THE HERMITE PROPERTY OF A CAUSAL WIENER ALGEBRA USED IN CONTROL THEORY

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Abstract. Let $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \text{Re}(s) \geq 0\}$ and let $\mathcal{A}$ denote the Banach algebra

$$\mathcal{A} = \left\{ s(\in \mathbb{C}_+) \mapsto \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \;\middle|\; \begin{array}{l} f_a \in L^1(0,\infty),\ (f_k)_{k\geq 0} \in \ell^1, \\ 0 = t_0 < t_1 < t_2 < \dots \end{array} \right\}$$

equipped with pointwise operations and the norm:

$$\|f\| = \|f_a\|_{L^1} + \|(f_k)_{k\geq 0}\|_{\ell^1}, \quad f(s) = \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in \mathbb{C}_+).$$

(Here $\widehat{f_a}$ denotes the Laplace transform of f_a.) It is shown that, endowed with the Gelfand topology, the maximal ideal space of $\mathcal{A}$ is contractible. In particular, the ring $\mathcal{A}$ is Hermite. The algebra $\mathcal{A}$ arises in control theory, and the Hermite property has useful consequences in the problem of stabilization of linear systems; see [3, Corollary 4.14]. The following statements are equivalent for $f \in \mathcal{A}^{n\times k}$, $k < n$:

(1) There exists a $g \in \mathcal{A}^{k\times n}$ such that $gf = I_k$ on $\mathbb{C}_+$.
(2) There exist $F, G \in \mathcal{A}^{n\times n}$ such that $GF = I_n$ on $\mathbb{C}_+$ and $F_{ij} = f_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq k$.
(3) There exists a $\delta > 0$ such that $f(s)^* f(s) \geq \delta^2 I_k$, $s \in \mathbb{C}_+$.

1. Introduction

The aim of this paper is to show that the maximal ideal space $M(\mathcal{A})$ of the algebra $\mathcal{A}$ (defined below), is contractible. We also apply this result to the problem of completing a left invertible matrix with entries in $\mathcal{A}$ to an isomorphism, which has useful consequences in control theory.

Throughout the article, we will use the following notation:

$$\mathbb{C}_+ := \{s \in \mathbb{C} \mid \text{Re}(s) \geq 0\}.$$

Definition 1.1. Let $\mathcal{A}$ denote the Banach algebra

$$\mathcal{A} = \left\{ f : \mathbb{C}_+ \to \mathbb{C} \;\middle|\; \begin{array}{l} f(s) = \widehat{f_a}(s) + \displaystyle\sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in \mathbb{C}_+), \\ f_a \in L^1(0,\infty),\ (f_k)_{k\geq 0} \in \ell^1, 0 = t_0 < t_1 < t_2 < \dots \end{array} \right\}$$

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equipped with pointwise operations and the norm:

$$\|f\| = \|f_a\|_{L^1} + \|(f_k)_{k\geq 0}\|_{\ell^1}.$$

Here $\widehat{f_a}$ denotes the Laplace transform of f_a.

The above algebra arises as a natural class of transfer functions of stable distributed parameter systems in control theory; see [1], [3], [4].

Notation 1.2. Let $M(\mathcal{A})$ denote the maximal ideal space of $\mathcal{A}$, that is the set of all nonzero homomorphisms $\varphi : \mathcal{A} \to \mathbb{C}$. We equip $M(\mathcal{A})$ with the weak-$*$ topology (that is, the Gelfand topology).

In Proposition 1.4 below, we recall the known characterization of $M(\mathcal{A})$; see for example [1, Lemma A.1, p. 658]. But first we give the following definition.

Definition 1.3. $\chi : \mathbb{R} \to \mathbb{C}$ is a *character* if

$$|\chi(t)| = 1 \quad \text{and} \quad \chi(t+\tau) = \chi(t)\chi(\tau) \text{ for all } t, \tau \in \mathbb{R}.$$

Proposition 1.4. *$M(\mathcal{A})$ is the set of the following three types of nonzero homomorphisms on $\mathcal{A}$:*

$$f \longmapsto f(s), \quad s \in \mathbb{C}_+$$

$$f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \longmapsto f_0$$

$$f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \longmapsto \sum_{k=0}^{\infty} f_k e^{-\sigma t_k}\chi(t_k), \quad \sigma \geq 0 \text{ and } \chi \text{ is a character.}$$

In the above, $f \in \mathcal{A}$, $f_a \in L^1(0,\infty)$ and $(f_k)_{k\geq 0} \in \ell^1$.

Notation 1.5.

(1) The homomorphism $f \mapsto f(s)$ $(f \in \mathcal{A})$, corresponding to point evaluation at $s \in \mathbb{C}_+$ will be denoted henceforth by $\underline{s}$. The set of all such homomorphisms will be denoted by $\underline{\mathbb{C}_+}$.

(2) The homomorphism

$$\mathcal{A} \ni \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \longmapsto f_0$$

will be denoted by φ_∞.

(3) We define

$$U := M(\mathcal{A}) \setminus \underline{\mathbb{C}_+}.$$

We will show that $M(\mathcal{A})$ is contractible. We recall the notion of contractibility below:

Definition 1.6. A topological space X is said to be *contractible* if there exists a continuous map $R : X \times [0,1] \to X$ and a $x_0 \in X$ such that

$$\text{for all } x \in X, \; R(x,0) = x, \text{ and}$$
$$\text{for all } x \in X, \; R(x,1) = x_0.$$

Our main result is the following:

Theorem 1.7. *$M(\mathcal{A})$ is contractible.*

In particular, by a result proved in V. Ya. Lin [2, Theorem 3, p. 127], the above implies that the ring $\mathcal{A}$ is Hermite. Before stating this result, we recall the definition of a Hermite ring:

Definition 1.8. Let R be a ring with an identity element. A matrix $f \in R^{n\times k}$ is called *left invertible* if there exists a $g \in R^{k\times n}$ such that $gf = I$.

The ring R is called a *Hermite ring* if for all $k, n \in \mathbb{N}$ with $k < n$ and all left invertible matrices $f \in R^{n\times k}$, there exist $F, G \in R^{n\times n}$ such that $GF = I_n$ and $F_{ij} = f_{ij}$ for all $1 \le i \le n$ and $1 \le j \le k$.

Corollary 1.9. *$\mathcal{A}$ is a Hermite ring.*

The motivation for proving that $\mathcal{A}$ is a Hermite ring arises from control theory, where it plays an important role in the problem of stabilization of linear systems. Indeed, $\mathcal{A}$ being Hermite implies that if a transfer function G has a right (or left) coprime factorization, then G has a doubly coprime factorization, and the standard Youla parameterization yields all stabilizing controllers for G. For further details on the relevance of the Hermite property in control theory, see [3, Corollary 4.14, p. 296] and [4, Theorem 66, p. 347].

The corona theorem for $\mathcal{A}$ gives an analytic test for left invertibility (see [1]):

Proposition 1.10. *Let $f \in \mathcal{A}^{n\times k}$. Then the following are equivalent:*

1. *There exists a $g \in \mathcal{A}^{k\times n}$ such that $gf = I_k$ on $\mathbb{C}_+$.*
2. *There exists a $\delta > 0$ such that $f(s)^* f(s) \ge \delta^2 I_k$, $s \in \mathbb{C}_+$.*

Combining this with the fact the $\mathcal{A}$ is a Hermite ring now yields the following:

Corollary 1.11. *Let $k < n$ and $f \in \mathcal{A}^{n\times k}$. Then the following are equivalent:*

1. *There exists a $g \in \mathcal{A}^{k\times n}$ such that $gf = I_k$ on $\mathbb{C}_+$.*
2. *There exist $F, G \in \mathcal{A}^{n\times n}$ such that $GF = I_n$ on $\mathbb{C}_+$ and $F_{ij} = f_{ij}$, $1 \le i \le n$, $1 \le j \le k$.*
3. *There exists a $\delta > 0$ such that $f(s)^* f(s) \ge \delta^2 I_k$, $s \in \mathbb{C}_+$.*

In Section 3, we will give the proof of Theorem 1.7, but before doing that, in Section 2, we first prove a few technical results we will need in the sequel.

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2. Preliminaries

In this section, we prove a few technical results we will need in order to prove our main result.

First we prove that the subset $U := M(\mathcal{A}) \setminus \underline{\mathbb{C}_+}$ is closed in $M(\mathcal{A})$.

Lemma 2.1. *$\underline{\mathbb{C}_+}$ is open in $M(\mathcal{A})$.*

Proof. We observe that if $\cdot^g$ denotes the Gelfand transform, then $\varphi \in \underline{\mathbb{C}_+}$ iff there exists a $f_a \in L^1(0,\infty)$ such that

$$\left|\left(\widehat{f_a}\right)^g(\varphi)\right| = \left|\varphi\left(\widehat{f_a}\right)\right| > 0.$$

Thus $\underline{\mathbb{C}_+}$ is a union of open sets:

$$\underline{\mathbb{C}_+} = \bigcup_{f_a \in L^1(0,\infty)} \left\{\varphi \in M(\mathcal{A}) \mid \left|\left(\widehat{f_a}\right)^g(\varphi)\right| > 0\right\},$$

and is consequently open. □

Next we show that there is a one-to-one correspondence between $\mathbb{C}_+$ and $\underline{\mathbb{C}_+}$, and moreover their topologies coincide.

Lemma 2.2. *$\underline{\mathbb{C}_+}$ is homeomorphic to $\mathbb{C}_+$.*

Proof. The map

$$\underline{\cdot} : \mathbb{C}_+ \to \underline{\mathbb{C}_+} \quad \text{given by} \quad s \mapsto \underline{s}$$

is clearly onto. It is also one-to-one, since if

$$\underline{s_1} = \underline{s_2},$$

then in particular, their action on the Laplace transform of $e^{-t} \in L^1(0,\infty)$ must be identical:

$$\underline{s_1}\left(\widehat{e^{-t}}\right) = \underline{s_1}\left(\frac{1}{s+1}\right) = \frac{1}{s_1+1} = \frac{1}{s_2+1} = \underline{s_2}\left(\frac{1}{s+1}\right) = \underline{s_2}\left(\widehat{e^{-t}}\right),$$

and so $s_1 = s_2$. Thus $\underline{\cdot}$ is invertible.

Let (s_α) be a net such that $s_\alpha \to s_0$. Since $f \in \mathcal{A}$ is continuous in $\mathbb{C}_+$, it follows that $f(s_\alpha) \to f(s_0)$, that is,

$$\underline{s_\alpha}(f) \to \underline{s_0}(f).$$

But the choice of f was arbitrary, and so

$$\underline{s_\alpha} \to \underline{s_0} \text{ in } \underline{\mathbb{C}_+}.$$

Finally we prove the continuity of the inverse. Let $(\underline{s_\alpha})$ be a net such that $\underline{s_\alpha} \to \underline{s_0}$. In particular, since $e^{-t} \in L^1(0,\infty)$, we must have

$$\underline{s_\alpha}\left(\widehat{e^{-t}}\right) = \underline{s_\alpha}\left(\frac{1}{s+1}\right) = \frac{1}{s_\alpha+1} \to \frac{1}{s_0+1} = \underline{s_0}\left(\frac{1}{s+1}\right) = \underline{s_0}\left(\widehat{e^{-t}}\right),$$

which yields $s_\alpha \to s_0$ in $\mathbb{C}_+$. □

We will also need the following.

Lemma 2.3. *If $(\underline{s_\alpha})$ is a net in $\underline{\mathbb{C}_+}$ such that it is convergent in $M(\mathcal{A})$ to $\varphi \in U$, then $(s_\alpha) \to \infty$.*

Proof. In particular, for $e^{-t} \in L^1(0,\infty)$, we have

$$\underline{s_\alpha}(\widehat{e^{-t}}) = \underline{s_\alpha}\left(\frac{1}{s+1}\right) = \frac{1}{s_\alpha + 1} \to 0 = \varphi\left(\frac{1}{s+1}\right) = \varphi(\widehat{e^{-t}}),$$

and so $1/(s_\alpha+1) \to 0$. Thus $s_\alpha \to \infty$. □

The following lemma gives a useful criterion for convergence to an element in U.

Lemma 2.4. *Let $\varphi \in U$ and let (φ_α) be a net in $M(\mathcal{A})$ such that*

(1) *For all $f_a \in L^1(0,\infty)$, $\varphi_\alpha(\widehat{f_a}) \to 0$, and*
(2) *for all $T > 0$, $\varphi_\alpha(e^{-sT}) \to \varphi(e^{-sT})$.*

Then $\varphi_\alpha \to \varphi$ in U.

Proof. From the hypothesis, we see that for every $f_a \in L^1(0,\infty)$ and for every exponential polynomial

$$p = \sum_{k=0}^{N} f_k e^{-\cdot\, t_k}, \quad 0 = t_0 \le t_1 \le \cdots \le t_N,$$

we have $\varphi_\alpha(\widehat{f_a} + p) \to \varphi(\widehat{f_a} + p)$. Let

$$f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \in \mathcal{A}$$

be given and let $\epsilon > 0$. Choose an exponential polynomial p such that

$$\|f - \widehat{f_a} - p\| = \left\| \sum_{k=0}^{\infty} f_k e^{-st_k} - p \right\| \le \frac{\epsilon}{4}.$$

Since $\varphi_\alpha(\widehat{f_a} + p) \to \varphi(\widehat{f_a} + p)$, there exists an α_* such that for all $\alpha \ge \alpha_*$,

$$\left|\varphi_\alpha(\widehat{f_a} + p) - \varphi(\widehat{f_a} + p)\right| < \frac{\epsilon}{2}.$$

Then for all $\alpha \ge \alpha_*$, we have

$$\begin{aligned}
|\varphi_\alpha(f) - \varphi(f)| &= \left|\varphi_\alpha(\widehat{f_a} + p + f - \widehat{f_a} - p) - \varphi(\widehat{f_a} + p + f - \widehat{f_a} - p)\right| \\
&\le \left|\varphi_\alpha(\widehat{f_a} + p) - \varphi(\widehat{f_a} + p)\right| + \left|(\varphi_\alpha - \varphi)(f - \widehat{f_a} - p)\right| \\
&< \frac{\epsilon}{2} + \|\varphi_\alpha - \varphi\| \|f - \widehat{f_a} - p\| \\
&\le \frac{\epsilon}{2} + (\|\varphi_\alpha\| + \|\varphi\|)\frac{\epsilon}{4} \\
&\le \frac{\epsilon}{2} + (1+1)\frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

Hence for all $f \in \mathcal{A}$, $\varphi_\alpha(f) \to \varphi(f)$. Consequently, (φ_α) converges in the weak-$*$ topology on $M(\mathcal{A})$. □

3. Contractibility of $M(\mathcal{A})$

In this section we will prove our main result. Before giving the proof, we explain the main idea behind it: The maximal ideal space can be partitioned into the following three subsets:

$$\underline{\mathbb{C}_+}, \quad \{\varphi_\infty\}, \quad U \setminus \{\varphi_\infty\}.$$

We will construct a continuous contraction $R : M(\mathcal{A}) \times [0,1] \to M(\mathcal{A})$ which takes the identity map to the constant map (identically equal to φ_∞), via translations along $[0,\infty]$: On $\underline{\mathbb{C}_+}$, R acts as follows:

$$\underline{s} \mapsto \underline{s - \log(1-t)}.$$

So if $f \in \mathcal{A}$, then the action of $\underline{s - \log(1-t)}$ on f gives

$$\widehat{f_a}(s - \log(1-t)) + f_0 + \sum_{k=1}^{\infty} f_k e^{-(s-\log(1-t))t_k},$$

and when t becomes 1, formally this goes to

$$0 + f_0 + \sum_{k=0}^{\infty} f_k \cdot 0 = f_0 = \varphi_\infty(f).$$

In this manner the part $\underline{\mathbb{C}_+}$ of the maximal ideal space will be shown to contractible to φ_∞.

On the other hand, we will define the action of R on $U \setminus \{\varphi_\infty\}$ as follows: if

$$\varphi(f) = \sum_{k=0}^{\infty} f_k e^{-\sigma t_k} \chi(t_k), \quad f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \in \mathcal{A},$$

then

$$(R(\varphi,t))(f) = \sum_{k=0}^{\infty} f_k e^{-(\sigma-\log(1-t))t_k} \chi(t_k),$$

and once again, when t becomes 1, this goes to

$$f_0 + \sum_{k=1}^{\infty} f_k \cdot 0 \cdot \chi(t_k) = f_0 = \varphi_\infty(f).$$

In this way, we will show that the part $U \setminus \{\varphi_\infty\}$ of the maximal ideal space is also contractile to φ_∞.

We now give the proof our main result.

Proof of Theorem 1.7. Let $R : M(\mathcal{A}) \times [0,1] \to M(\mathcal{A})$ be defined as follows:

(1) If $\underline{s} \in \underline{\mathbb{C}_+}$, then

$$R(\underline{s}, t) = \underline{s - \log(1-t)} \quad \text{for } t \in [0,1), \quad \text{and} \quad R(\underline{s}, 1) = \varphi_\infty.$$

(2) For all $t \in [0,1]$, $R(\varphi_\infty, t) = \varphi_\infty$.

(3) Let $\varphi \in U \setminus \{\varphi_\infty\}$. Then there exists a $\sigma \geq 0$ and a character χ such that

$$\varphi(f) = \sum_{k=0}^{\infty} f_k e^{-\sigma t_k} \chi(t_k), \quad f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \in \mathcal{A}.$$

With this notation, we define

$$(R(\varphi, t))(f) = \sum_{k=0}^{\infty} f_k e^{-(\sigma - \log(1-t))t_k} \chi(t_k) \quad \text{for } t \in [0,1), \quad \text{and}$$

$$R(\varphi, 1) = \varphi_\infty.$$

We prove below that R is continuous. First note that any net $(\varphi_\alpha, t_\alpha)$ in $M(\mathcal{A})$ can be partitioned into three subnets:

$\underline{1}^\circ$ One with terms $(\varphi_\alpha, t_\alpha) \in \{\varphi_\infty\} \times [0,1]$,
$\underline{2}^\circ$ another one with terms $(\varphi_\alpha, t_\alpha) \in (U \setminus \{\varphi_\infty\}) \times [0,1]$,
$\underline{3}^\circ$ and finally one with terms $(\underline{s_\alpha}, t_\alpha) \in \underline{\mathbb{C}_+} \times [0,1]$.

So it is enough to prove that for each of the nets of the above type, if $(\varphi_\alpha, t_\alpha)$ is convergent to (φ, t) in $M(\mathcal{A}) \times [0,1]$, then $(R(\varphi_\alpha, t_\alpha))$ converges to $R(\varphi, t)$ in $M(\mathcal{A})$.

$\underline{1}^\circ$ We have $R(\varphi_\alpha, t_\alpha) = R(\varphi_\infty, t_\alpha) = \varphi_\infty$. Moreover, $\varphi_\infty = \varphi_\alpha \to \varphi$, and so $\varphi = \varphi_\infty$. Thus $R(\varphi, t) = \varphi_\infty$. Hence $R(\varphi_\alpha, t_\alpha) = \varphi_\infty = R(\varphi, t)$.

$\underline{2}^\circ$ By Lemma 2.1, U is closed, and so $\varphi \in U$. Thus

$$(R(\varphi_\alpha, t_\alpha))\big(\widehat{f_a}\big) = 0 = (R(\varphi, t))\big(\widehat{f_a}\big) \quad \text{for all } f_a \in L^1(0, \infty).$$

We break this subnet into two further subnets: First consider the case when t_α is identically 1. Then $t_\alpha \to t$ gives $t = 1$. Thus we have

$$R(\varphi_\alpha, t_\alpha) = R(\varphi_\alpha, 1) = \varphi_\infty = R(\varphi, 1) = R(\varphi, t).$$

Now consider the case that each $t_\alpha \in [0,1)$. Thus if

$$\varphi_\alpha(f) = \sum_{k=0}^{\infty} f_k e^{-\sigma_\alpha t_k} \chi_\alpha(t_k), \quad f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \in \mathcal{A},$$

then

$$(R(\varphi_\alpha, t_\alpha))(e^{-sT}) = e^{-(\sigma_\alpha - \log(1-t_\alpha))T} \chi_\alpha(T), \quad T > 0.$$

We now consider the following two cases:

(1) $\varphi \neq \varphi_\infty$. Let

$$\varphi(f) = \sum_{k=0}^{\infty} f_k e^{-\sigma t_k} \chi(t_k), \quad f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \in \mathcal{A}.$$

Since $\varphi_\alpha \to \varphi$, it follows in particular for $T > 0$,

$$\varphi_\alpha(e^{-sT}) = e^{-\sigma_\alpha T} \chi(T) \longrightarrow e^{-\sigma T} \chi(T) = \varphi(e^{-sT}).$$

If $t < 1$, then

$$e^{-(\sigma_\alpha - \log(1-t_\alpha))T} \chi_\alpha(T) \longrightarrow e^{-(\sigma - \log(1-t))T} \chi(T),$$

that is, $(R(\varphi_\alpha, t_\alpha))(e^{-sT}) \to (R(\varphi, t))(e^{-sT})$.

On the other hand, if $t = 1$, then

$$e^{-(\sigma_\alpha - \log(1-t_\alpha))T}\chi_\alpha(T) \longrightarrow 0 \cdot e^{-\sigma T}\chi(T) = \varphi_0(e^{-sT}),$$

that is, $(R(\varphi_\alpha, t_\alpha))(e^{-sT}) \to (R(\varphi, 1))(e^{-sT})$.

(2) $\varphi = \varphi_\infty$. As $\varphi_\alpha \to \varphi = \varphi_\infty$, for $T > 0$, $\varphi_\alpha(e^{-sT}) \to \varphi_\infty(e^{-sT}) = 0$, that is, $e^{-\sigma_\alpha T}\chi_\alpha(T) \to 0$. Thus

$$e^{-(\sigma_\alpha - \log(1-t_\alpha))T}\chi_\alpha(T) \longrightarrow 0 = \varphi_\infty(e^{-sT}),$$

that is, $(R(\varphi_\alpha, t_\alpha))(e^{-sT}) \to (R(\varphi_\infty, t))(e^{-sT})$.

The result now follows from Lemma 2.4.

$\underline{3}^\circ$ We break this subnet into two further subnets: First consider the case when t_α is identically 1. Then $t_\alpha \to t$ gives $t = 1$. Thus we have

$$R(\underline{s_\alpha}, t_\alpha) = R(\underline{s_\alpha}, 1) = \varphi_\infty = R(\varphi, 1) = R(\varphi, t).$$

Now consider the case that each $t_\alpha \in [0, 1)$. Then

$$R(\underline{s_\alpha}, t_\alpha) = \underline{s_\alpha - \log(1 - t_\alpha)}.$$

We now consider the following three cases:

(1) $\varphi = \underline{s}$. If $t \in [0, 1)$, then

$$R(\underline{s}, t) = \underline{s - \log(1 - t)}.$$

Since $\underline{s_\alpha} \to \underline{s}$, it follows from Lemma 2.2 that $s_\alpha \to s$ in $\mathbb{C}_+$. Moreover, the map $-\log(1 - \cdot) : [0, 1) \to [0, \infty)$ is continuous, and so $-\log(1 - t_\alpha) \to -\log(1 - t)$. It follows that

$$s_\alpha - \log(1 - t_\alpha) \longrightarrow s - \log(1 - t) \quad \text{in } \mathbb{C}_+.$$

Thus by Lemma 2.2 again,

$$R(\underline{s_\alpha}, t_\alpha) = \underline{s_\alpha - \log(1 - t_\alpha)} \longrightarrow \underline{s - \log(1 - t)} = R(\underline{s}, t).$$

If on the other hand, $t = 1$, then

$$R(\underline{s}, t) = \varphi_\infty.$$

Since $t_\alpha \to 1$, $-\log(1-t_\alpha) \to +\infty$. Thus $\text{Re}(s_\alpha - \log(1-t_\alpha)) \to +\infty$. If

$$f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \in \mathcal{A},$$

then $\widehat{f_a}(s_\alpha - \log(1 - t_\alpha)) \to 0$ and

$$\left\| \sum_{k=1}^{\infty} f_k e^{-(s_\alpha - \log(1-t_\alpha))t_k} \right\| \le \|f\| e^{t_1 \log(1-t_\alpha)} \longrightarrow 0.$$

Hence for all $f \in \mathcal{A}$,

$$(\underline{s_\alpha - \log(1 - t_\alpha)})(f) = f(s_\alpha - \log(1 - t_\alpha)) \longrightarrow f_0 = \varphi_\infty(f).$$

But the choice of f was arbitrary. Consequently,

$$R(\underline{s_\alpha}, t_\alpha) = \underline{s_\alpha - \log(1-t_\alpha)} \longrightarrow \varphi_\infty = R(\underline{s}, t).$$

(2) $\varphi = \varphi_\infty$. Then

$$R(\varphi, t) = R(\varphi_\infty, t) = \varphi_\infty.$$

Since

$$\underline{s_\alpha} \longrightarrow \varphi = \varphi_\infty,$$

by Lemma 2.3 it follows that $s_\alpha \to \infty$. So $s_\alpha - \log(1-t_\alpha) \to \infty$. (This is obvious if $t_\alpha \to 1$. But otherwise, $-\log(1-t_\alpha) \to -\log(1-t)$.) Hence

$$\widehat{f_a}(s_\alpha - \log(1-t_\alpha)) \longrightarrow 0 = \varphi_\infty(\widehat{f_a}) \quad \text{for all } f_a \in L^1(0,\infty).$$

Also, for $T > 0$, we have

$$\underline{s_\alpha}(e^{-sT}) = e^{-s_\alpha T} \longrightarrow 0 = \varphi_\infty(e^{-sT}).$$

Since $t_k \log(1-t_\alpha) \leq 0$, it follows that $e^{-(s_\alpha - \log(1-t_\alpha))T} \to 0$, that is,

$$(\underline{s_\alpha - \log(1-t_\alpha)})(e^{-sT}) \longrightarrow \varphi_\infty(e^{-sT}).$$

From Lemma 2.4, it follows that

$$R(\underline{s_\alpha}, t_\alpha) = \underline{s_\alpha - \log(1-t_\alpha)} \longrightarrow \varphi_\infty = R(\varphi, t).$$

(3) $\varphi \neq \varphi_\infty$. Since

$$\underline{s_\alpha} \longrightarrow \varphi \in U \setminus \{\varphi_\infty\},$$

by Lemma 2.3 it follows that $s_\alpha \to \infty$. So $s_\alpha - \log(1-t_\alpha) \to \infty$. (This is obvious if $t_\alpha \to 1$. But otherwise, $-\log(1-t_\alpha) \to -\log(1-t)$.) Hence

$$\widehat{f_a}(s_\alpha - \log(1-t_\alpha)) \longrightarrow 0 = \varphi(\widehat{f_a}) \quad \text{for all } f_a \in L^1(0,\infty).$$

Let

$$\varphi(f) = \sum_{k=0}^{\infty} f_k e^{-\sigma t_k} \chi(t_k), \quad f = \widehat{f_a} + \sum_{k=0}^{\infty} f_k e^{-\cdot\, t_k} \in \mathcal{A}.$$

If $t = 1$, then

$$R(\varphi, t) = R(\varphi, 1) = \varphi_\infty.$$

As $\underline{s_\alpha} \to \varphi$, we have for $T > 0$,

$$\underline{s_\alpha}(e^{-sT}) = e^{-s_\alpha T} \longrightarrow e^{-\sigma T}\chi(T) = \varphi(e^{-sT}).$$

Since $\log(1-t_\alpha) \to -\infty$, it follows that

$$e^{-(s_\alpha - \log(1-t_\alpha))T} \longrightarrow 0 \cdot e^{-\sigma T}\chi(T),$$

that is,

$$(\underline{s_\alpha - \log(1-t_\alpha)})(e^{-sT}) \longrightarrow \varphi_\infty(e^{-sT}).$$

From Lemma 2.4, we can now conclude that

$$R(\underline{s_\alpha}, t_\alpha) = \underline{s_\alpha - \log(1 - t_\alpha)} \longrightarrow \varphi_\infty = R(\varphi, 1).$$

On the other hand, if $t < 1$, then for $T > 0$,

$$(R(\varphi, t))(e^{-sT}) = e^{-(\sigma - \log(1-t))T}\chi(T).$$

Since $\underline{s_\alpha} \to \varphi$,

$$\underline{s_\alpha}(e^{-sT}) = e^{-s_\alpha T} \longrightarrow e^{-\sigma T}\chi(T) = \varphi(e^{-sT}).$$

So

$$e^{-(s_\alpha - \log(1-t_\alpha))T} \longrightarrow e^{-(\sigma - \log(1-t))T}\chi(T),$$

that is,

$$(\underline{s_\alpha - \log(1 - t_\alpha)})(e^{-sT}) \longrightarrow (R(\varphi, t))(e^{-sT}).$$

From Lemma 2.4, we can now conclude that

$$R(\underline{s_\alpha}, t_\alpha) = \underline{s_\alpha - \log(1 - t_\alpha)} \longrightarrow R(\varphi, 1).$$

This completes the proof. □

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