vol. VI. John Wiley and Sons, Inc., New York, 1965.

[7] H. Render. The maximal ideal space of $H^\infty(\mathbb{D})$ with respect to the Hadamard product. *Proceedings of the American Mathematical Society*, no. 5, 127:1409-1411, 1999.

[8] H. Render and A. Sauer. Algebras of holomorphic functions with Hadamard multiplication. *Studia Mathematica*, no. 1, 118:77-100, 1996.

[9] R. Mortini and R. Rupp. Totally reducible elements in rings of analytic functions. *Communications in Algebra*, no. 6, 20:1705-1713, 1992.

[10] L. Schwartz. *Mathematics for the Physical Sciences.* Hermann, Paris; Addison-Wesley Publishing Company, Reading, Massachussetts-London-Don Mills, Ontario, 1966.

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