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Complex Analysis and Operator Theory

Tolokonnikov's Lemma for Real H^∞ and the Real Disc Algebra

Kalle Mikkola and Amol Sasane

Abstract. We prove Tolokonnikov's Lemma and the inner-outer factorization for the real Hardy space $H^\infty_{\mathbb{R}}$, the space of bounded holomorphic (possibly operator-valued) functions on the unit disc all of whose matrix-entries (with respect to fixed orthonormal bases) are functions having real Fourier coefficients, or equivalently, each matrix entry f satisfies $\overline{f(\overline{z})} = f(z)$ for all $z \in \mathbb{D}$.

Tolokonnikov's Lemma for $H^\infty_{\mathbb{R}}$ means that if f is left-invertible, then f can be completed to an isomorphism; that is, there exists an F, invertible in $H^\infty_{\mathbb{R}}$, such that $F = \begin{bmatrix} f & f_c \end{bmatrix}$ for some f_c in $H^\infty_{\mathbb{R}}$. In control theory, Tolokonnikov's Lemma implies that if a function has a right coprime factorization over $H^\infty_{\mathbb{R}}$, then it has a doubly coprime factorization in $H^\infty_{\mathbb{R}}$. We prove the lemma for the real disc algebra $A_{\mathbb{R}}$ as well. In particular, $H^\infty_{\mathbb{R}}$ and $A_{\mathbb{R}}$ are Hermite rings.

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1. Introduction

In this paper, we prove Tolokonnikov's Lemma for the real Hardy space $H^\infty_{\mathbb{R}}$, which is defined below, and for the real disc algebra $A_{\mathbb{R}}$, the subspace of $H^\infty_{\mathbb{R}}$ functions having a continuous extension to the closed unit disc (Section 4).

We begin by introducing some notation. We will denote the complex plane by $\mathbb{C}$, the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ by $\mathbb{D}$, and the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ by $\mathbb{T}$.

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Throughout this article X and Y denote separable complex Hilbert spaces with fixed orthonormal bases.[1] $\mathcal{L}(X,Y)$ denotes the complex Banach space of bounded linear operators from X to Y, equipped with the operator norm.

If M is a matrix (possibly infinite), then $\overline{M}$ denotes the matrix obtained from M by taking the complex conjugate of each entry of M; that is, $\overline{M}_{jk} := \overline{M_{jk}}$ for every j,k.

$H^\infty(\mathcal{L}(X,Y))$ denotes the Banach space of functions $f : \mathbb{D} \to \mathcal{L}(X,Y)$ that are holomorphic and bounded, equipped with the supremum norm

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} \|f(z)\|_{\mathcal{L}(X,Y)}\,.$$

We denote by $H^2(X)$ the Hilbert space of functions $f : \mathbb{D} \to X$ that are holomorphic in $\mathbb{D}$ such that

$$\|f\|_2 := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^2 d\theta \right)^{\frac{1}{2}} < \infty\,.$$

Definition 1.1. We denote by $H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$ the set comprising functions $f \in H^\infty(\mathcal{L}(X,Y))$ such that

$$f(z) = \overline{f(\overline{z})} \quad \text{for all} \quad z \in \mathbb{D}\,.$$

$H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$ is a Banach space over $\mathbb{R}$ with the supremum norm $\|\cdot\|_\infty$. We call the elements of $H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$ *real* (or *real-symmetric*).

Analogously, we define the real Hilbert space $H^2_{\mathbb{R}}(X)$.

An operator $A \in \mathcal{L}(X,Y)$ is called *real* if it is real as a constant function in $H^\infty_{\mathbb{R}}$, or equivalently, if its matrix entries are real numbers.

Tolokonnikov's Lemma says that if $X \subset Y$, $\dim(X) < \infty$ and $f \in H^\infty(\mathcal{L}(X,Y))$, then the following two statements are equivalent:

1. (*Left invertibility*) There exists $g \in H^\infty(\mathcal{L}(Y,X))$ such that $gf \equiv I_X$.
2. (*Complementation to an isomorphism*) There exists $F \in H^\infty(\mathcal{L}(Y))$ such that $F|_X = f$ and $F^{-1} \in H^\infty(\mathcal{L}(Y))$.

For a proof of Tolokonnikov's Lemma, see for instance §10 in Appendix 3 of [3], and also [6,7]. When X and Y are both finite dimensional, then this lemma simply says that the ring H^∞ (of scalar functions, with pointwise addition and multiplication) is Hermite. For background on Hermite rings, we refer the reader to [2]. Tolokonnikov's Lemma was generalized to the case when X is not necessarily finite dimensional by Sergei Treil in [8]. Following his proof, we establish the following result in this article.

[1] This allows us to identify any $A \in \mathcal{L}(X,Y)$ with the corresponding (possibly infinite) matrix and makes $\overline{A}$ well defined. One easily verifies that $\overline{\alpha A + B} = \overline{\alpha}\overline{A} + \overline{B}$, $\overline{AB} = \overline{A}\,\overline{B}$, $\overline{(\overline{A})} = A$, $\overline{A}^* = \overline{A^*}$, $\overline{A^{-1}} = (\overline{A})^{-1}$, and $A^*_{jk} := \overline{A_{kj}}$ for every j,k, when $\alpha \in \mathbb{C}$ and A and B are linear operators or vectors of compatible dimensions. Moreover, for any $\mathcal{L}(X,Y)$-valued function f we have $f \in H^\infty$ iff $\overline{f(\overline{\cdot})} \in H^\infty$.

Theorem 1.2. *Let X, Y be complex Hilbert spaces, and let $f \in H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$. Then the following are equivalent:*

1. *There exists a $g \in H^\infty_{\mathbb{R}}(\mathcal{L}(Y,X))$ such that $gf \equiv I_X$.*
2. *There exists a complex Hilbert space X_c and there exists a function $F \in H^\infty_{\mathbb{R}}(\mathcal{L}(X \oplus X_c, Y))$ such that $F^{-1} \in H^\infty_{\mathbb{R}}(\mathcal{L}(Y, X \oplus X_c))$ and $F(z)|_X = f(z)$ for all $z \in \mathbb{D}$ (and a.e. on $\mathbb{T}$).*

If X is a finite dimensional subspace of Y, then in Theorem 1.2 the space X_c can be chosen to be the orthogonal complement of X in Y, but the same is not true when X is infinite dimensional.

Using Theorem 1.2, we also obtain an analogous result for the real disk algebra $A_{\mathbb{R}}$.

The motivation for proving Tolokonnikov's Lemma in the real case arises from *control theory*, where it plays an important role in the problem of stabilization of linear systems. Indeed, Tolokonnikov's Lemma implies that if a transfer function G has a right (or left) coprime factorization, then G has a doubly coprime factorization, and the standard Youla parameterization yields all stabilizing controllers for G. For background on the relevance of Tolokonnikov's Lemma in control theory and further details, we refer the reader to Vidyasagar [9].

In applications in control theory, the linear systems and transfer functions have *real* coefficients, and so in this context it is important to consider *real* functions, since otherwise the controllers obtained would be physically meaningless. With the real doubly coprime factorization provided by Theorem 1.2 one can parameterize all real stabilizing controllers by restricting to real parameters in the Youla formula; see, for example, Curtain and Zwart [1], Staffans [5] or Vidyasagar [9].

It is known (see for instance Wick [11]) that a real version of the operator-valued Corona Theorem holds true: if $g \in H^\infty(\mathcal{L}(Y,X))$ is the left inverse of a real function $f \in H^\infty(\mathcal{L}(X,Y))$, then so is the *symmetrization* $\widetilde{g} := 1/2(g + \overline{g(\overline{\cdot})}) \in H^\infty_{\mathbb{R}}(\mathcal{L}(Y,X))$ (see Lemma 3.1(1)). We remark that also real versions of (the operator-valued cases) of the Hartman Theorem, Nehari Theorem and Adamjan–Arov–Krein Theorem can be obtained analogously (the symmetrization of a solution is again a solution if the original data is real), although for the latter two theorems the verification requires further technical details, which we omit.

However, Tolokonnikov's Lemma does not allow for such a straightforward real variant, because the symmetrization of an invertible matrix need not be invertible!

In order to prove our main result (Theorem 1.2 above), we will need a real version of the operator-valued canonical inner-outer factorization in the Hardy space H^∞, which we prove first in Section 2, and in Section 3 we give the proof of Theorem 1.2. Finally, as a corollary to Theorem 1.2, we also derive a real version of Tolokonnikov's Lemma for the operator-valued analogue of the real disc algebra in Section 4. To our knowledge, our results are new even in the finite-dimensional case.

2. Inner-outer factorization

In this section we will prove a real version of the canonical inner-outer factorization for operator-valued functions from the Hardy class. We start with some observations on real and complex spaces, operators and functions.

Real functions can be characterized in several ways.

Lemma 2.1. *A function $f \in H^\infty(\mathcal{L}(X,Y))$ is real iff $f(z)$ is real for each $z \in (-1,1)$. An equivalent condition is that $\widehat{f}(n)$ is real for each $n \in \mathbb{N}$. If f is real, then so is f'.*

Proof. 1° If $f = \overline{f(\overline{\cdot})}$ on nondiscrete subset of $\mathbb{D}$, then $f = \overline{f(\overline{\cdot})}$ on $\mathbb{D}$, hence then f is real. But $\overline{f(\overline{\cdot})} = \overline{f}$ on $(-1,1)$, so the first equivalence holds.

2° If f is real, then $f'(z) = \lim_{\mathbb{R}\ni h\to 0} h^{-1}(f(z+h) - f(z))$ is real for $z \in (-1,1)$, because $f(z+h)$ and $f(z)$ are real; thus, then f is real, by 1°.

3° If f is real, then so is $\widehat{f}(n) = f^{(n)}(0)/n!$ $\forall n \in \mathbb{N}$, by 2° and induction. Conversely, if $\widehat{f}(n)$ is real for each n, then $f(z) = \sum_{n\in\mathbb{N}} \widehat{f}(n) z^n$ is real for each $z \in (-1,1)$. □

Definition 2.2. If $(V, \langle\cdot,\cdot\rangle)$ is a real Hilbert space, then by $V_{\mathbb{C}}$ we denote the complex Hilbert space $V+iV$, with the natural scalar multiplication (where $i(v+i\widetilde{v}) := -\widetilde{v}+iv$), and inner product given by $\langle w+i\widetilde{w}, v+i\widetilde{v}\rangle_{\mathbb{C}} := \langle w,v\rangle + \langle \widetilde{w},\widetilde{v}\rangle + i\langle \widetilde{w}, v\rangle - i\langle w,\widetilde{v}\rangle$.

We omit the proof of the following obvious claim.

Lemma 2.3. *If B is an orthonormal basis of a real Hilbert space V, then B is also an orthonormal basis of the complex Hilbert space $V_{\mathbb{C}}$.*

Recall that every $f \in H^\infty(\mathcal{L}(X,Y))$ has a boundary function (the nontangential limit a.e.) that we denote by the same symbol $f : \mathbb{T} \to \mathcal{L}(X,Y)$. (See for example [4].)

Definition 2.4. We call $f \in H^\infty(\mathcal{L}(X,Y))$ *inner* if $f(z)^* f(z) = I$ a.e. on $\mathbb{T}$, or equivalently, if multiplication by f is an isometry from $H^2(X)$ to $H^2(Y)$.

$f \in H^\infty(\mathcal{L}(X,Y))$ is called *outer* if $\{f\varphi \mid \varphi \in H^2(X)\}$ is dense in $H^2(Y)$.

An equivalent, equally standard definition is given in [4]. We now establish the inner-outer factorization.

Theorem 2.5. *If $f \in H^\infty(\mathcal{L}(X,Y))$, then there exists a Hilbert space W and functions $f_{\mathrm{inn}} : \mathbb{D} \to \mathcal{L}(W,Y)$, $f_{\mathrm{out}} : \mathbb{D} \to \mathcal{L}(X,W)$ such that $f = f_{\mathrm{inn}} f_{\mathrm{out}}$, $f_{\mathrm{inn}} \in H^\infty(\mathcal{L}(W,Y))$ is inner, and $f_{\mathrm{out}} \in H^\infty(\mathcal{L}(X,W))$ outer.*

If f is real, then we can also ensure that f_{inn} and f_{out} are real.

Proof. Except for the last claim about making the factors real, the result is given in §1.6.4.(b) of [4]. We first outline the construction of f_{inn} and W as in [4], and then we prove that f_{inn} and f_{out} can be made real whenever f is real.

The closure $E := \mathrm{clos}(fH^2(X))$ is an S-invariant subspace of $H^2(Y)$. Here S denotes the shift operator, $S\varphi(z) = z\varphi(z)$, $\varphi \in H^2(Y)$. Define $W = E \ominus SE$, and let f_{inn} be defined as follows: $f_{\mathrm{inn}}(\zeta)\varphi = \varphi(\zeta)$, $\zeta \in \mathbb{D}$, $\varphi \in W$. Then $f_{\mathrm{inn}} \in H^\infty(\mathcal{L}(W,Y))$ is inner and $E = f_{\mathrm{inn}} H^2(W)$. Moreover, f_{out} defined by $f_{\mathrm{out}}(\zeta) = f_{\mathrm{inn}}(\zeta)^* f(\zeta)$ for $\zeta \in \mathbb{T}$ is outer.

Let $A := \mathrm{clos}(f H^2_{\mathbb{R}}(X))$ and $V := A \ominus SA = \{\varphi \in A \mid \langle \varphi, S\psi\rangle = 0\ \forall \psi \in A\}$. Then A and V are real Hilbert spaces. Let $B = \{\varphi_1, \varphi_2, \varphi_3, \dots\}$ be an orthonormal basis of V. One easily verifies that

$$\begin{aligned} W &= (A + iA) \ominus S(A + iA) \\ &= \left\{\varphi + i\widetilde{\varphi} \in A + iA \mid \langle \varphi + i\widetilde{\varphi}, S\psi + iS\widetilde{\psi}\rangle = 0 \ \ \forall \psi, \widetilde{\psi} \in A\right\} \\ &= V + iV. \end{aligned}$$

By Lemma 2.3, it follows that B is also an orthonormal basis of W; we fix it as the canonical basis of W (the one used in the definition of "real").

Any $\varphi \in W$ can be written as $\sum_k w_k \varphi_k$ with $(w_k)_{k\geq 1} \in \ell^2$. Here ℓ^2 denotes the space of square-summable sequences with values in $\mathbb{C}$. Now

$$f_{\mathrm{inn}}(z)\varphi = \varphi(z) = \sum_k w_k \varphi_k(z) \in Y\,,$$

and so

$$\big(f_{\mathrm{inn}}(z)\varphi\big)_j = \varphi(z)_j = \sum_k w_k \big[\varphi_k(z)\big]_j \in \mathbb{C} \tag{2.1}$$

for all j. But also

$$\big(f_{\mathrm{inn}}(z)\varphi\big)_j = \sum_k \big[f_{\mathrm{inn}}(z)\big]_{jk} w_k\,. \tag{2.2}$$

Since (2.1) and (2.2) hold for all $\varphi \in W$, we obtain $[f_{\mathrm{inn}}(z)]_{jk} = [\varphi_k(z)]_j$. Thus for all j, k,

$$\overline{\big[f_{\mathrm{inn}}(\bar z)\big]_{jk}} = \overline{\big[\varphi_k(\bar z)\big]_j} = \big[\varphi_k(z)\big]_j = \big[f_{\mathrm{inn}}(z)\big]_{jk}$$

since each φ_k is real. Consequently, $\overline{f_{\mathrm{inn}}(\bar z)} = f_{\mathrm{inn}}(z)$, and so f_{inn} is real. Finally, $f_{\mathrm{out}} = f_{\mathrm{inn}}^* f$ is real as well. □

Remarks 2.6. We make the following observations:

1. In Theorem 2.5 above, we have $\dim Y \geq \dim W \leq \dim X$ (since $f_{\mathrm{inn}}(z)^* f_{\mathrm{inn}}(z) = I$ for some $z \in \mathbb{T}$ and $f_{\mathrm{out}}(0)$ has dense range).
2. We write $f \in H^2_{\mathrm{strong}}(\mathcal{L}(X,Y))$ if $f : \mathbb{D} \to \mathcal{L}(X,Y)$ is such that $fx \in H^2(Y)$ for all $x \in X$. Such a function is called *outer* if the set $\{fp \mid p$ is an X-valued polynomial$\}$ is dense in $H^2(Y)$ [4, p. 14]. With the alternative assumption that $f \in H^2_{\mathrm{strong}}(\mathcal{L}(X,Y))$, Theorem 2.5 still holds except that then $f_{\mathrm{out}} \in H^2_{\mathrm{strong}}(\mathcal{L}(W,Y))$ (instead of H^∞). In this case the proof requires slight modifications; for example, we need to replace $fH^2(X)$ (respectively, $fH^2_{\mathbb{R}}(X)$) by the set of polynomials contained in $H^2(X)$ (respectively, in $H^2_{\mathbb{R}}(X)$).

3. Note also that if we want to have the inner and outer factors real with respect to some other fixed orthonormal basis of W instead of the one constructed in the proof, we just have to replace f_{inn} and f_{out} by $f_{\mathrm{inn}}U$ and $U^{-1}f_{\mathrm{out}}$, where $U \in \mathcal{L}(W)$ is a unitary operator that maps the new basis to the original basis. Analogously, for any set Q of cardinality $\dim W$, we can replace W by $\ell^2(Q)$ in Theorem 2.5 by applying a suitable unitary $U \in \mathcal{L}(\ell^2(Q), W)$ as above.

3. Tolokonnikov's Lemma for $H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$

We call $f \in H^\infty(\mathcal{L}(X,Y))$ *left invertible in* H^∞ if $gf \equiv I_X$ for some $g \in H^\infty(\mathcal{L}(Y,X))$. We call $f \in H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$ *left invertible in* $H^\infty_{\mathbb{R}}$ if $gf \equiv I_X$ for some $g \in H^\infty_{\mathbb{R}}(\mathcal{L}(Y,X))$.

We shall need a real version of Nikolski's Lemma [8], which we show in item 2 below.

Lemma 3.1. *Let $f \in H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$. Then*

1. *f is left invertible in $H^\infty_{\mathbb{R}}$ iff f is left invertible in H^∞.*
2. *If*
$$\exists \delta > 0 \quad \textit{such that} \quad \forall z \in \mathbb{D} \quad f(z)^* f(z) \geq \delta^2 I_X\,, \tag{3.1}$$
then f is left invertible in H^∞ iff there exists a $\mathcal{P} \in H^\infty_{\mathbb{R}}(\mathcal{L}(Y))$ whose values are projections (not necessarily orthogonal) onto $f(z)X$ for all $z \in \mathbb{D}$ (and a.e. on $\mathbb{T}$).

Remark 3.2. Condition (3.1) is necessary for left invertibility in H^∞. It is also sufficient if $\dim X < \infty$, but not in general (see [8]).

Proof. 1. In order to prove the first claim, we note that if g is a left inverse of f, then we can make g real by symmetrization, that is, if $g \in H^\infty(\mathcal{L}(Y,X))$ and $gf \equiv I$, then with $\widetilde{g} \in H^\infty_{\mathbb{R}}(\mathcal{L}(Y,X))$ defined by $\widetilde{g}(z) := \frac{1}{2}(g(z) + \overline{g(\overline{z})})$, $z \in \mathbb{D}$, we have $\widetilde{g}f \equiv I$.

2. It was shown in [8] (see Lemma 6.1, attributed to N. Nikolski) that if the condition (3.1) holds, then f is left invertible in H^∞ iff there exists a $\mathcal{P} \in H^\infty(\mathcal{L}(Y))$ whose values are projections onto $f(z)X$ for all $z \in \mathbb{D}$ (and a.e. on $\mathbb{T}$). Moreover, in the proof of the 'only if' part, the projection valued function $\mathcal{P}$ was constructed from a left inverse g of f as follows: $\mathcal{P}(z) = f(z)g(z)$, $z \in \mathbb{D}$. In our case, we can choose the g to be real, and so we can ensure that $\mathcal{P}$ is real as well. □

We now prove the real version of Tolokonnikov's Lemma.

Proof of Theorem 1.2*:* Let g be a left inverse of f in $H^\infty_{\mathbb{R}}$. Set
$$\mathcal{Q} := I - \mathcal{P}\,,$$
where $\mathcal{P}$ is as in Lemma 3.1, and write $\mathcal{Q} = \mathcal{Q}_{\mathrm{inn}}\mathcal{Q}_{\mathrm{out}}$ as in Theorem 2.5, with $X_c := W$. Then
$$F = \begin{bmatrix} f & \mathcal{Q}_{\mathrm{inn}} \end{bmatrix} \in H^\infty\big(\mathcal{L}(X \oplus X_c, Y)\big)$$

has the inverse

$$G = \begin{bmatrix} g\mathcal{P} \\ \mathcal{Q}_{\text{out}} \end{bmatrix} \in H^\infty\big(\mathcal{L}(Y, X \oplus X_c)\big),$$

as shown at the end of [8]. But F and G are real, so we are done. $\square$

4. Tolokonnikov's Lemma for $A_{\mathbb{R}}(\mathcal{L}(X,Y))$

As a corollary to Tolokonnikov's Lemma for $H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$, we derive a real version of it also for the analogue of the real disc algebra, defined below.

Definition 4.1. We denote by $A_{\mathbb{R}}(\mathcal{L}(X,Y))$ the set comprising functions $f \in H^\infty_{\mathbb{R}}(\mathcal{L}(X,Y))$ such that f has a continuous extension to $\text{clos}(\mathbb{D})$.

Corollary 4.2. *Let X, Y be complex Hilbert spaces, and let $f \in A_{\mathbb{R}}(\mathcal{L}(X,Y))$. Then following are equivalent:*

1. *There exists a $\delta > 0$ such that for all $z \in \mathbb{D}$, $f(z)^* f(z) \geq \delta^2 I_X$.*
2. *There exists a $g \in A_{\mathbb{R}}(\mathcal{L}(Y,X))$ such that $gf \equiv I_X$.*
3. *There exists a complex Hilbert space X_c and there exists a function $F \in A_{\mathbb{R}}(\mathcal{L}(X \oplus X_c, Y))$ such that $F^{-1} \in A_{\mathbb{R}}(\mathcal{L}(Y, X \oplus X_c))$ and $F(z)|_X = f(z)$ for all $z \in \text{clos}(\mathbb{D})$.*

Remark 4.3. If $\dim(X) < \infty$, then condition (1) holds iff $f(z)$ is one-to-one for every $z \in \text{clos}(\mathbb{D})$.

Proof. The equivalence of (1) with (2) is known (see [10]), and (3)$\Rightarrow$(2) is trivial. So we simply prove (2)$\Rightarrow$(3).

If (2) above holds, then $g \in H^\infty_{\mathbb{R}}(\mathcal{L}(Y,X))$, and by Tolokonnikov's Lemma for $H^\infty_{\mathbb{R}}$, there exists a $\widetilde{F} = \begin{bmatrix} f & f_c \end{bmatrix} \in H^\infty_{\mathbb{R}}(\mathcal{L}(X \oplus X_c, Y))$ such that $G := \widetilde{F}^{-1} \in H^\infty_{\mathbb{R}}(\mathcal{L}(Y, X \oplus X_c))$, and $\widetilde{F}(z)|_X = f(z)$ for all $z \in \mathbb{D}$.

Given a function w on $\mathbb{D}$ and an $r \in (0,1)$, we define the dilation of w, denoted by w_r, as follows:

$$w_r(z) = w(rz), \quad z \in \mathbb{D}_r := r^{-1}\mathbb{D}.$$

It is clear that if w is real, then so is w_r.

On $\mathbb{D}_r$, we have $G_r \widetilde{F}_r = G_r \begin{bmatrix} f_r & f_{c,r} \end{bmatrix} \equiv I_{X \oplus X_c}$. In particular, on $\text{clos}(\mathbb{D})$, there holds that

$$G_r \begin{bmatrix} f & f_{c,r} \end{bmatrix} = I_{X \oplus X_c} + G_r \begin{bmatrix} f_r - f & 0 \end{bmatrix} =: h.$$

But $\|G_r\|_\infty \leq \|G\|_\infty$. Moreover, since f is uniformly continuous on $\text{clos}(\mathbb{D})$, it follows that $\lim_{r \nearrow 1} \|f_r - f\|_\infty = 0$. Hence we can choose a r close enough to 1 such that h is invertible in $A_{\mathbb{R}}(\mathcal{L}(X \oplus X_c))$. Defining

$$F = \begin{bmatrix} f & f_{c,r} \end{bmatrix} \in A_{\mathbb{R}}\big(\mathcal{L}(X \oplus X_c, Y)\big),$$

we see that F has inverse $h^{-1} G_r \in A_{\mathbb{R}}(\mathcal{L}(Y, X \oplus X_c))$, and $F(z)|_X = f(z)$ for all $z \in \text{clos}(\mathbb{D})$. $\square$

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Kalle Mikkola
Institute of Mathematics
Helsinki University of Technology
Box 1100, 02015 HUT
Finland
e-mail: `Kalle.Mikkola@iki.fi`

Amol Sasane
Department of Mathematics
London School of Economics
Houghton Street
London WC2A 2AE
United Kingdom
e-mail: `A.J.Sasane@lse.ac.uk`

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