NONCOHERENCE OF SOME RINGS OF FUNCTIONS

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Abstract. Let $\mathbb{D}$, $\mathbb{T}$ denote the unit disc and unit circle, respectively, in $\mathbb{C}$, with center 0. If $S \subset \mathbb{T}$, then let A_S denote the set of complex-valued functions defined on $\mathbb{D} \cup S$ that are analytic in $\mathbb{D}$, and continuous and bounded on $\mathbb{D} \cup S$. Then A_S is a ring with pointwise addition and multiplication. We prove that if the intersection of S with the set of limit points of S is not empty, then the ring A_S is not coherent.

1. Introduction

In this paper, we investigate the coherence of some rings of analytic functions. We first recall the notion of coherence.

Definition 1.1. Let R be a commutative ring with identity element e, and let $R^n = R \times \cdots \times R$ (n times). Let $f = (f_1, \dots, f_n) \in R^n$. An element $(g_1, \dots, g_n) \in R^n$ is called a *relation on* f if $g_1 f_1 + \cdots + g_n f_n = 0$. The set of all relations on $f \in R^n$, denoted by $f^{\perp}$, is a R-submodule of the R-module R^n. The ring R is called *coherent* if for each $f \in R^n$, $f^{\perp}$ is finitely generated, that is, there exists a $d \in \mathbb{N}$ and there exist $g_j \in f^{\perp}$, $j \in \{1, \dots, d\}$, such that for all $g \in f^{\perp}$, there exist $r_j \in R$, $j \in \{1, \dots, d\}$ such that $g = r_1 g_1 + \cdots + r_d g_d$.

In [5], McVoy and Rubel showed that while H^∞ is coherent, the disk algebra A is not coherent. In this paper, we prove the noncoherence of some rings that lie between A and H^∞. The rings that we consider are introduced below:

Definition 1.2. Let the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$ be denoted by $\mathbb{D}$, and the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ be denoted by $\mathbb{T}$. If S be a subset of $\mathbb{T}$, then

$$A_S = \{f : \mathbb{D} \cup S \to \mathbb{C} \mid f \text{ is analytic in } \mathbb{D} \text{ and } f \text{ is continuous and bounded on } \mathbb{D} \cup S\},$$

with pointwise addition and multiplication: if $f, g \in A_S$, then

$$(f+g)(z) = f(z) + g(z) \quad \text{and} \quad (f \cdot g)(z) = f(z)g(z), \quad \text{for all } z \in \mathbb{D} \cup S.$$

Equipped with the supremum norm,

$$\|f\|_\infty = \sup_{z \in \mathbb{D} \cup S} |f(z)| \quad \text{for } f \in A_S,$$

the ring A_S forms a Banach algebra. We note that if $S = \mathbb{T}$, then $A_{\mathbb{T}}$ is the usual disk algebra A, while if $S = \emptyset$, then one obtains the Hardy space H^∞.

The spaces A_S considered here have been studied earlier (see for instance [1], where among other things, it was shown that the corona theorem holds for A_S). These classes are also of interest in control theory, where they form natural families

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of irrational transfer functions (see [10], where it was shown that A_S is a Hermite domain, although not a Bézout domain, and the stable rank and topological stable rank of A_S are 1 and 2, respectively, and the consequences of these properties in control theory were elaborated).

In this note, we study the coherence of the rings A_S. The relevance of the coherence property in control theory can be found in [7], [8]. For an expository background on coherent rings, we refer the reader to [2]. Noncoherence has been discussed in the context of other rings, for instance, see [6], [4].

Following the ideas of McVoy and Rubel from [5], used for proving the noncoherence of A, we show that if S is a subset of the unit circle such that the intersection of S with the set of limit points of S is not empty (that is, $S \cap S' \neq \emptyset$, where S' denotes the set of limit points of S), then A_S is not coherent. If $S = \emptyset$, then $S \cap S' = \emptyset$, and $A_S = H^\infty$: in [5], it is shown that H^∞ is coherent. In the case when S is not empty, and $S \cap S' = \emptyset$, I do not know if A_S is coherent or not, and this is an open problem.

2. Noncoherence of A_S for S intersecting S'

In this section we prove the following.

Theorem 2.1. *If S is a subset of $\mathbb{T}$ such that the intersection of S with the set of limit points of S is not empty, then the ring A_S is not coherent.*

Proof. Let z_0 be a limit point of S that belongs to S, and let $(z_n)_{n\in\mathbb{N}}$ be any sequence in $S \setminus \{z_0\}$ with distinct terms, and with limit z_0:

$$\forall n \in \mathbb{N},\ \ z_n \in S \setminus \{z_0\}; \qquad \forall m, n \in \mathbb{N},\ \ z_m \neq z_n; \qquad \lim_{n\to\infty} z_n = z_0 \in S. \tag{2.1}$$

Let B_1 be the Blaschke product

$$B_1(z) = \prod_{n=1}^{\infty} \frac{|\alpha_n|}{\alpha_n} \frac{\alpha_n - z}{1 - \alpha_n^* z}, \quad z \in \mathbb{D}, \tag{2.2}$$

where

$$\alpha_n = \left(1 - \frac{1}{(n+1)^2}\right) z_0, \quad n \in \mathbb{N}.$$

Let B_2 be the Blaschke product

$$B_2(z) = \prod_{n=1}^{\infty} \frac{|\beta_n|}{\beta_n} \frac{\beta_n - z}{1 - \beta_n^* z}, \quad z \in \mathbb{D}, \tag{2.3}$$

where

$$\beta_n = \left(1 - \frac{1}{\sqrt{2}(n+1)^2}\right) z_0, \quad n \in \mathbb{N}.$$

If $m, n \in \mathbb{N}$, then $\alpha_m \neq \beta_n$ by construction. We also observe that $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are convergent with the same limit z_0. The functions B_1, B_2 are analytic in an open set Ω containing $\overline{\mathbb{D}} \setminus \{z_0\}$, and the infinite products converge uniformly on compact subsets contained in Ω (see for instance Exercise 12 on page 317 of [9]).

Define $f_i : \overline{\mathbb{D}} \to \mathbb{C}$, $i \in \{1, 2\}$, as follows:

$$f_i(z) = \begin{cases} (z - z_0) B_i(z) & \text{if } z \in \overline{\mathbb{D}} \setminus \{z_0\}, \\ 0 & \text{if } z = z_0. \end{cases} \tag{2.4}$$

Then $f_1, f_2 \in A$.

Lemma 2.2. *Let S be a subset of $\mathbb{T}$, and let $z_0 \in S$. Suppose that f_i, $i \in \{1,2\}$ are defined by (2.4). If $g_1, g_2 \in A_S$ are such that for all $z \in \mathbb{D} \cup S$, $g_1(z)f_1(z) + g_2(z)f_2(z) = 0$, then $g_1(z_0) = 0 = g_2(z_0)$.*

Proof. For all $z \in \mathbb{D}$, we have $|g_1(z)||z-z_0||B_1(z)| = |g_2(z)||z-z_0||B_2(z)|$, and in particular, $|g_1(\beta_n)||\beta_n - z_0||B_1(\beta_n)| = 0$. As $B_1(\beta_n) \neq 0$ and $|\beta_n - z_0| \neq 0$, it follows that $g_1(\beta_n) = 0$. Since $g_1 \in A_S$, we obtain

$$g_1(z_0) = g_1\left(\lim_{n\to\infty} \beta_n\right) = \lim_{n\to\infty} g_1(\beta_n) = 0.$$

Similarly, $g_2(z_0) = 0$. □

We shall prove that the module of relations on (f_1, f_2) is not finitely generated. Assume, on the contrary, that the module of relations on (f_1, f_2) is generated by $(g_{j,1}, g_{j,2}) \in A_S^2$, $j \in \{1, \dots, d\}$. From Lemma 2.2, it follows that for all $i \in \{1,2\}$ and all $j \in \{1, \dots, d\}$,

$$\lim_{n\to\infty} g_{j,i}(z_n) = g_{j,i}\left(\lim_{n\to\infty} z_n\right) = g_{j,i}(z_0) = 0. \tag{2.5}$$

Let K denote the compact set defined as follows: $K = \{z_n \mid n \in \mathbb{N}\} \cup \{z_0\}$. Define the function $\varphi : \mathbb{D} \cup S \to \mathbb{C}$ by

$$\varphi(z) = \left(\max\{|z-z_0|, |g_{1,1}(z)|, \dots, |g_{d,1}(z)|, |g_{1,2}(z)|, \dots, |g_{d,2}(z)|\}\right)^{\frac{1}{2}}.$$

Then it can be seen that the restriction of φ to K is a continuous function on K. Also, we note that

$$\forall n \in \mathbb{N}, \quad \varphi(z_n)^2 \geq |z_n - z_0| > 0, \tag{2.6}$$

and moreover, using (2.5), it follows that

$$\lim_{n\to\infty} \varphi(z_n) = 0. \tag{2.7}$$

K is a closed set of Lebesgue measure 0 on the unit circle, and $\varphi : K \to \mathbb{C}$ is a continuous function on K. We now recall the following result (see for instance the theorem on page 81 of [3])

Theorem 2.3 (Rudin)**.** *Let K be a closed set of Lebesgue measure zero on the unit circle, and let F be any continuous complex-valued function on K. Then there exists a function in the disk algebra A whose restriction to K is F.*

An application of this theorem yields the existence of a function $\Phi \in A$ whose restriction to K is φ. Observe that $\Phi(z_0) = \varphi(z_0) = 0$. Define $\Gamma_i : \mathbb{D} \cup S \to \mathbb{C}$, $i \in \{1,2\}$ as follows:

$$\Gamma_i(z) = \begin{cases} \Phi(z)B_i(z) & \text{if } z \neq z_0, \\ 0 & \text{if } z = z_0. \end{cases}$$

Then we claim that $\Gamma_1, \Gamma_2 \in A_S$, and this can be seen as follows:

Let $i \in \{1,2\}$. As Φ and B_i are analytic in $\mathbb{D}$, it follows that Γ_i is also analytic in $\mathbb{D}$.

If $z \in \mathbb{D} \cup S$, then $|\Phi(z)| \leq \|\Phi\|_\infty < +\infty$. If $z \in \mathbb{D}$, then $|B_i(z)| \leq 1$. If $z \in S \setminus \{z_0\}$, then each factor in the partial product of (2.2) or (2.3) has modulus 1, and using the (uniform) convergence of B_i on compact subsets contained in Ω, it follows that $|B_i(z)| = 1$ for all $z \in S \setminus \{z_0\}$. Consequently Γ_i is bounded on $\mathbb{D} \cup S$.

Finally, continuity of Γ_i on $(\mathbb{D} \cup S) \setminus \{z_0\}$ follows from the continuity of B_i and that of Φ on $(\mathbb{D} \cup S) \setminus \{z_0\}$. Continuity of Γ_i at z_0 follows by observing that it is the product of a bounded function and the continuous function Φ which converges to 0. So we conclude that $\Gamma_1, \Gamma_2 \in A_S$.

Let $g_1 := \Gamma_2$ and $g_2 := -\Gamma_1$. Then (g_1, g_2) is a relation on (f_1, f_2). Indeed, if $z \in (\mathbb{D} \cup S) \setminus \{z_0\}$, then

$$g_1(z)f_1(z) + g_2(z)f_2(z) = \Phi(z)B_2(z)(z-z_0)B_1(z) - \Phi(z)B_1(z)(z-z_0)B_2(z) = 0.$$

If $z = z_0$, then $g_1(z)f_1(z) + g_2(z)f_2(z) = 0 \cdot 0 + 0 \cdot 0 = 0$.

However, we now show that there cannot exist functions $h_1, \dots, h_d$ in A_S such that

$$h_1(g_{1,1}, g_{1,2}) + \cdots + h_d(g_{1,d}, g_{2,d}) = (g_1, g_2). \tag{2.8}$$

If (2.8) holds for some $h_1, \dots, h_d$ in A_S, then for all $n \in \mathbb{N}$,

$$h_1(z_n)g_{1,1}(z_n) + \cdots + h_d(z_n)g_{d,1}(z_n) = g_1(z_n) = \Gamma_2(z_n) = \Phi(z_n)B_2(z_n) = \varphi(z_n)B_2(z_n).$$

Hence for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
|\varphi(z_n)| &= |\varphi(z_n)B_2(z_n)| \leq |h_1(z_n)||g_{1,1}(z_n)| + \cdots + |h_d(z_n)||g_{d,1}(z_n)| \\
&\leq d \cdot \max\{\|h_1\|_\infty, \dots, \|h_d\|_\infty\} \cdot \max\{|g_{1,1}(z_n)|, \dots, |g_{d,1}(z_n)|\} \\
&\leq M \cdot \max\{|z_n - z_0|, |g_{1,1}(z_n)|, \dots, |g_{d,1}(z_n)|, |g_{1,2}(z_n)|, \dots, |g_{d,2}(z_n)|\} \\
&= M \cdot |\varphi(z_n)|^2,
\end{aligned}$$

where $M := d \cdot \max\{\|h_1\|_\infty, \dots, \|h_d\|_\infty\}$. Consequently, using (2.6), we have

$$\forall n \in \mathbb{N}, \quad 1 \leq M|\varphi(z_n)|. \tag{2.9}$$

But from (2.7), the right hand side of (2.9) tends to 0 as $n \to \infty$, and so we arrive at the contradiction that $1 \leq 0$. This completes the proof. $\square$

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