# ON THE PRE-BÉZOUT PROPERTY OF WIENER ALGEBRAS ON THE DISC AND THE HALF-PLANE

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Abstract. Let $\mathbb{D}$ denote the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$, and $\mathbb{C}_+$ denote the right half-plane $\{s \in \mathbb{C} \mid \text{Re}(s) \geq 0\}$.

(1) Let $W^+(\mathbb{D})$ be the Wiener algebra of the disc, that is the set of all absolutely convergent Taylor series in the open unit disk $\mathbb{D}$, with pointwise operations.
(2) Let $W^+(\mathbb{C}_+)$ be the set of all functions defined in the right half-plane $\mathbb{C}_+$ that differ from the Laplace transform of a function $f_a \in L^1(0, \infty)$ by a constant. Equipped with pointwise operations, $W^+(\mathbb{C}_+)$ forms a ring.

We show that the rings $W^+(\mathbb{D})$ and $W^+(\mathbb{C}_+)$ are pre-Bézout rings.

## 1. Introduction

The aim of this paper is to show that the rings $W^+(\mathbb{D})$ and $W^+(\mathbb{C}_+)$ (defined below) are pre-Bézout.

We first recall the notion of a pre-Bézout ring.

**Definition 1.1.** Let $R$ be a commutative, unital ring.

(1) An element $d \in R$ is called a *greatest common divisor* of $a, b \in R$ if it is a divisor of $a$ and $b$ and if $k$ is another divisor, then $k$ divides $d$.
(2) The ring $R$ is said to be *pre-Bézout* if for every $a, b \in R$ for which there exists a greatest common divisor $d$, there exist $x, y \in R$ such that $d = xa + yb$.

Michael von Renteln [12, Theorem 2.4, p. 54] proved that the disc algebra $A(\mathbb{D})$ (the ring of continuous functions on the closed unit disc $\overline{\mathbb{D}}$, which are analytic in the open unit disc $\mathbb{D}$, with the usual pointwise operations) is pre-Bézout. The first author of the present paper [7] showed the pre-Bézout property for the Sarason algebra $QA = (C(\mathbb{T}) + \widetilde{C(\mathbb{T})}) \cap H^\infty(\mathbb{D})$ of bounded analytic functions having quasicontinuous boundary values. (Here $\widetilde{C(\mathbb{T})}$ denotes the set of harmonic conjugates of continuous functions on $\mathbb{T}$.) Note that the algebra $H^\infty(\mathbb{D})$ (of all bounded and analytic functions in the open unit disc, with pointwise operations) is not pre-Bézout [12, Remark, p. 54]. In this article, we will show that the rings $W^+(\mathbb{D})$ and $W^+(\mathbb{C}_+)$ (defined below) are pre-Bézout.

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Throughout the article, we will use the following notation:

$$\begin{aligned} \mathbb{D} &:= \{z \in \mathbb{C} \mid |z| < 1\} \\ \overline{\mathbb{D}} &:= \{z \in \mathbb{C} \mid |z| \leq 1\} \\ \mathbb{C}_+ &:= \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}. \end{aligned}$$

**Definition 1.2.**

(1) The *Wiener algebra of the disc*, $W^+(\mathbb{D})$, is the set of all functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ such that $f$ is analytic in $\mathbb{D}$ and $\sum_{n=0}^{\infty} |a_n| < \infty$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Equipped with pointwise operations and the norm $\|f\|_{W^+} := \sum_{n=0}^{\infty} |a_n|$, $W^+(\mathbb{D})$ is a Banach algebra.

(2) Let $W^+(\mathbb{C}_+)$ denote the set of all functions $F : \mathbb{C}_+ \to \mathbb{C}$ such that $F(s) = \widehat{f_a}(s) + f_0$ $(s \in \mathbb{C}_+)$, where $f_a \in L^1(0, \infty)$, $f_0 \in \mathbb{C}$, and $\widehat{f_a}$ denotes the Laplace transform of $f_a$ given by

$$\widehat{f_a}(s) = \int_0^{\infty} e^{-st} f_a(t) dt, \quad s \in \mathbb{C}_+.$$

Equipped with pointwise operations and the norm

$$\|F\|_{W^+} = \|f_a\|_{L^1} + |f_0|,$$

$W^+(\mathbb{C}_+)$ is a Banach algebra.

We note that $W^+(\mathbb{C}_+)$ is contained in the set of all holomorphic functions on the right half-plane that admit continuous extensions to the imaginary axis and have a limit at infinity. We will call $W^+(\mathbb{C}_+)$ the *Wiener-Laplace algebra*.

**Remark 1.3.**

(1) From the application point of view, the above algebras also arise as a natural classes of transfer functions of stable distributed parameter systems in control theory; see [10].

(2) We use the notation $W^+(\mathbb{C}_+)$ in order to highlight the similarity with $W^+(\mathbb{D})$. Indeed, $W^+(\mathbb{D})$ is isomorphic to the algebra of summable sequences $\ell^1(\mathbb{N})$ with convolution, pointwise addition, and the $\ell^1(\mathbb{N})$ norm. Now instead of this "discrete" convolution algebra, we consider "distributed" summable functions $L^1(0, \infty)$, again with convolution, pointwise addition, and the $L^1(0, \infty)$-norm, and attach the identity element $\delta$ (=Dirac distribution) to it, we obtain the convolution algebra $L^1(0, \infty) + \mathbb{C}\delta$. Then $W^+(\mathbb{C}_+)$ is isomorphic to the algebra $L^1(0, \infty) + \mathbb{C}\delta$ via Laplace transformation.

Our main results are the following:

**Theorem 1.4.** *The ring $W^+(\mathbb{D})$ is pre-Bézout.*

**Theorem 1.5.** *The ring $W^+(\mathbb{C}_+)$ is pre-Bézout.*

**Remark 1.6.**

(1) The relevance of the pre-Bézout property in control theory is the following:

Suppose $R$ is a pre-Bézout ring and we have a plant whose transfer function $p$ belongs to the field of fractions of $R$. Then $p$ has a weakly coprime factorization iff $p$ has a coprime factorization; see [9, Proposition, p. 54].

(2) We recall that a commutative ring $R$ is called *Bézout* if every finitely generated ideal in $R$ is principal.

Neither of our algebras $W^+(\mathbb{D})$ nor $W^+(\mathbb{C}_+)$ are Bézout. That $W^+(\mathbb{D})$ is not Bézout can be shown by considering the ideal $(f, g)$, where

$$f = (1 - z)^3 \quad \text{and} \quad g = (1 - z)^3 e^{-\frac{1+z}{1-z}};$$

see [8, Remark after Theorem 1, p. 224]. On the other hand the fact that $W^+(\mathbb{C}_+)$ is not Bézout follows from a general result which says that if $R$ is any subring of the ring $H^\infty$ (of bounded analytic functions in the open right half-plane Re$(s)> 0$, with pointwise addition and multiplication), such that $R$ contains the Laplace transforms of functions from $L^1(0, \infty)$, then $R$ has a finitely generated ideal which is not principal; see [6, Theorem].

In Sections 3 and 4 we will give the proofs of Theorems 1.4 and 1.5, respectively, but before doing that, in Section 2, we first give a few preliminaries.

## 2. Preliminaries

It is well known that the maximal ideals (or kernels of multiplicative linear functionals) of $W^+(\mathbb{D})$ have the form

$$\mathfrak{m}_a = \{f \in W^+(\mathbb{D}) \mid f(a) = 0\}$$

for some $a \in \overline{\mathbb{D}}$. Similarly, the set of maximal ideals in $W^+(\mathbb{C}_+)$ coincides with the set of ideals of the form $\mathfrak{M}_{s_0}$ and $\mathfrak{M}_\infty$, where

$$\mathfrak{M}_{s_0} = \{F \in W^+(\mathbb{C}_+) \mid F(s_0) = 0\}, \quad s_0 \in \mathbb{C}_+,$$

and $\mathfrak{M}_\infty$ is given by the kernel of the homomorphism $\varphi : W^+(\mathbb{C}_+) \to \mathbb{C}$ defined by

$$F = \widehat{f_a} + f_0 \overset{\varphi}{\mapsto} f_0 \quad (f_a \in L^1(0, \infty),\ f_0 \in \mathbb{C}).$$

That is,

$$\mathfrak{M}_\infty = \{F \in W^+(\mathbb{C}_+) \mid \exists f_a \in L^1(0, \infty) \text{ such that } F = \widehat{f_a}\} = \widehat{L^1(0, \infty)}.$$

Since every maximal ideal is closed, all the sets $\mathfrak{m}_\alpha$, $|\alpha| = 1$, are commutative Banach subalgebras of $W^+(\mathbb{D})$. Similarly, $\mathfrak{M}_{i\beta}$, $\beta \in \mathbb{R}$, and $\mathfrak{M}_\infty$ are commutative Banach subalgebras of $W^+(\mathbb{C}_+)$. Obviously these algebras have no identity element. But there is a substitute, namely the notion of the bounded approximate identity, which will be useful in the sequel.

**Definition 2.1.** Let $R$ be a commutative Banach algebra (without identity element). We say that $R$ has a *bounded approximate identity* if there exists a bounded sequence $(e_n)_n$ of elements $e_n$ in $R$ such that for any $f \in R$,

$$\lim_n \|e_n f - f\| = 0.$$

We will also need the following technical result:

**Proposition 2.2** (Varopoulos, [14])**.** *Let $R$ be a Banach algebra with a bounded left approximate identity. Then for every sequence $(a_n)_{n\geq 1}$ in $R$ converging to $0$, there exists a sequence $(b_n)_{n\geq 1}$ in $R$ converging to $0$, as well as an element $c \in R$ such that for all $n \geq 1$, $a_n = cb_n$.*

**Lemma 2.3.** *Let $R$ be a commutative integral domain with identity $1$. If $d$ $(\neq 0)$ is a greatest common divisor of $f_1, \dots, f_n$, then $1$ is a greatest common divisor of $\frac{f_1}{d}, \dots, \frac{f_n}{d}$.*

*Proof.* Clearly 1 divides $\frac{f_1}{d}, \dots, \frac{f_n}{d}$. If $h$ is a divisor of $\frac{f_1}{d}, \dots, \frac{f_n}{d}$, then $\frac{f_k}{d} = hg_k$, for some $g_k \in R$, $k = 1, \dots, n$. So $dh$ is common divisor of $f_1, \dots, f_n$, and as $d$ is the greatest common divisor of $f_1, \dots, f_n$, $dh$ divides $d$, that is, $dhk = d$ for some $k \in R$. Since $R$ is an integral domain and $d \neq 0$, we obtain $hk = 1$, that is, $h$ divides 1, proving the claim. □

## 3. $W^+(\mathbb{D})$ is pre-Bézout

Let $z_0 \in \mathbb{T} := \{z \in \mathbb{D} \mid |z| = 1\}$. Consider the maximal ideal

$$\mathfrak{m}_{z_0} := \{f \in W^+(\mathbb{D}) \mid f(z_0) = 0\}.$$

We will use the following result on the existence of a bounded approximate identity for $\mathfrak{m}_{z_0}$. Without loss of generality, we take $z_0 = 1$.

**Proposition 3.1** (Faivyševskij, [2])**.** *Let $n \in \mathbb{N}$, $(r_n)_{n\in\mathbb{N}}$ be any sequence such that $r_n \searrow 1$, and*

$$e_n(z) := \frac{z-1}{z-r_n}.$$

*Then $(e_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\mathfrak{m}_1$.*

A rather lengthy proof of the above result in the case when $r_n = 1 + \frac{1}{n}$ can be found in [5, Lemma 1]. For the reader's convenience we present a short proof here.

*Proof.* A simple calculation gives that $||\frac{z-1}{z-1-\epsilon}||_{W^+} \leq 2$. Since the partial sums $S_n - S_n(1)$ for $f$ approximate $f \in \mathfrak{m}_1$, it suffices to consider $q(z) = (z-1)p(z)$, where $p \in \mathbb{C}[z]$. But

$$\left\|\frac{z-1}{z-1-\epsilon}\, q - q\right\|_{W^+} = \left\|\epsilon\, \frac{q}{z-1-\epsilon}\right\|_{W^+} = \epsilon \left\|\frac{z-1}{z-1-\epsilon}\, p\right\|_{W^+} \leq 2\epsilon\, ||p||_{W^+}.$$

□

We will also use the following fact proved on page 301 of the Proof of the Theorem in [13].

**Proposition 3.2** (M. von Renteln). *Let $f \in W^+(\mathbb{D})$ and $z_0 \in \mathbb{D}$ be such that $f(z_0) = 0$. Then $\frac{f}{z-z_0} \in W^+(\mathbb{D})$.*

We will also need the corona theorem for $W^+(\mathbb{D})$; see for example [13, Theorem]:

**Proposition 3.3.** *If $f_1, \dots, f_n \in W^+(\mathbb{D})$ are such that*

$$\text{for all } z \in \overline{\mathbb{D}}, \ \ |f_1(z)| + \cdots + |f_n(z)| > 0,$$

*then there exist $g_1, \dots, g_n \in W^+(\mathbb{D})$ such that*

$$\text{for all } z \in \overline{\mathbb{D}}, \ \ g_1(z)f_1(z) + \cdots + g_n(z)f_n(z) = 1.$$

**Lemma 3.4.** *Suppose that $f_1, \dots, f_n \in W^+(\mathbb{D})$ and $d$ is a greatest common divisor of $f_1, \dots, f_n$. If $z_0 \in \overline{\mathbb{D}}$ is a common zero of $f_1, \dots, f_n$, then $d(z_0) = 0$ as well.*

*Proof.* If $z_0 \in \mathbb{D}$, then let $m$ be the least integer among the multiplicities of $z_0$ as a zero respectively of $f_1, \dots, f_n$. By Proposition 3.2, $(z - z_0)^m$ is a divisor of $f_1, \dots, f_n$. But since $d$ is the greatest common divisor of $f_1, \dots, f_n$, it follows that $(z - z_0)^m$ divides $d$.

If on the other hand $z_0 \in \mathbb{T}$, then $f_1, \dots, f_n \in \mathfrak{m}_{z_0}$, where

$$\mathfrak{m}_{z_0} := \{f \in W^+(\mathbb{D}) \mid f(z_0) = 0\}.$$

By Proposition 3.1, $\mathfrak{m}_{z_0}$ has a bounded approximate identity. Applying Proposition 2.2, with $(a_n)_{n\geq 1} := (f_1, \dots, f_n, 0, 0, 0, \dots)$, we get the existence of an element $c \in \mathfrak{m}_{z_0}$, and $g_1, \dots, g_n \in \mathfrak{m}_{z_0}$ such that $f_k = cg_k$, $k = 1, \dots, n$. So we have a common divisor $c$ of $f_1, \dots, f_n$. Since $c(z_0) = 0$, and $d$ is a greatest common divisor, we have that $c$ divides $d$ and hence $d(z_0) = 0$, too. □

*Proof of Theorem 1.4.* Let $f_1, \dots, f_n \in W^+(\mathbb{D})$ have a greatest common divisor $d$ $(\neq 0)$. By the algebraic result in Lemma 2.3, it follows that 1 is a greatest common divisor of $\frac{f_1}{d}, \dots, \frac{f_n}{d}$. Lemma 3.4 gives

$$\left|\frac{f_1}{d}\right| + \cdots + \left|\frac{f_n}{d}\right| > 0 \quad \text{in } \overline{\mathbb{D}}.$$

By Proposition 3.3, it follows that there exist $g_1, \dots, g_n \in W^+(\mathbb{D})$ such that

$$g_1\frac{f_1}{d} + \cdots + g_n\frac{f_n}{d} = 1,$$

and so $g_1f_1 + \cdots + g_nf_n = d$, completing the proof of the theorem. □

## 4. $W^+(\mathbb{C}_+)$ is pre-Bézout

We will first prove that the maximal ideals $\mathfrak{M}_{i\beta}$, $\beta \in \mathbb{R}$, and $\mathfrak{M}_\infty$ in $W^+(\mathbb{C}_+)$ have a bounded approximate identity. To this end, we need the following lemma.

**Lemma 4.1.** *Suppose $F \in \mathfrak{M}_0$. Then for each $\epsilon > 0$, there exists $P \in \mathfrak{M}_0$ such that $P = \widehat{p_a} + p_0$, where $p_a \in L^1(0, \infty)$ has compact support, $p_0 \in \mathbb{C}$, and $\|F - P\|_{W^+} < \epsilon$.*

*Proof.* Let $\epsilon > 0$ be given. Let $F = \widehat{f_a} + f_0$, where $f_a \in L^1(0, \infty)$ and $f_0 \in \mathbb{C}$. Choose a compactly supported $p_a \in L^1(0, \infty)$ such that

$$\|p_a - f_a\|_{L^1} < \frac{\epsilon}{2}.$$

Set

$$P := \widehat{p_a} + \underbrace{\left( - \int_0^\infty p_a(t) dt \right)}_{=: p_0}.$$

Then $P \in W^+(\mathbb{C}_+)$ and

$$P(0) = \widehat{p_a}(0) + p_0 = \int_0^\infty p_a(t) dt - \int_0^\infty p_a(t) dt = 0.$$

So $P \in \mathfrak{M}_0$. We have

$$\begin{aligned} \left| f_0 + \int_0^\infty p_a(t) dt \right| &= \left| f_0 + \int_0^\infty f_a(t) dt + \int_0^\infty \big(p_a(t) - f_a(t)\big) dt \right| \\ &= \left| f_0 + \widehat{f_a}(0) + \int_0^\infty \big(p_a(t) - f_a(t)\big) dt \right| \\ &\leq |F(0)| + \|p_a - f_a\|_{L^1} < 0 + \frac{\epsilon}{2} = \frac{\epsilon}{2}. \end{aligned}$$

Thus

$$\|F - P\|_{W^+} = \|f_a - p_a\|_{L^1} + \left| f_0 + \int_0^\infty p_a(t) dt \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

**Theorem 4.2.**

(a) *Let $\mathfrak{M}_0 := \{F \in W^+(\mathbb{C}_+) \mid F(0) = 0\}$ and*

$$E_n := \frac{s}{s + \frac{1}{n}}, \quad n \in \mathbb{N}.$$

*Then $(E_n)_{n \in \mathbb{N}}$ is a bounded approximate identity for $\mathfrak{M}_0$.*

(b) *Let $\mathfrak{M}_\infty = \widehat{L_1(0, \infty)}$ and*

$$U_n = \widehat{n \mathbf{1}_{[0, \frac{1}{n}]}}, \quad n \in \mathbb{N},$$

*where $\mathbf{1}_{[0, \frac{1}{n}]}(t)$ is 1 if $t \in [0, \frac{1}{n}]$, and 0 otherwise. Then $(U_n)_{n \geq 1}$ is a bounded approximate identity for $\mathfrak{M}_\infty$.*

*Proof.* (b) The existence of a bounded approximate identity for $\mathfrak{M}_\infty$ follows [1, Theorem 6.5, p. 105]. The above example is easy to check.

(a) We note that

$$\|E_n\|_{W^+} = \left\| 1 + \widehat{\big(-\frac{1}{n} e^{-\frac{t}{n}}\big)} \right\|_{W^+} = |1| + \left\| -\frac{1}{n} e^{-\frac{t}{n}} \right\|_{L^1} = 1 + 1 = 2,$$

and so the sequence is bounded.

Given $F \in W^+(\mathbb{C}_+)$, and $\epsilon > 0$ arbitrarily small, in view of Lemma 4.1, we can find a $P \in \mathfrak{M}_0$ such that $P = \widehat{p_a} + p_0$, where $p_a \in L^1(0,\infty)$ has compact support, $p_0 \in \mathbb{C}$, and $\|F - P\|_{W^+} < \epsilon$. Then

$$\|E_n F - F\|_{W^+} \leq \|E_n P - P\|_{W^+} + \|E_n\|_{W^+} \|F - P\|_{W^+} + \|F - P\|_{W^+}.$$

So it is enough to prove that

$$\lim_{n\to\infty} \|E_n P - P\|_{W^+} = 0$$

for all $P \in \mathfrak{M}_0$ such that $P = \widehat{p_a} + p_0$, where $p_a \in L^1(0,\infty)$ has compact support, and $p_0 \in \mathbb{C}$. We do this below.

We have

$$E_n P - P = \frac{s + \frac{1}{n} - \frac{1}{n}}{s + \frac{1}{n}} P - P = -\frac{1}{n} \frac{1}{s + \frac{1}{n}} P = -\frac{1}{n} \left( (\widehat{e^{-t/n} * p_a}) + p_0 \widehat{e^{-t/n}} \right).$$

Let $c$ be given by

$$c(t) := \int_0^t e^{-\frac{t-\tau}{n}} p_a(\tau) d\tau + p_0 e^{-\frac{t}{n}}.$$

Then $c \in L^1(0,\infty)$. Let $T > 0$ be such that $\operatorname{supp}(p_a) \subset [0,T]$. We have

$$\|E_n P - P\|_{W^+} = \frac{1}{n} \|c\|_{L^1} = \frac{1}{n} \int_0^\infty |c(t)| dt = \underbrace{\frac{1}{n} \int_0^T |c(t)| dt}_{(I)} + \underbrace{\frac{1}{n} \int_T^\infty |c(t)| dt}_{(II)}.$$

We estimate $(I)$ as follows:

$$\begin{aligned} (I) = \frac{1}{n} \int_0^T |c(t)| dt &= \frac{1}{n} \int_0^T \left| \int_0^t e^{-\frac{t-\tau}{n}} p_a(\tau) d\tau + p_0 e^{-\frac{t}{n}} \right| dt \\ &\leq \frac{1}{n} \int_0^T \left[ \int_0^t e^{-\frac{t-\tau}{n}} |p_a(\tau)| d\tau + |p_0| e^{-\frac{t}{n}} \right] dt \\ &\leq \frac{1}{n} \underbrace{\int_0^T \big[ \int_0^t 1 \cdot |p_a(\tau)| d\tau + |p_0| \cdot 1 \big] dt}_{(III)}. \end{aligned}$$

Since the integral $(III)$ does not depend on $n$, we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^T |c(t)| dt = 0.$$

Furthermore,

$$
\begin{aligned}
(II) &= \frac{1}{n}\int_T^\infty |c(t)|dt = \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\int_0^t e^{\frac{\tau}{n}} p_a(\tau)d\tau + p_0\right|dt \\
&= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\int_0^\infty e^{\frac{\tau}{n}} p_a(\tau)d\tau + p_0\right|dt \qquad (\text{since } \operatorname{supp}(p_a)\subset[0,T]) \\
&= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\widehat{p_a}\Big(-\frac{1}{n}\Big) + p_0\right|dt
\end{aligned}
$$

Since $p_a$ has compact support in $[0,T]$, $\widehat{p_a}$ is an entire function by the Payley-Wiener theorem; see for instance [11, Theorem 7.2.3, p. 122]. Consequently,

$$
\begin{aligned}
(II) &= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\widehat{p_a}\Big(-\frac{1}{n}\Big) + p_0\right|dt \\
&= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}dt \cdot \left|\widehat{p_a}\Big(-\frac{1}{n}\Big) + p_0\right| \\
&= e^{-\frac{T}{n}}\left|\widehat{p_a}\Big(-\frac{1}{n}\Big) + p_0\right| \stackrel{n\to\infty}{\longrightarrow} 1\cdot|\widehat{p_a}(0)+p_0| = |P(0)| = 0.
\end{aligned}
$$

This completes the proof of the case (a). □

The case of $\mathfrak{M}_{i\beta}$ works in a similar manner.

**Theorem 4.3.** *Let* $F \in W^+(\mathbb{C}_+)$*, and let* $s_0 \in \mathbb{C}$ *be such that* $\mathrm{Re}(s_0) > 0$ *and* $F(s_0) = 0$*. Then* $\frac{F}{s-s_0} \in W^+(\mathbb{C}_+)$*.*

*Proof.* Let $F = \widehat{f_a} + f_0$, where $f_a \in L^1(0,\infty)$ and $f_0 \in \mathbb{C}$. Since $F(s_0) = 0$, we have

$$
F(s_0) = \int_0^\infty e^{-s_0\tau} f_a(\tau)d\tau + f_0 = 0. \tag{1}
$$

Let $c$ be defined by

$$
c(t) = \begin{cases} f_0 e^{s_0 t} + \displaystyle\int_0^t e^{s_0(t-\tau)} f_a(\tau)d\tau & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{2}
$$

We have

$$\begin{aligned}\int_0^\infty |c(t)|dt &= \int_0^\infty e^{\mathrm{Re}(s_0)t}\left| f_0 + \int_0^t e^{-s_0\tau} f_a(\tau)d\tau \right| dt \\ &= \int_0^\infty e^{\mathrm{Re}(s_0)t}\left| -\int_t^\infty e^{-s_0\tau} f_a(\tau)d\tau \right| dt \quad \text{(using (1))} \\ &\leq \int_0^\infty e^{\mathrm{Re}(s_0)t} \int_t^\infty e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)|d\tau dt \\ &= \int_0^\infty \int_t^\infty e^{\mathrm{Re}(s_0)t} e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)|d\tau dt \\ &= \int_0^\infty \int_0^\tau e^{\mathrm{Re}(s_0)t} e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)|dt d\tau \\ &= \int_0^\infty e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)| \int_0^\tau e^{\mathrm{Re}(s_0)t} dt d\tau \\ &= \int_0^\infty e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)| \frac{e^{\mathrm{Re}(s_0)\tau}-1}{\mathrm{Re}(s_0)} d\tau \\ &\leq \frac{1}{\mathrm{Re}(s_0)} \int_0^\infty |f_a(\tau)|d\tau < \infty.\end{aligned}$$

So $c \in L^1(0,\infty)$.

Let $\beta \in \mathbb{C}$ be such that $\mathrm{Re}(\beta)>\mathrm{Re}(s_0)$. Then we have from (2) that

$$e^{-\beta t}c(t) = \begin{cases} f_0 e^{(s_0-\beta)t} + \left( \left[e^{(s_0-\beta)x}u\right] * \left[e^{-\beta x} f_a\right]\right)(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{3}$$

where $u$ denotes the step function, given by $u(t) = 1$ for $t > 0$ and $u(t) = 0$ otherwise.

Recall the fact that if $g_a \in L^1(0,\infty)$, then for a complex number $\alpha$ such that $\mathrm{Re}(\alpha)> 0$, $(\widehat{e^{-\alpha t}g_a})(s) = \widehat{g_a}(s+\alpha)$ (for $s \in \mathbb{C}_+$). Using this, we obtain

$$\big(\widehat{e^{-\beta t}c}\big)(s) = \widehat{c}(s+\beta) \quad \text{and} \quad \big(\widehat{e^{-\beta t}f_a}\big)(s) = \widehat{f_a}(s+\beta) \quad (s \in \mathbb{C}_+).$$

Since the Laplace transform of a convolution is the product of the Laplace transforms (see for instance [4, Proposition 14.1]), we have

$$\left(\widehat{\left[e^{(s_0-\beta)x}u\right] * \left[e^{-\beta x} f_a\right]}\right)(s) = \frac{1}{s+\beta-s_0}\cdot \widehat{f_a}(s+\beta) \quad (s \in \mathbb{C}_+).$$

Using these facts, we see by taking Laplace transform on both sides of (3) that

$$\widehat{c}(s+\beta) = \frac{f_0}{s+\beta-s_0} + \frac{1}{s+\beta-s_0}\cdot \widehat{f_a}(s+\beta) = \frac{F(s+\beta)}{s+\beta-s_0} \quad (s \in \mathbb{C}_+).$$

So for all $s$ such that $\mathrm{Re}(s)>\mathrm{Re}(s_0)$, we have

$$\widehat{c}(s) = \frac{F(s)}{s-s_0}.$$

By the identity principle, the above holds in $\mathbb{C}_+$. So $\frac{F}{s-s_0} = \widehat{c} \in W^+(\mathbb{C}_+)$. $\square$

In our proof of Theorem 1.5 will need the corona theorem for $W^+(\mathbb{C}_+)$:

**Proposition 4.4.** *If $F_1, \dots, F_n \in W^+(\mathbb{C}_+)$ are such that there exists a $\delta > 0$ such that*

$$\text{(4)} \qquad \textit{for all } s \in \mathbb{C}_+, \ \ |F_1(s)| + \dots + |F_n(s)| > \delta > 0,$$

*then there exist $G_1, \dots, G_n \in W^+(\mathbb{C}_+)$ such that*

$$\text{(5)} \qquad \textit{for all } s \in \mathbb{C}_+, \ \ G_1(s)F_1(s) + \dots + G_n(s)F_n(s) = 1.$$

*Proof.* That (5) implies (4) is easy to see. The reverse implication follows from the classical result (see [3, p.112]) that the maximal ideals of $W^+(\mathbb{C}_+)$ are given by $\mathfrak{M}_\infty$ and $\mathfrak{M}_{s_0}$, where $s_0 \in \mathbb{C}_+$. Indeed, suppose that $F_1, \dots, F_n \in W^+(\mathbb{C}_+)$ satisfy (4), but that the ideal $(F_1, \dots, F_n) \neq (1)$. Then there exists a maximal ideal $\mathfrak{M}$ that contains $(F_1, \dots, F_n)$. We now consider the two possible cases:

(i) If $\mathfrak{M} = \mathfrak{M}_{s_0}$ for some $s_0 \in \mathbb{C}_+$, then $(F_1, \dots, F_n) \subset \mathfrak{M}_{s_0}$ yields that $F_1(s_0) = \dots = F_n(s_0) = 0$, which contradicts (4).

(ii) Now suppose that $\mathfrak{M} = \mathfrak{M}_{s_0}$. Let $F_k = \widehat{f_{k,a}} + f_{k,0}$, where $f_{k,a} \in L^1(0, \infty)$ and $f_{k,0} \in \mathbb{C}$, $k = 1, \dots, n$. Since $(F_1, \dots, F_n) \subset \mathfrak{M}_\infty$, we have $f_{1,0} = \dots = f_{n,0} = 0$. Hence $F_k = \widehat{f_{k,a}}$, $k = 1, \dots, n$. Passing to the limit $s \to \infty$ in (4), we obtain the contradiction that $0 \geq \delta$.

Consequently $(F_1, \dots, F_n) = (1)$, and so (5) holds for some $G_1, \dots, G_n \in W^+(\mathbb{C}_+)$. $\square$

**Lemma 4.5.** *Suppose that $F_1, \dots, F_n \in W^+(\mathbb{C}_+)$ and that $D$ is a greatest common divisor of $F_1, \dots, F_n$. If $s_0 \in \mathbb{C}_+$ is a common zero of $F_1, \dots, F_n$, then $D(s_0) = 0$ as well.*

*Proof.* If $\operatorname{Re}(s_0) > 0$, then let $m$ be the least integer among the multiplicities of $s_0$ as a zero respectively of $F_1, \dots, F_n$. By Theorem 4.3, $(s - s_0)^m$ is a divisor of $F_1, \dots, F_n$. But since $D$ is the greatest common divisor of $F_1, \dots, F_n$, it follows that $(s - s_0)^m$ divides $D$.

If on the other hand $\operatorname{Re}(s_0) = 0$, then $F_1, \dots, F_n \in \mathfrak{M}_{s_0}$, where

$$\mathfrak{M}_{s_0} := \{F \in W^+(\mathbb{C}_+) \mid F(s_0) = 0\}.$$

By Theorem 4.2, $\mathfrak{M}_{s_0}$ has a bounded approximate identity. Applying Proposition 2.2, with $(a_n)_{n \geq 1} := (F_1, \dots, F_n, 0, 0, 0, \dots)$, we get the existence of an element $C \in \mathfrak{M}_{s_0}$, and $G_1, \dots, G_n \in \mathfrak{M}_{s_0}$ such that $F_k = CG_k$, $k = 1, \dots, n$. So we have a common divisor $C$ of $F_1, \dots, F_n$. Since $D$ is a greatest common divisor, $C$ divides $D$ and so $D(s_0) = 0$, too. $\square$

**Lemma 4.6.** *Suppose that $F_1, \dots, F_n \in \mathfrak{M}_\infty$ and that $D \in W^+(\mathbb{C}_+)$ is a greatest common divisor of $F_1, \dots, F_n$. Then $D \in \mathfrak{M}_\infty$ as well.*

*Proof.* Applying Theorem 4.2 and Proposition 2.2 to the sequence $(a_n) = (F_1, F_2, \dots, F_n, 0, 0, 0 \dots)$ we get a common divisor $C \in \mathfrak{M}_\infty$ of the $F_j$'s, $j = 1, \dots, n$. Hence $C$ divides $D$; that is $D = QC$ for some $Q \in W^+(\mathbb{C}_+)$. Therefore $D \in \mathfrak{M}_\infty$. □

*Proof of Theorem 1.5.* Let $F_1, \dots, F_n \in W^+(\mathbb{C}_+)$ have a greatest common divisor $D$. By the algebraic result in Lemma 2.3, it follows that 1 is a greatest common divisor of $\frac{F_1}{D}, \dots, \frac{F_n}{D}$.

Lemma 4.5 gives

$$\left|\frac{F_1}{D}\right| + \dots + \left|\frac{F_n}{D}\right| > 0 \text{ in } \mathbb{C}_+. \tag{6}$$

Since $F_k/D \in W^+(\mathbb{C}_+)$ for each $k$, $F_k/D = \widehat{h_k} + \alpha_k$, where $h_k \in L^1(0,\infty)$ and $\alpha_k \in \mathbb{C}$. Since $h_k \in L^1(0,\infty)$, we have

$$\lim_{\substack{s\to\infty \\ s\in\mathbb{C}_+}} \widehat{h_k}(s) = 0. \tag{7}$$

We consider the two possible cases:

(i) All the $\alpha_k$'s are zero. Then by Lemma 4.6, it follows that 1 is the Laplace transform of an element in $L^1(0,\infty)$, which is a contradiction.

(ii) At least one of the $\alpha_k$'s is not zero. Then $|\alpha_1| + \dots + |\alpha_n| > 0$. So for $s \in \mathbb{C}_+$ such that $|s| > R$ with a large enough $R$, (7) gives the existence of a $\delta > 0$ such that

$$\left|\frac{F_1}{D}\right| + \dots + \left|\frac{F_n}{D}\right| > \delta > 0,$$

while on the compact set $K$ consisting of $s \in \mathbb{C}_+$ with $|s| \le R$, $|\frac{F_1}{D}| + \dots + |\frac{F_n}{D}|$ is at least as large as its minimum value on $K$, which is positive by (6). So by Proposition 4.4, it follows that there exist $G_1, \dots, G_n \in W^+(\mathbb{C}_+)$ such that

$$G_1\frac{F_1}{D} + \dots + G_n\frac{F_n}{D} = 1,$$

and so $G_1F_1 + \dots + G_nF_n = D$.

This completes the proof of the theorem. □

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