# REDUCIBILITY IN $A_{\mathbb{R}}(K)$, $C_{\mathbb{R}}(K)$ AND $A(K)$

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Abstract. Let $K$ denote a compact real symmetric subset of $\mathbb{C}$ and let $A_{\mathbb{R}}(K)$ denote the real Banach algebra of all real symmetric continuous functions on $K$ which are analytic in the interior $K^{\circ}$ of $K$, endowed with the supremum norm. We characterize all unimodular pairs $(f, g)$ in $A_{\mathbb{R}}(K)^2$ which are reducible.

In addition, for an arbitrary compact $K$ in $\mathbb{C}$, we give a new proof (not relying on Banach algebra theory or elementary stable rank techniques) of the fact that the Bass stable rank of $A(K)$ is 1.

Finally, we also characterize all compact real symmetric sets $K$ such that $A_{\mathbb{R}}(K)$, respectively $C_{\mathbb{R}}(K)$, has Bass stable rank 1.

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## 1. Introduction

The concept of stable rank of a ring was introduced by H. Bass [2] to study some stabilisation questions in algebraic K-theory. We recall this notion below:

**Definition 1.1.** Let $\mathcal{A}$ be a commutative ring with an identity element, denoted by 1. Let $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. An element $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ is called *unimodular* if there exists a $b = (b_1, \dots, b_n) \in \mathcal{A}^n$ such that

$$\sum_{k=1}^{n} b_k a_k = 1.$$

We denote by $U_n(\mathcal{A})$ the set of unimodular elements of $\mathcal{A}^n$.

We say that $a = (a_1, \dots, a_n) \in U_n(\mathcal{A})$ is *reducible* (in $\mathcal{A}$), if there exist $h_1, \dots, h_{n-1} \in \mathcal{A}$ such that $(a_1 + h_1 a_n, \dots, a_{n-1} + h_{n-1} a_n) \in U_{n-1}(\mathcal{A})$.

The *Bass stable rank of* $\mathcal{A}$, denoted by bsr $\mathcal{A}$, is the least $n \in \mathbb{N}$ such that every $a \in U_{n+1}(\mathcal{A})$ is reducible, and it is infinite if no such integer $n$ exists.

The Bass stable rank of several complex Banach algebras of analytic functions is well known: for example if $K$ compact in $\mathbb{C}$, then the Bass stable rank of $A(K)$ is 1, where $A(K)$ denotes the set of all continuous functions on $K$ that are analytic in the interior $K^\circ$ of $K$; see [5, Theorem 2.3] and [13].

In [17], Brett Wick considered reducibility questions in the *real* Banach algebra $A_{\mathbb{R}}(\overline{\mathbb{D}})$ consisting of those elements of the disk algebra $A(\overline{\mathbb{D}})$ which have real Fourier coefficients, or equivalently, those elements from the disk algebra that satisfy the symmetry condition $f(z) = (f(z^*))^*$ for all $z \in \overline{\mathbb{D}}$. (Throughout this article, we use the following notation.)

**Notation 1.2.** We use $z^*$ to denote the complex conjugate of $z$, and we use $\overline{\Omega}$ to denote the closure of the set $\Omega \subset \mathbb{C}$.

Bass and topological stable ranks of $A_{\mathbb{R}}(\overline{\mathbb{D}})$ play an important role in *control theory* in the problem of stabilization of linear systems. We refer the reader to [11] and [16] for background on the connection between stable rank and control theory.

In this article, we study the reducibility of corona pairs, in some real Banach algebras of "real symmetric" functions. We define these in Definition 1.4 below.

**Definition 1.3.** Let $K$ denote a compact subset of $\mathbb{C}$ and let $A(K)$ denote the complex Banach algebra of all continuous functions on $K$ which are analytic in the interior $K^\circ$ of $K$, endowed with the supremum norm:

$$\|f\|_\infty = \sup_{z \in K} |f(z)|,$$

whereas $R(K)$ denotes the uniform closure of all rational functions with poles off $K$.

**Definition 1.4.** If $K$ is *real symmetric* (that is, $z \in K$ if and only if $z^* \in K$), we use the symbol $A_{\mathbb{R}}(K)$ (respectively $R_{\mathbb{R}}(K)$) to denote the set of functions $f$ belonging to $A(K)$, (respectively $R(K)$) that are *real symmetric*, that is,

$$f(z) = (f(z^*))^* \quad (z \in K).$$

Moreover, $C_{\mathbb{R}}(K)$ denotes the set of complex-valued, bounded, continuous functions $f$ defined on $K$, that satisfy $f(z) = (f(z^*))^*$ $(z \in K)$.

$\mathbb{R}[z]$ denotes the set of all polynomial functions with real coefficients, while $\mathbb{R}(z)$ denotes the set of all rational functions which are ratios of polynomials from $\mathbb{R}[z]$.

## 2. BSR, TSR $R_{\mathbb{R}}(K) \leq 2$

In this section we prove that if $\mathbb{C} \setminus K$ has only finitely many connected components, then tsr $A_{\mathbb{R}}(K) \leq 2$ and so bsr $A_{\mathbb{R}}(K) \leq$(tsr $A_{\mathbb{R}}(K) \leq$)2. We will do this by first computing the topological stable rank (defined below) and using the known fact that the Bass stable rank is bounded above by the topological stable rank (Proposition 2.2).

**Definition 2.1.** [12] Let $\mathcal{A}$ denote a commutative unital Banach algebra. The *topological stable rank of* $\mathcal{A}$, denoted by tsr $\mathcal{A}$, is the minimum $n \in \mathbb{N}$ such that $U_n(\mathcal{A})$ is dense in $\mathcal{A}^n$, and it is infinite if no such integer exists.

We recall the following result [4, Theorem 3, p. 293]:

**Proposition 2.2.** *Let $\mathcal{A}$ be a commutative unital real (or complex) Banach algebra. If $U_n(\mathcal{A})$ is a dense subset of $\mathcal{A}^n$, then* bsr $\mathcal{A} \leq n$.

We will use the following fact several times in some of our proofs.

**Lemma 2.3.** *Let $\mathcal{A}$ be a ring such that $\mathbb{R}[z] \subset \mathcal{A} \subset A_{\mathbb{R}}(K)$. If the Bass stable rank of $\mathcal{A}$ is 1, then $K \cap \mathbb{R}$ is totally disconnected.*

*Proof.* If $K \cap \mathbb{R}$ is not totally disconnected, then there exists a closed connected subset $L$ of $K \cap \mathbb{R}$ which is not a singleton, so two different real numbers $a, b$ belong to $L \subset \mathbb{R}$. But then the interval $[a, b]$ is contained in $L$. (If not, we have $c \in \mathbb{R} \setminus L$ such that $a < c < b$. Since $L$ is closed, it follows that for a sufficiently small $r$, we have that $(c - r, c + r) \subset \mathbb{R} \setminus L$, and so $L$ splits into the disjoint closed union $L = ([a, c - r] \cap L) \cup ([c + r, b] \cap L)$.) But then the unimodular pair $\left(z - \frac{a+b}{2}, (z - a)(z - b)\right)$ is not reducible by the intermediate value theorem for real continuous functions on the interval $[a, b]$, a contradiction. Hence $K \cap \mathbb{R}$ is totally disconnected. $\square$

**Theorem 2.4.** *Let $K$ be a real symmetric subset of $\mathbb{C}$.*

1. *The topological stable rank of $R_{\mathbb{R}}(K)$ is at most 2.*
2. *The topological stable rank of $R_{\mathbb{R}}(K)$ is equal to 1 if and only if $K^\circ = \emptyset$ and $K \cap \mathbb{R}$ is totally disconnected.*

*Proof.* $\underline{1}^\circ$ We will show that $U_2(R_{\mathbb{R}}(K))$ is dense in $R_{\mathbb{R}}(K)^2$. Take $(f,g) \in R_{\mathbb{R}}(K)^2$ and approximate $f, g$ by real symmetric rational functions $r, s$, respectively. Since $r \in \mathbb{R}(z)$, we have the following representation for $r$:

$$r(z) = \frac{C \prod(z - r_j) \prod(z - w_j)(z - w_j^*)}{q},$$

where $C, r_j$ are real numbers, $q \in \mathbb{R}[z]$ has no zeros in $K$ and $w_j$ denote the non-real zeros of $r$. If $r$ and $s$ have a common root in $K$, then we replace $r_j, w_j, w_j^*$ by $r_j + \epsilon, w_j + \epsilon, w_j^* + \epsilon$ with a sufficiently small real $\epsilon$ so that the new real symmetric rational function $\widetilde{r}$ has no common root with $s$ in $K$. Thus $(\widetilde{r}, s) \in U_2(R_{\mathbb{R}}(K))$ is near $(f,g)$. So tsr $R_{\mathbb{R}}(K) \le 2$.

$\underline{2}^\circ$ Suppose that $K^\circ = \emptyset$ and $K \cap \mathbb{R}$ is totally disconnected. We must show that $U_1(R_{\mathbb{R}}(K))$ is dense in $R_{\mathbb{R}}(K)$. Let $f \in R_{\mathbb{R}}(K)$. Given $\epsilon > 0$, by the definition of $R_{\mathbb{R}}(K)$ we can find a real symmetric rational function $r$ with poles off $K$ such that $\|f - r\|_\infty < \epsilon/2$. Since $r \in \mathbb{R}(z)$ has poles off $K$, it again has the following representation:

$$r(z) = \frac{C \prod(z - r_j) \prod(z - w_j)(z - w_j^*)}{q},$$

where $C, r_j$ are real numbers, $q \in \mathbb{R}[z]$ has no zeros in $K$ and $w_j$ denote the non-real zeros of $r$. If $r$ has any zeros in $K$, then since $K \cap \mathbb{R}$ is totally disconnected, we can replace $r_j$ by $r_j + \delta$ with sufficiently small $\delta > 0$, such that $r_j + \delta \in \mathbb{R} \setminus K$. Since $K^\circ$ is void we can replace all non-real zeros $w_j, w_j^*$ by $w_j + \rho, w_j^* + \rho^*$, where $|\rho|$ is small such that the new real symmetric rational function $\widetilde{r}$ has no zeros in $K$ and moreover $\|r - \widetilde{r}\|_\infty < \epsilon/2$. Since $\widetilde{r} \in \mathbb{R}(z)$ has zeros and poles off $K$, it is invertible in $R_{\mathbb{R}}(K)$, and we also have $\|f - \widetilde{r}\|_\infty < \epsilon$.

Suppose now that the topological stable rank of $R_{\mathbb{R}}(K) = 1$, that is $U_1(R_{\mathbb{R}}(K))$ is dense in $R_{\mathbb{R}}(K)$. Then by Proposition 2.2, it follows that the Bass stable rank of $R_{\mathbb{R}}(K) = 1$ as well. By Lemma 2.3, $K \cap \mathbb{R}$ is totally disconnected.

If $K^\circ$ is not empty we show that $U_1(R_{\mathbb{R}}(K))$ is not dense in $R_{\mathbb{R}}(K)$, a contradiction. Note that $U_1(R_{\mathbb{R}}(K))$ is the set of units in $R_{\mathbb{R}}(K)$, and $f$ is invertible as an element in $R_{\mathbb{R}}(K)$ only if it has no zero in $K$. Now consider $z_0$ in the interior $K^\circ$ of $K$, and let the open disk $D(z_0, r)$ be contained in $K^\circ$. But by Hurwitz's theorem, the uniform limit of a sequence of nowhere-vanishing analytic functions on a connected open set $U$ is either identically zero or has no zeros in $U$; see [1, Theorem 2, p.178]. So taking any function in $R_{\mathbb{R}}(K)$ with finitely many zeros in $D(z_0, r)$, say $(z - z_0)(z - z_0^*)$, we see that it cannot be the uniform limit of a sequence in $U_1(R_{\mathbb{R}}(K))$. So tsr $R_{\mathbb{R}}(K) > 1$. ☐

In light of Theorem 2.4, Proposition 2.2 yields the following:

**Corollary 2.5.** *Let $K$ denote a real symmetric compact subset of $\mathbb{C}$. The Bass stable rank of $R_{\mathbb{R}}(K)$ is at most 2.*

Of course it is natural to ask for conditions for Bass stable rank 1.

**Lemma 2.6.** *Let $K$ denote a real symmetric compact subset of $\mathbb{C}$ such that $K \cap \mathbb{R}$ is totally disconnected. Then the set of elements $u \cdot r$, where $u \in R_{\mathbb{R}}(K)^{-1}$ and the real symmetric rational function $r \in R_{\mathbb{R}}(K)$ has only non-real zeros, is dense in $R_{\mathbb{R}}(K)$.*

*Proof.* Let $f \in R_{\mathbb{R}}(K)$. Given $\epsilon > 0$, by the definition of $R_{\mathbb{R}}(K)$ we can find a real symmetric rational function $r$ with poles off $K$ such that $\|f - r\|_\infty < \epsilon/2$. Since $r \in \mathbb{R}(z)$ has poles off $K$, it again has the following representation:

$$r(z) = \frac{C \prod(z - r_j) \prod(z - w_j)(z - w_j^*)}{q},$$

where $C, r_j$ are real numbers, $q \in \mathbb{R}[z]$ has no zeros in $K$ and $w_j$ denote the non-real zeros of $r$. If $r$ has any zeros in $K$, then since $K \cap \mathbb{R}$ is totally disconnected, we can replace $r_j$ by $r_j + \delta$ with sufficiently small $\delta > 0$, such that $r_j + \delta \in \mathbb{R} \setminus K$. The new real symmetric rational function $\widetilde{r}$ has only non-real zeros in $K$ and has the form $u \cdot r$ from the assertion. Moreover $\|r - \widetilde{r}\|_\infty < \epsilon/2$. Hence we conclude $\|f - \widetilde{r}\|_\infty < \epsilon$. □

**Theorem 2.7.** *Let $K$ denote a real symmetric compact subset of $\mathbb{C}$. The Bass stable rank of $R_{\mathbb{R}}(K)$ is 1 if and only if $K \cap \mathbb{R}$ is totally disconnected.*

*Proof.* If the Bass stable rank of $R_{\mathbb{R}}(K))$ is equal to 1, then by Lemma 2.3, $K \cap \mathbb{R}$ is totally disconnected. Assuming that $K \cap \mathbb{R}$ is totally disconnected we must show that every unimodular pair $(f, g)$ is reducible. For unimodular $(f, g) \in U_2(R_{\mathbb{R}}(K))$ there exist $\alpha, \beta \in R_{\mathbb{R}}(K)$ such that

$$\alpha(z)f(z) + \beta(z)g(z) = 1 \quad (z \in K).$$

We now approximate $\alpha$ by functions of the form $u \cdot r$, where $u \in R_{\mathbb{R}}(K)^{-1}$ and $r$ has only non-real zeros; see Lemma 2.6. To be precise

$$\|u \cdot r - \alpha\|_\infty \cdot \|f\|_\infty < 1/2.$$

This gives

$$|u(z)r(z)f(z) + \beta(z)g(z)| = |1 + (u(z)r(z) - \alpha(z))f(z)| \geq 1 - 1/2 = 1/2$$

for all $z \in K$. Hence

$$u \cdot r \cdot f + \beta \cdot g =: U \in R_{\mathbb{R}}(K)^{-1}.$$

**Claim:** $(ur, g)$ is reducible, that is, there exists $h \in R_{\mathbb{R}}(K)$ such that $ur + hg \in R_{\mathbb{R}}(K)^{-1}$.
To this end we look at the product representation

$$r(z) = \frac{C \prod(z - w_j)(z - w_j^*)}{q},$$

where $C, r_j$ are real numbers, $q \in \mathbb{R}[z]$ has no zeros in $K$ and $w_j$ denote the non-real zeros of $r$. It is enough to show that $(\prod(z-w_j)(z-w_j^*), g)$ is reducible. For the moment we will work with the complex Banach algebra $R(K)$. The Bass stable rank of $R(K)$ is 1, see Corach and Suarez [6, Theorem 3.1]. Fix a non-real zero $w$ of $r$. Then the unimodular pair $((z-w), g)$ is reducible in $R(K)$, i.e. there exists a $k \in R(K)$ such that $z-w+k(z)g(z)$ is invertible in $R(K)$. By symmetrization we conclude that $z-w^*+k(z^*)^*g(z)$ is also invertible in $R(K)$. Multiplying both results shows that

$$\nu_w := (z-w)(z-w^*)+\Bigg(\underbrace{k(z)(z-w^*)+k(z^*)^*(z-w)+k(z)k(z^*)^*g(z)}_{=:k_w(z)}\Bigg)g(z)$$

is invertible in $R(K)$. But $\nu_w, k_w$ are real symmetric and consequently, by taking the product of the $\nu_w$ corresponding to each non-real zero $w$, we see that $(ur, g)$ is reducible in $R_{\mathbb{R}}(K)$. Starting from

$$ur + hg = v \in R_{\mathbb{R}}(K)^{-1}$$

we conclude

$$urf + hfg = vf.$$

Recalling now that

$$urf + \beta g = U \in R_{\mathbb{R}}(K)^{-1}$$

gives us

$$vf + (\beta - hf)g = U \in R_{\mathbb{R}}(K)^{-1}.$$

This shows that $(f, g)$ is reducible. $\square$

Now we make the assumption that $\mathbb{C}\setminus K$ has only finitely many connected components. Then the real symmetric rational functions with poles off $K$ are dense in $A_{\mathbb{R}}(K)$ (and so $A_{\mathbb{R}}(K) = R_{\mathbb{R}}(K)$). Indeed, given $f \in A_{\mathbb{R}}(K)$ and $\epsilon > 0$, Mergelyan's theorem gives the existence of a rational function $\widetilde{r}$ with poles off $K$ such that

$$\|f - \widetilde{r}\|_\infty < \epsilon/2.$$

The desired *real symmetric* rational function $r$ can now be obtained simply by symmetrization:

$$r(z) := \frac{\widetilde{r}(z) + (\widetilde{r}(z^*))^*}{2} \quad (z \in K).$$

Then $r$ has poles off $K$ and $\|f - r\|_\infty < \epsilon$.

**Corollary 2.8.** *Let $K$ denote a real symmetric compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ has only finitely many connected components. Then Bass stable rank and topological stable rank of $A_{\mathbb{R}}(K)$ is at most 2.*

## 3. Preliminaries

3.1. **Lemmas on zero and level sets.** In this subsection, we collect some technical lemmas on zero sets and level sets.

**Definition 3.1.** For $g \in A(K)$ the *zero set* $Z_g$ of $g$ is

$$Z_g := \{z \in K \mid g(z) = 0\},$$

and for $\delta > 0$ the *level set* $Z_g(\delta)$ of $g$ is

$$Z_g(\delta) := \{z \in K \mid |g(z)| \leq \delta\}.$$

Of course the inclusion $Z_g \subset Z_g(\delta)$ holds.

The following property of level sets and zero sets will play an important role in the sequel.

**Lemma 3.2.** *Let $K$ denote a compact subset of $\mathbb{C}$. For every function $g \in A(K)$ and every $\delta > 0$ the following holds.*

1. *Every component of $\mathbb{C} \setminus Z_g(\delta)$ contains a component of $\mathbb{C} \setminus K$.*
2. *Every component of $\mathbb{C} \setminus Z_g$ contains a component of $\mathbb{C} \setminus K$.*

*These assertions also hold if $K^\circ = \emptyset$.*

*Proof.* Obviously, there is only one unbounded component $G_\infty$ of the complement, because $Z_g(\delta)$ (respectively $Z_g$) is compact. But then the unbounded component of $\mathbb{C} \setminus K$ belongs to $G_\infty$.

(1): Let $G$ denote a bounded component of $\mathbb{C} \setminus Z_g(\delta)$.

**Claim:** If there exists a bounded component $G$ of the complement $\mathbb{C} \setminus Z_g(\delta)$, then we must have $G \cap (\mathbb{C} \setminus K) \neq \emptyset$.

Assuming the contrary, there exists a bounded component of $\mathbb{C} \setminus Z_g(\delta)$ such that $G \subset K$. If $K^\circ = \emptyset$, then no such open $G$ exists, so we are done. If $K^\circ \neq \emptyset$, then we proceed as follows. Being in the complement of the level set, we must have $|g(z)| \geq \delta$ for all $z \in \partial G \subset K$. On the other hand, $|g(z)| \leq \delta$ for all $z \in \partial G \subset K$, because

$$\partial G \subset \partial(\mathbb{C} \setminus Z_g(\delta)) = \partial Z_g(\delta) \subset Z_g(\delta).$$

This gives $|g(z)| = \delta$ for all $z \in \partial G$. The maximum modulus theorem now shows that in fact we must have $G \subset Z_g(\delta)$, a contradiction. Hence no such bounded component of the complement of $Z_g(\delta)$ can exist. Thus $G$ must intersect a component $C$ of $\mathbb{C} \setminus K$. By connectedness we now conclude $C \subset G$, proving the assertion.

(2): The proof for $\mathbb{C} \setminus Z_g$ is entirely similar. □

In order to facilitate handling zero sets, we prove the following result, in which we enclose the zero set by finitely many closed sets.

**Lemma 3.3.** *Let $K$ denote a real symmetric compact subset of $\mathbb{C}$, and let $U$ denote an open real symmetric neighborhood of $K$ in $\mathbb{C}$. If $g \in A_{\mathbb{R}}(K)$, then for all $\delta > 0$, there exist finitely many closed sets $H_1, \dots, H_N \subset U$ lying symmetrically with respect to the real axis, that is, $H_j = H_k^*$ for certain $j, k$, with the following properties:*

(1) $Z_g \subset \bigcup_{j=1}^N H_j$ *and* $(\bigcup_{j=1}^N H_j) \cap K \subset Z_g(\delta)$.
(2) $H_j \cap H_k = \emptyset$ *($j \neq k$).*
(3) 1° *If no real zero of $g$ belongs to $H_j$ then $H_j \cap K \cap \mathbb{R} = \emptyset$, $H_j \cap K$ belongs entirely to the upper (respectively lower) half plane and $H_j \cap K = H_k^* \cap K$ for some $j \neq k$.*
    2° *If at least one real zero belongs to $H_j$ (that is, $x_0 \in Z_g \cap H_j \cap \mathbb{R}$), then $H_j = H_j^*$ holds and $H_j$ is connected.*
(4) *If the zero $z_0$ belongs to $H_j$ then there exists a disc $D$ with center $z_0$ such that $D \cap K \subset H_j$.*

Before we prove this lemma, we make the following observations:

**Remarks 3.4.**

(1) A construction of the covering sets in $K$ is possible if the components of the (relatively) open sets $H := \{z \in K \mid |g(z)| < \delta\}$ are open. This is the case if $H$ is locally connected, for example if $K$ is bounded by finitely many pairwise disjoint Jordan curves.
(2) A similar result is true in case $g \in A(K)$, where $K$ is compact but not necessarily real symmetric. The corresponding covering of the zero set intersected with $K$ belongs to $Z_g(\delta)$ and consists of pairwise disjoint, connected sets. Assertions (1), (2) and (4) remain true.

*Proof.* We extend the real symmetric continuous function $g$ from $K$ likewise to the closure $\overline{U}$, the extension being denoted by $g_0$. The zero set $Z_g \subset K$ is compact, and so finitely many components $K_j$, $j = 1, \dots, M$, of the open set $H := \{z \in U \mid |g_0(z)| < \delta\}$ will suffice to cover $Z_g$. Since $H$ is symmetric with respect to the real axis, its components are symmetric as well. Unfortunately, the closures $\overline{K_j}$ need not be disjoint. However, we may take the closed connected components of $\bigcup_{j=1}^N \overline{K_j}$, and there are at most $M$ such components. These components are symmetric as well.

To ensure all four assertion we must eventually truncate the closed sets $\overline{K_j}$:

1° If no real zero of $g$ belongs to the set $\overline{K_j}$, then $|g(z)| \geq \rho_j > 0$ for all $z \in (\overline{K_j} \cap K \cap \mathbb{R}) \times (|\mathrm{Im}(z)| \leq \delta_j)$ for a sufficiently small $\delta_j > 0$. Hence no zero of $g$ belongs to $z \in (\overline{K_j} \cap K \cap \mathbb{R}) \times (|\mathrm{Im}(z)| \leq \delta_j)$. We truncate as follows: $H_j := \overline{K_j} \cap K \cap (\mathrm{Im}(z) \geq \delta_j)$ (and a corresponding reflected set $H_j^*$ in the lower half plane). The closed set $\overline{K_j} \cap K$ splits in two closed sets belonging entirely to the upper (respectively lower) half plane.

2° If at least one real zero of $g$ belongs to $\overline{K_j}$, then we don't truncate, that is, $H_j := \overline{K_j}$. By symmetry we have $H_j = H_j^*$ and $H_j = \overline{K_j}$ is connected, because $K_j$ is.

All the zeros of $g$ belong to exactly one closed set $K_j$, $j = 1, \dots, N$, by construction.

To prove the last assertion take a small disc $D$ with center $z_0 \in Z_g \cap H_j$ such that $D \subset \{|\mathrm{Im}(z)| \geq \delta_j\}$ in case 1° above and $|g(z)| < \delta$ holds for all $z \in D \cap K$. By the construction, $D \cap K \subset \bigcup_{j=1}^N H_j$. Because the sets $H_j$ are compact and pairwise disjoint, they have a positive distance from each other. So choosing the radius of the disc $D$ small enough gives $D \cap K \subset H_j$. □

3.2. **Factorization theorem for units.** We begin with the following definition sign-functions, and prove Theorem 3.6 on units, which will be needed later.

**Definition 3.5.** A *sign-function* $\chi \in A_{\mathbb{R}}(K)$ is a function satisfying $\chi^2 = 1$ on $K$. (Note that $K$ may be disconnected.)

**Theorem 3.6** (Units). *Let $K$ denote a real symmetric compact subset of $\mathbb{C}$ and let $\mathcal{A}$ denote one of the algebras $A_{\mathbb{R}}(K), C_{\mathbb{R}}(K)$ respectively. For any unit $u \in \mathcal{A}^{-1}$ we have two factorizations:*

(F1) *$u = p \cdot \exp(H)$, where $p$ denotes a real symmetric invertible rational function $p \in \mathcal{A}^{-1}$ and a function $H \in C(K)$*

(F2) *$u = p \cdot \chi \cdot \exp(h)$, where $p$ denotes a real symmetric invertible rational function $p \in \mathcal{A}^{-1}$, $\chi \in \mathcal{A}$ is a sign-function, and $h$ is a real symmetric function in $\mathcal{A}$.*

*The rational function $p$ in* (F1) *is the same as that in* (F2).

*Proof.* First of all we prove the theorem in case $\mathcal{A} = C_{\mathbb{R}}(K)$.

(F1): We prove the existence of a real symmetric rational function $p$ with poles off $K$ and $H \in C(K)$ such that $u = p \cdot \exp(H)$.

Without symmetry this would be the assertion of Theorem 4.29 in [3]. The need for symmetry causes some difficulty in the proof in [3], and so we include it in modified form:

By an affine transformation, preserving symmetry, we may assume that

$$K \subset (0,1) \times (-1,1) =: Q.$$

Tietze's Theorem gives a continuous extension $f_0 : Q \to \mathbb{C}$ of $u$ from $K$ to $Q$. It can be chosen to be real symmetric. Let

$$L := f_0^{-1}(\{0\}).$$

This is a closed subset of $Q$ disjoint from $K$, so by compactness there exists an $r > 0$ such that

$$|z - w| \geq r \quad (z \in K,\ w \in L).$$

Let $m$ be a positive integer such that

$$m > \frac{\sqrt{2}}{r}$$

and consider the squares

$$Q_{j,k} := \left[\frac{j-1}{m}, \frac{j}{m}\right] \times \left[\frac{k-1}{m}, \frac{k}{m}\right] \text{ with center } p_{j,k} := \frac{j-1/2}{m} + i\frac{k-1/2}{m}$$

for all $j, k \in \{1, \dots, m\}$, and their reflections

$$Q_{j,k} := \left[\frac{j-1}{m}, \frac{j}{m}\right] \times \left[\frac{k+1}{m}, \frac{k}{m}\right] \text{ with center } p_{j,k} := \frac{j-1/2}{m} + i\frac{k+1/2}{m}$$

for all $k = -m, \dots, -1$ and $j = 1, \dots, m$.

As will be seen in a moment, the two symmetrically situated squares $Q_{j,-1}$ and $Q_{j,1}$ play a different role.

Hence we define the rectangles $R_j := Q_{j,-1} \cup Q_{j,1}$ with center $p_j := \frac{j-1/2}{m}$, $j = 1, \dots, m$. We define

$$\begin{aligned} \mathcal{K} &:= \{(j,k) \mid 1 \le j, |k| \le m \text{ and } Q_{j,k} \cap K \neq \emptyset\} \\ \mathcal{K}_\emptyset &:= \{(j,k) \mid 1 \le j, |k| \le m \text{ and } Q_{j,k} \cap K = \emptyset\}\,. \end{aligned}$$

By symmetry we have either the case that both $(j,-1)$ and $(j,1)$ belong to $\mathcal{K}$, or the case that both $(j,-1)$ and $(j,1)$ belong to $\mathcal{K}_\emptyset$, hence we have

$$R_j \subset \bigcup_{(j,k)\in\mathcal{K}} Q_{j,k} \quad \text{or} \quad R_j \subset \bigcup_{(j,k)\in\mathcal{K}_\emptyset} Q_{j,k}, \quad (j = 1, \dots m).$$

We have that $K \subset K_1$, where $K_1$ is the closed set defined by

$$K_1 := \bigcup_{(j,k)\in\mathcal{K}} Q_{j,k},$$

and from the choice of $m$ and $r$ it also follows that

$$K_1 \subset Q \setminus L.$$

Note that either $R_j \subset K_1$ or $R_j \cap K_1 = \emptyset$ holds for $j = 1, \dots, m$. Let $f_1$ be the restriction of $f_0$ to $K_1$. Since $K_1$ is a union of squares $Q_{j,k}$, each interval $\left\{\frac{j}{m}\right\} \times \left[\frac{k-1}{m}, \frac{k}{m}\right]$ and each interval $\left[\frac{j-1}{m}, \frac{k}{m}\right] \times \left\{\frac{k}{m}\right\}$, $k = 1, \dots, m$, either lies wholly in $K_1$ or meets $K_1$ only at endpoints or does not meet $K_1$ at all. By symmetry this is also true for the reflected squares in the lower half plane. At each endpoint where $f_1$ is not already defined give it the value 1. Then for any interval $I = [a,b]$ of the above kind which does not lie wholly in $K_1$, $f_1(a), f_1(b)$ are non-zero complex numbers and $f_1$ is not defined in $(a,b)$. Extending the continuous function with values $\log f_1(a), \log f_1(b)$ from the compact set $\{a,b\}$ to $[a,b]$ gives a function which when exponentiated gives a continuous extension of $f_1$ to a map of $I$ into $\mathbb{C} \setminus \{0\}$.

In order to preserve symmetry we now proceed in a different manner than in [3]:

By symmetry we have a real symmetric, continuous, zero-free extension $f_2$ of $f_1$ to the closed set

$$K_2 := K_1 \cup \left( \bigcup_{j=1}^{m} \bigcup_{2 \le |k| \le m} \partial Q_{j,k} \right)$$

of the rectangle $Q$. The same is true for the boundaries of the rectangles $R_j$, $j = 1, \dots, m$. Note that the values on the boundary $\partial R_j$ are already defined by the values in $K_2$. We arrive at a symmetric, continuous, zero-free extension $f_3$ of $f_2$ to the closed set

$$K_3 := K_2 \cup \left( \bigcup_{j=1}^{m} \partial R_j \right)$$

of the rectangle $Q$. The definitions of $\mathcal{K}_\emptyset$ and $K_1$ then show that

$$K_3 \cap Q_{j,k} = \partial Q_{j,k} \quad \text{for } (j,k) \in \mathcal{K}_\emptyset \text{ and } |k| \ge 2$$

and

$$K_3 \cap R_j = \partial R_j \quad \text{for } (j,-1) \text{ and } (j,1) \in \mathcal{K}_\emptyset.$$

For each such $(j,k)$ there exists an integer $n_{j,k}$ such that $(z - p_{j,k})^{n_{j,k}} f_3(z)$ (respectively $(z - p_j)^{n_j} f_3(z)$) has a zero-free, continuous extension $F_{j,k}$ to $Q_{j,k}$ (respectively $R_j$), see [3, Theorems 4.23 and 4.24]. Note that we can use reflection to obtain $p_{n,-k} = p_{n,k}^*$ and $n_{j,-k} = n_{j,k}$. Hence we can consistently define $F_0$ on the rectangle $Q$ by

$$F_0(z) := f_3(z) \prod_{(j,k) \in \mathcal{K}_\emptyset,\, |k| \ge 2} (z - p_{jk})^{n_{j,k}} \prod_{(j,\pm 1) \in \mathcal{K}_\emptyset} (z - p_j)^{n_j}$$

for $z \in K_3$, and

$$F_0(z) := F_{j',k'}(z) \prod_{(j,k) \in \mathcal{K}_\emptyset \setminus (j'k'),\, |k| \ge 2} (z - p_{jk})^{n_{j,k}} \prod_{(j,\pm 1) \in \mathcal{K}_\emptyset,\, j \ne j'} (z - p_j)^{n_j}$$

for all $z \in Q_{j',k'}$ with $(j'k') \in \mathcal{K}_\emptyset$ and all $z \in R_{j'}, (j', \pm 1) \in \mathcal{K}_\emptyset$. This function is continuous and zero-free on $Q$, and so it has a continuous logarithm there. The restriction to $K$ gives the desired symmetric product form. This completes the proof of (F1).

(F2): Because the units $u$ and $p$ are real symmetric, we derive

$$\exp(H(z)) = \exp((H(z^*))^*) \quad (z \in K).$$

Hence for all $z \in K$, there exists an integer $k = k(z)$ such that

$$H(z) - (H(z^*))^* = 2k\pi i,$$

and so

$$\frac{H(z) + (H(z^*))^*}{2} = (H(z^*))^* + k\pi i.$$

But the difference in the first identity is a bounded continuous function on $K$, and so only finitely many integers $k_j$, $j = 1, \dots, m$ can occur. Thus $K$

splits in disjoint compact sets $K_j$, $j = 1, \dots, m$, and the sign-function $\chi$ is given by

$$\chi(z) := \exp(-k_j \pi i) \quad (z \in K_j, j = 1, \dots, m).$$

Defining $h \in C_{\mathbb{R}}(K)$ by

$$h(z) = \frac{H(z) + (H(z^*))^*}{2} \quad (z \in K),$$

we conclude that

$$\exp(H(z)) = \exp((H(z^*))^*) = \exp(h(z)) \cdot \chi(z) \quad (z \in K),$$

hence

$$u = p \cdot \exp(H) = p \cdot \chi \cdot \exp(h).$$

Note that as $u, p, h$ are real symmetric, $\chi$ is real symmetric as well.

The remaining case $\mathcal{A} = A_{\mathbb{R}}(K)$ now follows from the first case as follows. By the holomorphic inverse function theorem applied to $z \mapsto \exp(z)$, we see that it has a local holomorphic inverse around each point $z_0$, say $g_{z_0}$. Thus $z \mapsto h(z) = g_{z_0}(u(z)\chi(z)(p(z))^{-1})$ is holomorphic near $z_0$ as well. $\square$

3.3. **Lemma on relocation of poles.** In Sections 5 and 6, we will often use the following useful fact.

**Lemma 3.7.** *Let $K, L$ denote compact sets in $\mathbb{C}$ with $L \subset K$ and every component of $\mathbb{C} \setminus L$ contains a component of $\mathbb{C} \setminus K$. Suppose that $f \in C(K)$ is such that*

$$f(z) = p(z) \exp(k(z)) \quad (z \in L),$$

*where $k \in C(L)$ and $p$ is a rational function with poles and zeros off $L$. Then:*

(1) *There exists a rational function $\widetilde{p}$ and a $\widetilde{k} \in C(L)$ such that*

$$f(z) = \widetilde{p}(z) \exp(\widetilde{k}(z)) \quad (z \in L),$$

*and $\widetilde{p}$ has its poles and zeros off $K$.*

(2) *If $K, L, p, k$ are in addition real symmetric, then we can ensure that the $\widetilde{p}, \widetilde{k}$ constructed in* (1) *above are real symmetric as well.*

In other words, we can shift the poles and zeros of $p$ from $\mathbb{C} \setminus L$ to $\mathbb{C} \setminus K$. In our applications later, typically $L = Z_g$, where $g \in A(K)$.

*Proof.* Let $a$ denote a pole or zero of $p$ belonging to the component $G$ of $\mathbb{C} \setminus L$. By assumption every component of $\mathbb{C} \setminus L$ contains a component of $\mathbb{C} \setminus K$, and so there is a common point $b \in G \cap (\mathbb{C} \setminus K)$. Because $L$ does not separate $a$ and $b$ (that is, they lie in the same connected component $G$ of the complement of $L$), it follows from Eilenberg's theorem [3, Exercise 4.36] that there exists a logarithm $l \in C(L)$ such that

$$\frac{z-a}{z-b} = \exp(l(z)) \quad (z \in L).$$

Thus the claim in (1) follows.

If in addition $K, L, p$ are real symmetric, then in the above we have

$$\frac{z-a^*}{z-b^*} = \exp((l(z^*))^*) \quad (z \in L^* = L).$$

Consequently,

$$\frac{z-a}{z-b} \cdot \frac{z-a^*}{z-b^*} = \exp(l(z) + (l(z^*))^*)$$

for all $z \in L$. $\square$

## 4. Reducibility in real symmetric subalgebras of $A_{\mathbb{R}}(K)$

In this section, we will prove our main result in Theorem 4.1.

**Theorem 4.1.** *Let $K$ denote a real symmetric compact subset of $\mathbb{C}$. The following assertions are equivalent for any unimodular pair $(f, g) \in A_{\mathbb{R}}(K)^2$:*

(1) *There exists a sign-function $\chi \in A_{\mathbb{R}}(K)$, an invertible rational function $p \in A_{\mathbb{R}}(K)^{-1}$, a continuous function $l \in C(K)$ such that*

$$\text{for all } z \in Z_g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} = \exp(l(z)), \quad \text{and}$$

$$\text{for every real zero } z \text{ of } g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} > 0.$$

(2) *$(f, g)$ is reducible in $A_{\mathbb{R}}(K)$, that is, there exists a unit $u \in A_{\mathbb{R}}(K)^{-1}$ and there exists $k \in A_{\mathbb{R}}(K)$ such that $f + kg = u$.*

**Remarks 4.2.**

(1) There always exists a continuous logarithm $h$ for $\chi(z) \cdot f(z)/p(z)$ on $Z_g$ provided that $\mathbb{C} \setminus Z_g$ is connected, see [3, Corollary 4.33], and by Tietze's theorem, this can be extended to a continuous function on $K$.

(2) Since the complex algebra $A(K)$ has Bass stable rank 1 (see for instance [5, Theorem 2.3], [13] or Theorem 5.1), there always exists a $k \in A(K)$ and a unit $u \in A(K)^{-1}$ such that $f + kg = u$. Again we can deduce $u = \chi \cdot p \cdot \exp(v)$ for a sign-function $\chi$, and certain $p, v \in A(K)$ by the analogue of the unit representation. The important point is that for reducibility in $A_{\mathbb{R}}(K)$, we must have *real symmetric* functions $\chi, p, v$ and the positivity on real zeros of $g$.

*Proof.* (2)⇒(1): If there exists $k \in A_{\mathbb{R}}(K)$ and a unit $u \in A_{\mathbb{R}}(K)^{-1}$ such that $f + kg = u$, we use Theorem 3.6 to factor $u = p \cdot \chi \cdot \exp(l)$, where $p \in A_{\mathbb{R}}(K)^{-1}$, $l \in C_{\mathbb{R}}(K)$ and $\chi \in A_{\mathbb{R}}(K)$ is a sign-function. Obviously,

$$\text{for all } z \in Z_g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} = \exp(l(z)) \quad \text{and}$$

$$\text{for all real zeros of } g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} > 0.$$

(1)⇒(2): Now assume that there exists a sign-function $\chi \in A_{\mathbb{R}}(K)$, an invertible rational function $p \in A_{\mathbb{R}}(K)^{-1}$, a function $l \in C(K)$ such that

$$\text{for all } z \in Z_g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} = \exp(l(z)) \qquad \text{and}$$

$$\text{for all real zeros of } g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} > 0.$$

We abbreviate

$$f_0 := \frac{\chi \cdot f}{p}.$$

**Step 1**: There exist functions $h, k \in C_{\mathbb{R}}(K)$ with continuous partial derivatives in the interior $K^\circ$ of $K$, such that $f_0 + h \cdot g = \exp(k)$. Moreover, $\frac{\partial h}{\partial \overline{z}}, \frac{\partial k}{\partial \overline{z}}$ are bounded in $K^\circ$.

We think of $f_0, g$ as extended to a sufficiently small real symmetric neighborhood $U \supset K$. To be precise: $f, \chi$ are extended symmetrically to $U$.

Since $(f_0, g)$ is unimodular in $C_{\mathbb{R}}(K)$, there exists $\delta > 0$ and a sufficiently small real symmetric neighborhood $U \supset K$ such that $|f_0(z)| + |g(z)| \geq 4\delta$ for all $z \in U$. The level sets with respect to $U$ are denoted by

$$Z_g^U(\delta) := \{z \in U \mid |g(z)| \leq \delta\}.$$

By assumption, we have a continuous logarithm of $f_0$ on $Z_g$.

**Claim:** For sufficiently small $\delta > 0$ there exists $L \in C(U)$ such that

$$\frac{\chi(z) \cdot f(z)}{p(z)} = \exp(L(z)) \quad (z \in Z_g^U(2\delta))$$

and $L(z) = l(z)$ $(z \in Z_g)$.

Fix a symmetric, continuous extension $l_0$ of $l$ to $U$. For sufficiently small $\delta > 0$ we have

$$\operatorname{Re}\left(f_0(z)\exp(-l_0(z))\right) > 1/2 \quad (z \in Z_g^U(2\delta)).$$

Hence there exists a continuous logarithm $w$ of the function $f_0 \exp(-l_0)$; the principal branch of the logarithm will do. But then we have

$$f_0(z) = \exp(l_0(z) + w(z)) \quad (z \in Z_g^U(2\delta)).$$

This completes the proof of the Claim above.

By Lemma 3.3 (with $2\delta$ instead of $\delta$), there exist finitely many pairwise disjoint closed sets $H_1, \dots, H_N \subset U$ lying symmetrically with respect to the real axis, such that: $Z_g \subset \cup_{j=1}^N H_j$ and $|g(z)| \leq 2\delta$ holds there. Hence $|f_0(z)| \geq 2\delta$ in the union of this sets intersected with $K$.

In particular, from the Claim above, there exist functions $l_j$, continuous in the closed sets $H_j \subset Z_g^U(2\delta)$ such that

$$f_0(z) = \exp(l_j(z)), \quad (z \in H_j, \quad j = 1, \dots, N).$$

By assertion (3) of Lemma 3.3, we have $H_j \cap K \cap \mathbb{R} = \emptyset$ if no real zero of $g$ belongs to $H_j$. Moreover, $H_j \cap K$ belongs entirely to the upper (respectively lower) half plane. The desired logarithm is very easy to obtain for these sets, because they don't intersect the real line. By symmetry we have $H_j \cap K = H_k^* \cap K$ for some $j \neq k$. Hence we may redefine $l_j(z) = (l_k(z^*))^*$.

Thus only the case of a real zero $x_0$ of $g$ belonging to $H_j$ remains to be discussed. In this case $H_j$ is connected. By assumption $f_0(x_0) > 0$ holds for every real zero $x_0$ of $g$. Because $f_0$ is symmetric we derive

$$f_0(z) = \exp(l_j(z)) = \exp((l_j(z^*))^*) \quad (z \in H_j = H_j^*).$$

Because $H_j$ is connected and $l_j$ is continuous in $H_j$, there exists an integer $m$ such that

$$l_j(z) = (l_j(z^*))^* + 2m\pi i \quad (z \in H_j = H_j^*).$$

Restricting to the real zero $x_0 \in H_j \cap \mathbb{R}$ of $g$ gives $\mathrm{Im}\ l_j(x_0) = m\pi$. As $f_0(x_0) = \exp(l_j(x_0)) > 0$, the integer $m$ must be even. Now $l_j - m\pi i$ is the desired symmetric logarithm of $f_0$ on $H_j = H_j^*$.

Let $\chi_j$ denote a smooth real symmetric function being identically 1 on $H_j$ and identically 0 outside a neighborhood $W \subset U$ of $H_j$ sufficiently small such that this neighborhood doesn't intersect the other sets $H_k$ and $|g(z)| \leq 3\delta$ $(z \in K \cap W)$. Observe that the logarithm of $f_0$ exists and is bounded on $Z_g^U(2\delta)$. Define the function $k$ by

$$k := \sum_{j=1}^{N} \chi_j l_j.$$

Of course $\frac{\partial k}{\partial \bar{z}}$ is bounded in $K^\circ$. By construction $k \in C_{\mathbb{R}}(K)$ and $k(z) = l_j(z)$ $(z \in H_j \cap K, j = 1, \dots N)$. The desired function $h$ can now be defined as follows:

$$h(z) := \begin{cases} \frac{\exp k(z) - f_0(z)}{g(z)} & \text{for} \quad z \in K \setminus Z_g, \\ 0 & \text{for} \quad z \in Z_g. \end{cases}$$

This function belongs to $C_{\mathbb{R}}(K)$ because by Lemma 3.3 for the zero $z_0 \in H_j$ there exists a disc $D$ with center $z_0$ such that $D \cap K \subset H_j$, so we have $h(z) = 0$ for all $z \in D \cap K$. This implies also that $\frac{\partial h}{\partial \bar{z}}$ is bounded in $K^\circ$.

**Step 2:** There exist $h, k \in A_{\mathbb{R}}(K)$ such that $f_0 + hg = \exp(k)$.

With the functions from Step 1, we define the real symmetric continuous function on $K$

$$F := \frac{f_0}{f_0 + hg} = f_0 \exp(-k).$$

Clearly

$$\frac{F}{f_0} \cdot f_0 + \frac{1-F}{g} \cdot g = 1.$$

Of course, we have that

$$\frac{F}{f_0} = \exp(-k) \text{ and} \tag{1}$$

$$\frac{1-F}{g} = h \exp(-k) \tag{2}$$

are real symmetric, continuous in $K$, $\frac{\partial h}{\partial \overline{z}}, \frac{\partial k}{\partial \overline{z}}$ are bounded in the interior $K^\circ$ of $K$. However, $h, k$ are not necessarily analytic in $K^\circ$. Therefore we seek for a $u \in C_{\mathbb{R}}(K)$, which is continuously differentiable in $K^\circ$ such that

$$\frac{\partial}{\partial \overline{z}}\left[\frac{\exp(ug)}{f_0 + hg}\right] = 0,$$

which implies the analyticity of $\frac{F}{f_0}\exp(ug)$ and $F\exp(ug)$. This yields the inhomogeneous $\overline{\partial}$-equation

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{f_0 + hg} \cdot \frac{\partial h}{\partial \overline{z}} =: \mu.$$

As is well known, one solution $u$ to the $\overline{\partial}$-equation is given by

$$u(z) = \frac{1}{2\pi i}\int_{K^\circ} \frac{\mu(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} \quad (z \in K). \tag{3}$$

(See for instance [7, §1, Chap. VIII], where the result is given for the disc; in the general case, given a point $z_0 \in K^\circ$, we first consider a disc $\Delta$ centered around $z_0$, and then split the integral in (3) into an integral over $\Delta$ and over $K^\circ \setminus \Delta$.) It is easy to check that $u$ given by (3) is in fact real symmetric. It is continuous on $K$ because it is the convolution of the bounded function $\mu$ and a $L^1$-function.

We now "replace" the function $F$ by $F\exp(ug)$. By multiplying (1) by $\exp(ug)$, we obtain that the function

$$\alpha := \frac{F}{f_0}\exp(ug) = \exp\left[ug - k\right]$$

belongs to $A_{\mathbb{R}}(K)$, and $\alpha$ is an exponential. Using (2), we also see that the function

$$\beta := \frac{1 - F\exp(ug)}{g} = \left[h + f_0\frac{1 - \exp(ug)}{g}\right]\exp(-k)$$

is continuous up to all the boundary of $K$ and is analytic in the interior $K^\circ$. Since the identity

$$\alpha f_0 + \beta g = 1$$

holds, this completes Step 2.

Recalling the abbreviation $f_0 := \frac{\chi \cdot f}{p}$ we see that also $(f, g)$ is reducible in $A_{\mathbb{R}}(K)$. $\square$

We generalize this characterization to some subalgebras of $A_{\mathbb{R}}(K)$, restricting ourselves to compact symmetric subset $K$ of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ has finitely many components.

**Definition 4.3.** If $K$ denotes a compact subset of $\mathbb{C}$, then we say the *corona theorem* holds for $\mathcal{A}$ ($\subset A_{\mathbb{R}}(K)$) if the following is true for all $n \in \mathbb{N}$:

$(f_1, \dots, f_n) \in U_n(\mathcal{A})$ if and only if there exists a $\delta > 0$ such that for all $z \in K$, $\sum_{j=1}^n |f_j(z)| \geq \delta$, that is, if and only if the functions $f_1, \dots, f_n$ have no common zero in $K$.

That the corona theorem holds for $A_{\mathbb{R}}(K)$ follows easily from the corona theorem for the complex algebra $A(K)$ by symmetrization of the solution. We refer the reader to [9] for a constructive proof (not using any Gelfand theory nor Banach algebra theory) of the corona theorem for certain subalgebras of $A(K)$ under mild assumptions on $K$.

**Corollary 4.4.** *Let $K$ denote a compact real symmetric subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ has finitely many components. Let $\mathcal{A}$ denote a subalgebra of $A_{\mathbb{R}}(K)$ containing all real symmetric rational functions with poles off $K$, such that the corona theorem holds for $\mathcal{A}$. The following are equivalent for any unimodular pair $(f, g) \in U_2(\mathcal{A})$:*

(1) *There exists a sign-function $\chi \in A_{\mathbb{R}}(K)$, an invertible rational function $p \in \mathcal{A}^{-1}$ and a function $l \in C(K)$ such that*
$$\textit{for all } z \in Z_g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} = \exp(l(z)), \quad \textit{and}$$
$$\textit{for every real zero } z \textit{ of } g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} > 0.$$

(2) *$(f, g)$ is reducible in $\mathcal{A}$, that is, there exists a unit $u \in \mathcal{A}^{-1}$ and there exists a $k \in \mathcal{A}$ such that $f + kg = u$.*

*Proof.* (2)⇒(1): Let there exist a $k \in \mathcal{A} \subset A_{\mathbb{R}}(K)$ and a unit $u \in \mathcal{A}^{-1} \subset A_{\mathbb{R}}(K)^{-1}$ (because of the corona theorem) such that $f + kg = u$. Using Theorem 3.6, we can factor
$$u = p \cdot \chi \cdot \exp(l),$$
where the real symmetric rational $p \in A_{\mathbb{R}}(K)^{-1}$ also belongs to $\mathcal{A}^{-1}$ because of the corona theorem, the function $\chi \in A_{\mathbb{R}}(K)$ is a sign-function, and $l \in C_{\mathbb{R}}(K)$. Clearly,
$$\text{for all } z \in Z_g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} = \exp(l(z)), \quad \text{and}$$
$$\text{for all real zeros of } g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} > 0.$$

(1)⇒(2): Assume that there exists a sign-function $\chi \in A_{\mathbb{R}}(K)$, an invertible rational function $p \in \mathcal{A}^{-1} \subset A_{\mathbb{R}}(K)^{-1}$, and a function $l \in C(K)$ such that
$$\text{for all } z \in Z_g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} = \exp(l(z)), \quad \text{and}$$
$$\text{for all real zeros of } g, \quad \frac{\chi(z) \cdot f(z)}{p(z)} > 0.$$
Using Theorem 4.1 the pair $(f, g)$ is reducible in $A_{\mathbb{R}}(K)$, that is, there exists a unit $u \in A_{\mathbb{R}}(K)^{-1}$ and there exists a $k \in A_{\mathbb{R}}(K)$ such that
$$f + kg = u.$$

Using Mergelyan's theorem there exist rational functions $k_n$ with poles off $K$ converging uniformly to $k$ on $K$. Because $K$ is real symmetric, we can also approximate by the symmetrization of $k_n$, that is, the real rational functions $\widetilde{k_n}$ given by

$$\widetilde{k_n}(z) := \frac{k_n(z) + (k_n(z^*))^*}{2}$$

converge uniformly to $k$ too. Since $u$ is a unit in $A_{\mathbb{R}}(K)$, we must have

$$|u(z)| > \delta > 0 \quad (z \in K).$$

Choose a real symmetric rational function $k_n \in \mathcal{A}$ near $k$ such that

$$|u(z) - (k(z) - k_n(z))g(z)| > \delta/2 > 0 \quad (z \in K).$$

We conclude that $f + k_n g = u - (k - k_n)g$ (which belongs to the algebra $\mathcal{A}$) has no zeros in $K$ and so it is invertible because of the corona theorem. This completes the proof of the reducibility of $(f, g)$ in $\mathcal{A}$. $\square$

## 5. Bass stable rank of $A(K)$

The methods developed in the previous sections can be applied to prove that all unimodular pairs in $A(K)$ ($K$ compact in $\mathbb{C}$), are reducible, that is, the Bass stable rank of $A(K)$ is 1. This is well known (see [5, Theorem 2.3], [13]), but we present a proof which is independent of Banach algebra theory and elementary stable rank theory.

**Theorem 5.1.** *Let $K$ denote a compact set in $\mathbb{C}$. Then the stable rank of the algebra $A(K)$ is 1, that is, every unimodular pair $(f, g)$ is reducible in $A(K)$.*

*Proof.* Given an unimodular pair $(f, g)$ in $A(K)$, we must show the existence of $h \in A(K)$ and $u \in A(K)^{-1}$, such that $f + hg = u$. By unimodularity, we have a $\delta > 0$ such that $|f(z)| + |g(z)| \geq 4\delta$ for all $z \in K$. The function $f$ is continuous and zero-free in the zero set $Z_g$. Using the unit representation [3, Theorem 4.29] (Theorem 3.6 above without symmetry), we may write

$$f(z) = p(z) \cdot \exp(k(z)) \quad (z \in Z_g), \tag{4}$$

where $p$ denotes a rational function with poles off $Z_g$, and $k \in C(Z_g)$. By Lemma 3.2, every component of $\mathbb{C} \setminus Z_g$ contains a component of $\mathbb{C} \setminus K$. By Lemma 3.7, we can shift the poles and zeros of $p$ from $\mathbb{C} \setminus Z_g$ to $\mathbb{C} \setminus K$.

As in the justification of the Claim in Step 1 of the proof of Theorem 4.1, we extend (4) to the level set $Z_g(2\delta)$ for sufficiently small $\delta$, that is,

$$f(z) = p(z) \cdot \exp(k(z)) \quad (z \in Z_g(2\delta)), \tag{5}$$

where $p$ denotes a rational function with poles off $K$ and $k \in C(Z_g(2\delta))$.

The rest of the proof is now analogous to the proof of Theorem 4.1, that is we use Lemma 3.3 (without symmetry of course) to facilitate handling the zero set of $g$ with finitely many closed connected, pairwise disjoint subsets of $U \supset K$ lying within $Z_g^U(\delta)$. Then we use $\overline{\partial}$-equations to make the smooth solutions for reducibility analytic in $K$. $\square$

## 6. When is bsr $C_{\mathbb{R}}(K), A_{\mathbb{R}}(K) = 1$?

In Section 2, we gave a necessary and sufficient condition on $K$ so that the Bass stable rank of $R_{\mathbb{R}}(K)$ is 1. In this section we give a similar characterization for the algebras $C_{\mathbb{R}}(K)$ and $A_{\mathbb{R}}(K)$.

6.1. **Topological theorems.** We begin by proving two purely topological theorems, which are probably well known to the workers in the field; see [13] for a different characterization of the first one using the so called "boundary principle".

**Theorem 6.1.** *Let $K, L$ denote compact sets in $\mathbb{C}$ such that $L \subset K$. The following assertions are equivalent:*

(1) *Every continuous zero-free function $f \in C(L)$ can be extended to a continuous zero-free function $F \in C(K)$.*
(2) *Every component of $\mathbb{C} \setminus L$ contains a component of $\mathbb{C} \setminus K$.*

*Proof.* (1)$\Rightarrow$(2): By assumption, every continuous zero-free $f \in C(L)$ can be extended to a continuous zero-free function $F \in C(K)$. Assuming the contrary of (2), there exists a component $G$ of $\mathbb{C}\setminus L$ containing no component of $\mathbb{C} \setminus K$, that is, $G \subset K$. Of course, $G$ is not the unbounded component of $\mathbb{C} \setminus L$, because $G \subset K$. Fix a point $w \in G$. We conclude that $w \in K$ and $w \in \mathbb{C} \setminus L$. By assumption we can extend the continuous, zero-free function $f \in C(L)$ given by $f(z) := z - w$ to a continuous, zero-free function $F \in C(K)$. In particular, we have such an extension to the set $L \cup G \subset K$. This contradicts Theorem 4.31 in [3].

(2)$\Rightarrow$(1): Let $f \in C(L)$ be a function $f$ which is zero-free in the compact set $L$. Using the unit representation result [3, Theorem 4.29] (that is, Theorem 3.6 without symmetry), we may write

$$f(z) = p(z) \cdot \exp(k(z)) \quad (z \in L), \tag{6}$$

where $p$ denotes a rational function with poles and zeros off $L$, and $k \in C(L)$. By assumption every component of $\mathbb{C} \setminus L$ contains a component of $\mathbb{C} \setminus K$. Thus, applying Lemma 3.7, we can shift the poles and zeros of $p$ from $\mathbb{C} \setminus L$ to $\mathbb{C} \setminus K$, and so

$$f(z) = p(z) \cdot \exp(k(z)) \quad (z \in L), \tag{7}$$

where $p$ denotes a rational function with poles and zeros off $K$, and $k \in C(L)$. By Tietze's extension theorem we can extend $k$ continuously to $k_{\mathrm{e}} \in C(K)$. The desired extension $F$ is now given by

$$F(z) := p(z) \cdot \exp(k_{\mathrm{e}}(z)) \quad (z \in K).$$

This completes the proof. $\square$

So Lemma 3.2 now shows that given $g \in A(K)$, then every continuous, zero-free function can likewise be extended from the level set $Z_g(\delta)$ to $K$.

**Theorem 6.2.** *Let $K, L$ denote compact, real symmetric sets in $\mathbb{C}$ such that $L \subset K$. The following assertions are equivalent:*

(1) *Every continuous zero-free function $f \in C_{\mathbb{R}}(L)$ can be extended to a continuous zero-free function $F \in C_{\mathbb{R}}(K)$.*
(2) *Every component of $\mathbb{C} \setminus L$ contains a component of $\mathbb{C} \setminus K$ and every sign-function $\chi \in C_{\mathbb{R}}(L)$ can be extended to a continuous zero-free function $\chi_e \in C_{\mathbb{R}}(K)$.*

*Proof.* (1)⇒(2): By assumption, every continuous zero-free $f \in C_{\mathbb{R}}(L)$ can be extended to a continuous zero-free function $F \in C_{\mathbb{R}}(K)$. If there exists a component $G$ of $\mathbb{C} \setminus L$ containing no component of $\mathbb{C} \setminus K$, then $G \subset K$ (see the implication (1) ⇒ (2) in the proof of Theorem 6.1). Of course, $G$ is not the unbounded component of $\mathbb{C} \setminus L$, because $G \subset K$. Fix a point $w \in G$. We conclude that

$$w \in K \quad \text{and} \quad w \in \mathbb{C} \setminus L.$$

By assumption we can extend the continuous, zero-free function $f \in C_{\mathbb{R}}(L)$ given by $f(z) := (z - w)(z - w^*)$ to a continuous, zero-free function $F \in C_{\mathbb{R}}(K)$. Fix a number $r > \sup |G| = \sup |G^*|$ and let $D := \{z \in \mathbb{C} \mid |z| \leq r\}$. With these abbreviations we define the auxiliary function $H$ on $D$ by the formula:

$$H(z) := \begin{cases} (z-w)(z-w^*) & \text{for} \quad z \in D \setminus (G \cup G^*)\,, \\ F(z) & \text{for} \quad z \in G \cup G^*. \end{cases}$$

From $[D \setminus (G \cup G^*)] \cap (G \cup G^*) \subset \partial (G \cup G^*) \subset \partial G \cup \partial G^* \subset L \cup L^* = L$ it follows that $H$ is well defined, continuous and zero-free in the closed disk $D$ (note that $w \in G$), hence there exists a continuous logarithm $\phi$:

$$H(z) = \exp(\phi(z)) \quad (z \in D)\ .$$

In particular,

$$(z - w)(z - w^*) = \exp(\phi(z)) \quad (r - \varepsilon \leq |z| \leq r).$$

Taking the logarithm locally shows that $\phi$ is analytic, and so

$$\frac{1}{z - w} + \frac{1}{z - w^*} = \phi'(z) \quad (r - \varepsilon < |z| < r).$$

Integration along the circle $|z| = r - \varepsilon/2$ gives the contradiction $4\pi i = 0$. Hence no such component $G \subset K$ can exist. By (1) we can extend every sign-function $\chi \in C_{\mathbb{R}}(L)$, because it is zero-free in $L$. Thus we have proved all assertions in (2).

(2)⇒(1): Let $f \in C_{\mathbb{R}}(L)$ be a function which is zero-free in the compact set $L$. Using the unit representation Theorem 3.6, we may write

$$f(z) = p(z) \cdot \chi(z) \cdot \exp(k(z)) \quad (z \in L), \tag{8}$$

where $p$ is a real symmetric rational function with poles and zeros off $L$, $\chi \in C_{\mathbb{R}}(L)$ is a sign-function, and $k \in C_{\mathbb{R}}(L)$. By assumption every component of $\mathbb{C}\setminus L$ contains a component of $\mathbb{C}\setminus K$. Using Lemma 3.7, we can shift the poles and zeros of $p$ from $\mathbb{C}\setminus L$ to $\mathbb{C}\setminus K$ while respecting symmetry, that is,

$$f(z) = p(z)\cdot\chi(z)\cdot\exp(k(z)) \quad (z\in L), \tag{9}$$

where $p$ denotes a real symmetric rational function with poles and zeros off $K$, $\chi\in C_{\mathbb{R}}(L)$ is a sign-function, and $k\in C_{\mathbb{R}}(L)$. By Tietze's extension theorem we can extend $k$ continuously and real symmetric to $k_{\mathrm{e}}\in C_{\mathbb{R}}(K)$. By assumption (2) there exists a zero-free extension $\chi_{\mathrm{e}}\in C_{\mathbb{R}}(K)$ of $\chi$. The desired extension is now given by $F$, which is defined as follows:

$$F(z) := p(z)\cdot\chi_{\mathrm{e}}(z)\cdot\exp(k_{\mathrm{e}}(z)) \quad (z\in K).$$

This completes the proof. □

### 6.2. **A technical lemma.**

**Lemma 6.3.** *Let $K$ be real symmetric compact set in $\mathbb{C}$, and let $g\in C_{\mathbb{R}}(K)$ with nonempty zero set $Z_g\subset K$ be given. Moreover, assume that $K\cap\mathbb{R}$ is totally disconnected. Then for every sign-function $\chi\in C_{\mathbb{R}}(Z_g)$, there exists a zero-free real symmetric extension $\chi_{\mathrm{e}}\in C_{\mathbb{R}}(K)$.*

*Proof.* We split $K$ into the upper part $K^+$ belonging to the closed upper half plane and the lower part $K^-$ belonging to the closed lower half plane. Since $K\cap\mathbb{R}$ is totally disconnected, its covering dimension is 0; see [10].

If $K\cap\mathbb{R}$ is empty, it is easy to construct a real symmetric extension of $\chi$ to $K$: As a sign-function, we must have two sets $K_{-1}, K_1\subset K^+$ such that $\chi(z)=\pm1$, for $z\in K_{\pm1}$, respectively. Then we have the logarithm $l(z)=i\pi$ on $K_{-1}$ and $l(z)=0$ on $K_1$, and by Tietze's theorem we can extend $l$ continuously to $K^+$. Because $K\cap\mathbb{R}$ is empty, we may use reflection to the lower half plane to achieve a real symmetric extension of $l$ to $K$. Define $\chi_e\in C_{\mathbb{R}}(K)$ by

$$\chi_{\mathrm{e}}(z) := \begin{cases} \exp(l(z)) & \text{for} \quad z\in K^+,\\ \exp((l(z^*))^*) & \text{for} \quad z\in K^-.\end{cases}$$

So we may assume that $K\cap\mathbb{R}$ is nonempty and has dimension zero. With the notation $S^0 := \{-1,1\}$, we can apply Theorem III.2 in [10] to obtain a real valued continuous extension $\chi_0\in C(K\cap\mathbb{R})$ of the restriction of $\chi$ to $Z_g\cap\mathbb{R}$ with values in $S^0$. So we obtain the sign-function $\chi_0$. Hence we may extend the domain of $\chi$ by

$$\chi_1(z) := \begin{cases} \chi(z) & \text{for} \quad z\in Z_g\setminus\mathbb{R},\\ \chi_0(z) & \text{for} \quad z\in K\cap\mathbb{R}.\end{cases}$$

Using Tietze's theorem, we extend $\chi_1$ continuously to $K$. Take a continuous function $g_1$ vanishing exactly on $K\cap\mathbb{R}$. Then $Z_{gg_1} = Z_g\cup(K\cap\mathbb{R})$. It follows that the pair $(\chi_1, gg_1)$ is unimodular in the complex Banach algebra $C(K^+)$. But then the complement of the inversion set

$$I := \{\lambda\in\mathbb{C} \mid (\chi_1-\lambda, gg_1) \text{ is unimodular}\}$$

satisfies $\mathbb{C} \setminus I = \chi_1(Z_{gg_1}) \subset \{-1, 1\}$. Hence the complement of the inversion set $I$ is connected and $\lambda = 0$ belongs to it, so a result of Corach and Suarez (see for example [13, Proposition 1.3]) tells us that $(\chi_1, gg_1)$ is reducible. Thus there exist $k \in C(K^+)$, $U \in C(K^+)^{-1}$ such that $\chi_1 + kgg_1 = U$. In particular,

$$U(z) = \chi_0(z) \in \mathbb{R} \quad (z \in K \cap \mathbb{R}).$$

Hence the unit

$$\chi_e(z) := \begin{cases} U(z) & \text{for } \; z \in K^+ \\ (U(z^*))^* & \text{for } \; z \in K^- \end{cases}$$

is well-defined and is an extension of $\chi$. □

6.3. **When is bsr $A_{\mathbb{R}}(K) = 1$?** The following result answers a question posed in [14]:

**Theorem 6.4.** *Let $K$ denote a real symmetric compact set in $\mathbb{C}$. The following assertions are equivalent:*

(1) *The Bass stable rank of $A_{\mathbb{R}}(K)$ is 1.*
(2) *$K \cap \mathbb{R}$ is totally disconnected.*

*Proof.* (1)⇒(2): Suppose that every unimodular pair is reducible. By Lemma 2.3, it follows that $K \cap \mathbb{R}$ is totally disconnected.

(2)⇒(1): We must show that every unimodular pair $(f, g)$ is reducible. Unimodularity implies the existence of a $\delta > 0$ such that $|f(z)| + |g(z)| \geq \delta$ $(z \in K)$. Hence the real symmetric function $f$ restricted to the set $L = Z_g$ is zero-free. By Theorem 3.6 for the compact real symmetric set $Z_g$, there exists a real symmetric rational function $p$ with poles and zeros off $Z_g$, and a sign-function $\chi \in C_{\mathbb{R}}(Z_g)$ such that

$$f(z) = p(z) \cdot \chi(z) \cdot \exp(h(z)) \quad (z \in Z_g). \tag{10}$$

We think of $h$ as extended continuously to all of $K$ by Tietze's Theorem, i.e. $h \in C_{\mathbb{R}}(K)$. Using Lemma 3.2, every component of $\mathbb{C} \setminus Z_g$ contains a component of $\mathbb{C} \setminus K$. From Lemma 3.7, we can shift the poles and zeros of $p$ from $\mathbb{C} \setminus Z_g$ to $\mathbb{C} \setminus K$ while respecting symmetry, that is,

$$f(z) = p(z) \cdot \chi(z) \cdot \exp(h(z)) \quad (z \in Z_g), \tag{11}$$

where $p$ denotes a real symmetric rational function with poles and zeros off $K$, $\chi \in C_{\mathbb{R}}(Z_g)$ is a sign-function on $Z_g$ and $h \in C_{\mathbb{R}}(K)$. Since $K \cap \mathbb{R}$ is totally disconnected, Lemma 6.3 now shows the existence of a zero-free extension $\chi_e \in C_{\mathbb{R}}(K)$ of $\chi$. By Theorem 3.6 the real symmetric unit $\chi_e \in C_{\mathbb{R}}(K)$ can be factored as

$$\chi_e(z) = q(z) \cdot \psi(z) \cdot \exp(k(z)) \quad (z \in K),$$

where $q$ denotes a real symmetric rational function with poles and zeros off $K$, $\psi$ denotes a sign-function on $K$ and $k \in C_{\mathbb{R}}(K)$. Consequently,

$$\frac{f(z) \cdot \psi(z)}{p(z) \cdot q(z)} = \exp(h(z) + k(z)) \quad (z \in Z_g).$$

We note that $\exp(l(z)+k(z)) > 0$ for all $z \in Z_g \cap \mathbb{R}$ since $h+k \in C_{\mathbb{R}}(K)$. In each open component $G$ of $K$, $\psi$ is either identically $+1$ or identically $-1$, and hence analytic there. Thus in fact $\psi \in A_{\mathbb{R}}(K)$. Theorem 4.1 now implies that the unimodular pair $(f, g)$ is reducible. So the stable rank of $A_{\mathbb{R}}(K)$ is 1. □

6.4. **When is bsr $C_{\mathbb{R}}(K) = 1$?** Now we are in a position to calculate the Bass stable rank for $C_{\mathbb{R}}(K)$ for certain compact sets $K$. Surprisingly, the characterisation is not the same as for the *complex* Banach algebra $C(K)$. Indeed, a result of Vaserstein [15, Theorem 7, p.104] gives:

$$\text{bsr } C(K) = 1 \text{ if and only if } K^\circ = \emptyset.$$

In the case of the real algebra $C_{\mathbb{R}}(K)$ we have the following:

**Theorem 6.5.** *Let $K$ denote a real symmetric compact set in $\mathbb{C}$. The following assertions are equivalent:*

(1) *The Bass stable rank of $C_{\mathbb{R}}(K)$ is 1 .*
(2) *The interior $K^\circ$ of $K$ is empty and $K \cap \mathbb{R}$ is totally disconnected.*

*Proof.* (1)⇒(2): Suppose that every unimodular pair is reducible. Let $L$ denote a real symmetric compact set in $K$. Let $f_0$ denote any real symmetric zero-free continuous function on the real symmetric compact set $L \subset K$. Since each compact set in $\mathbb{C}$ is a $G_\delta$-set, there exists a continuous function $g_0 \in C(K)$ such that $Z_{g_0} = L$; see [8, p. 15]. But then

$$g(z) := g_0(z) \cdot (g_0(z^*))^*$$

defines a real symmetric function $g \in C_{\mathbb{R}}(K)$ such that $Z_g = L = L^*$. By Tietze's theorem, we can extend $f_0$ to a real symmetric continuous $f \in C_{\mathbb{R}}(K)$. Now the unimodular pair $(f, g)$ must be reducible, hence $f + hg = u$ for certain $h, u \in C_{\mathbb{R}}(K)$, $u$ zero-free in $K$. But $u$ is a real symmetric zero-free continuous extension of $f_0$ from $L$ to $K$. Theorem 6.2 now implies that every component of $\mathbb{C} \setminus L$ contains a component of $\mathbb{C} \setminus K$.

Next we will show that $K^\circ$ is empty. Assuming the contrary let $\Delta$ be an open disc such that $\overline{\Delta} \subset K$, and let $C$ be the boundary of $\Delta$. It is easy to see that we can arrange that $\Delta$ is contained in the upper half plane $\{z \mid \operatorname{Im}(z) > 0\}$. Then $L := C \cup C^*$ is real symmetric, compact, and $L \subset K$. Hence from the above, it follows that every component of $\mathbb{C} \setminus L$ contains a component of $\mathbb{C} \setminus K$. But one of the components of $\mathbb{C} \setminus L$ is $\Delta$, which would now contain a component of $\mathbb{C} \setminus K$, and hence a point outside $K$, a contradiction. This proves that $K^\circ$ is empty.

That $K \cap \mathbb{R}$ is totally disconnected follows from Lemma 2.3.

(2)⇒(1): By assumption, the interior $K^\circ$ is void, i.e. $A_{\mathbb{R}}(K) = C_{\mathbb{R}}(K)$, and $K \cap \mathbb{R}$ is totally disconnected. By Theorem 6.4, the Bass stable rank of $A_{\mathbb{R}}(K)$, and hence that of $C_{\mathbb{R}}(K)$, is 1. □

## 7. Open questions

We end this paper with some open questions. Corollary 2.8 says that if $K$ is a real symmetric compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ has finitely many connected components, then bsr $A_{\mathbb{R}}(K)$ is at most 2. We suspect that this might always be the case, and so we have the following questions:

(1) If $K$ is a real symmetric compact subset of $\mathbb{C}$, then is bsr $A_{\mathbb{R}}(K) \leq 2$?
(2) If $K$ is a real symmetric compact subset of $\mathbb{C}$, then what is tsr $A_{\mathbb{R}}(K)$?
(3) Find necessary and sufficient conditions for tsr $A_{\mathbb{R}}(K) = 1$.

In light of the results in this article, analogous questions for $C_{\mathbb{R}}(K)$ can also be posed:

(1) If $K$ is a real symmetric compact subset of $\mathbb{C}$, then is bsr $C_{\mathbb{R}}(K) \leq 2$?
(2) If $K$ is a real symmetric compact subset of $\mathbb{C}$, then what is tsr $C_{\mathbb{R}}(K)$?

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