Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Ideals of denominators in the disk-algebra

Raymond Mortini and Amol Sasane

Abstract

We show that there do not exist finitely generated, non-principal ideals of denominators in the disk-algebra $A(\mathbb{D})$. Our proof involves a new factorization theorem for $A(\mathbb{D})$ that is based on Treil's determination of the Bass stable rank for H^∞.

1. *Notation, background*

Let H^∞ be the uniform algebra of bounded analytic functions on the open unit disk $\mathbb{D}$ and let $A(\mathbb{D})$ denote the disk-algebra; that is the subalgebra of all functions in H^∞ that admit a continuous extension to the Euclidean closure $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ of $\mathbb{D}$.

Let $\gamma = n/d$ be a quotient of two functions n and d in $A = H^\infty$ or $A = A(\mathbb{D})$. It is well known that every ideal of denominators

$$\mathfrak{D}(\gamma) = \{f \in A : f\gamma \in A\}$$

in $A = H^\infty$ is a principal ideal, since H^∞ is a pseudo-Bézout ring; the latter means that each pair of functions in H^∞ has a greatest common divisor (see [**11**]). The situation in $A(\mathbb{D})$ is completely different, due to the fact that $A(\mathbb{D})$ does not enjoy the property of being a pseudo-Bézout ring. For example $1 - z$ and $(1 - z)\exp(-\frac{1+z}{1-z})$ do not have a greatest common divisor. Answering several questions of Frank Forelli [**3**, **4**], the first author could prove in his Habilitationsschrift [**8**] that any closed ideal in $A(\mathbb{D})$ is an ideal of denominators; that an ideal of denominators is closed if and only if $\gamma \in L^\infty(\mathbb{T})$; that the complement inside $\overline{\mathbb{D}}$ of the zero set

$$Z(\mathfrak{D}(\gamma)) = \bigcap_{f\in\mathfrak{D}(\gamma)} \{z \in \overline{\mathbb{D}} : f(z) = 0\}$$

of $\mathfrak{D}(\gamma)$ is the set of points a in $\overline{\mathbb{D}}$ for which there exists a neighborhood U in $\overline{\mathbb{D}}$ such that $|\gamma|$ admits a continuous extension to U; and that every ideal of denominators in $A(\mathbb{D})$ contains a function f whose zero set equals the zero set of the ideal (one then says that $\mathfrak{D}(\gamma)$ has the Forelli-property.) The proof of this last result was based on the approximation theorem of Carleman (see [**5**, p. 135]).

In the present note we shall be concerned with the question whether finitely generated, but non-principal ideals in $A(\mathbb{D})$ can be represented as ideals of denominators. It turns out that this is not the case. Our proof uses as main ingredient a deep result of Treil [**14**] that tells us that H^∞ has the Bass stable rank one. This is a generalization going far beyond the corona theorem and tells us that whenever (f, g) is a corona pair in H^∞, that is whenever $|f| + |g| \geq \delta > 0$ in $\mathbb{D}$, then there exists $h \in H^\infty$ such that $f + hg$ is invertible in H^∞. We actually need an extension of this found by the second author of this paper to algebras of the form

$$H^\infty_E = \{f \in H^\infty : f \text{ extends continously to } \mathbb{T} \setminus E\},$$

2000 *Mathematics Subject Classification* 46J15 (primary), 30D50 (secondary).

where E is a closed subset of the unit circle $\mathbb{T}$. That result on the Bass stable rank of H^∞_E will be used to prove a factorization property of functions in $A(\mathbb{D})$, which will be fundamental to achieve our main goal of characterizing the finitely generated ideals of denominators in $A(\mathbb{D})$.

From the applications point of view, there is also a control theoretic motivation for considering the question of finding out whether there are ideals of denominators which are finitely generated, but not principal. Indeed, [**10**, Theorem 1, p.30] implies that if a plant is internally stabilizable, then the corresponding ideal of denominators is generated by at most two elements, and moreover, if an ideal of denominators corresponding to a plant is principal, then the plant has a weak coprime factorization. In light of these two results, our result on the nonexistence of non-principal finitely generated ideal of denominators in the disk-algebra implies that every internally stabilizable plant over the disk-algebra has a weak coprime factorization. Finally, since the disk-algebra is pre-Bézout [**12**], it also follows that every plant having a weak coprime factorization, possesses a coprime factorization [**10**, Proposition, p. 54]. Consequently, every internally stabilizable plant over the disk-algebra has a coprime factorization.

2. *Factorization in $A(\mathbb{D})$*

Cohen's factorization theorem for commutative, non-unital Banach algebras X tells us that if X has a bounded approximate identity, then every $f \in X$ factors as $f = gh$, where both factors are in X (see e.g. [**1**, p.76]). For $A(\mathbb{D})$ this may be applied to every closed ideal of the form $X = \mathfrak{I}(E, A(\mathbb{D})) := \{f \in A(\mathbb{D}) : f|_E \equiv 0\}$, whenever E is a closed subset of $\mathbb{T}$ of Lebesgue measure zero (note that (e_n) with $e_n = 1 - p_E^n$ is such a bounded approximate identity, where p_E is a peak function in $A(\mathbb{D})$ associated with E; see [**7**, p. 80] for a proof of the existence of p_E). In the present paragraph we address the following question: Let $f \in A(\mathbb{D})$. Suppose that f vanishes on $E \subseteq \mathbb{T}$ and that E can be written as $E = E_1 \cup E_2$, where the E_j are closed, not necessarily disjoint.

(1) Do there exist factors f_j of f such that $f = f_1 f_2$ and such that f_j vanishes *only* on E_j?

A weaker version reads as follows:

(2) Do there exist factors f_j of f such that $f = f_1 f_2$ and such that f_1 vanishes *only* on E_1 and f_2 has the same zero set as f?

We will first answer question (2) above. The proof works along the model of [**8**, Proposition 2.3]. It uses the following lemma that is based on the approximation theorem of Carleman (see [**5**]):

LEMMA 2.1. *[**8**, Lemma 1.1] Let I be an open interval. Then for every continuous function u and every positive, continuous error function $\varepsilon(x) > 0$ on I there exists a C^1-function v on I such that $|u - v| < \varepsilon$ on I.*

We shall also give an answer to a variant of question (1) whenever the sets E_j are disjoint closed subsets in $\overline{\mathbb{D}}$. That result will be the main new ingredient to prove our result on the ideals of denominators.

In the sequel, let $Z(f)$ denote the zero set of a function.

THEOREM 2.2. *Let E be closed subset of $\mathbb{T}$ and suppose that $f|_E \equiv 0$ for some $f \in A(\mathbb{D})$, $f \not\equiv 0$. Then there exists a factor g of f that vanishes exactly on E. Moreover, g can be taken so that the quotient f/g vanishes everywhere where f does.*

Proof. We shall construct an outer function $g \in A(\mathbb{D})$ with $Z(g) = E$ such that $|f| \leq |g|^2$ on $\mathbb{T}$. Then, by the extremal properties for outer functions (see [**6**]), $|f| \leq |g|^2$ on $\overline{\mathbb{D}}$. Hence $|f/g| \leq |g|$ on $\overline{\mathbb{D}} \setminus E$. Clearly this quotient has a continuous extension (with value 0) at every point in E. Thus $f = gh$ for some $h \in A(\mathbb{D})$. To construct g, we write $\mathbb{T} \setminus E$ as a countable union of open arcs I_n. Note that f vanishes at the two (or in case E is a singleton, a single) boundary points of I_n. Let p_E be a peak function associated with E. Consider on $\mathbb{T}$ the continuous function $q = |f| + |1 - p_E|$. Then $|q| > 0$ on I_n, $Z(q) = E$ and $q = 0$ on the boundary points of I_n. If the outer function associated with q would be in $A(\mathbb{D})$, we were done. But we are not able to prove that. So we need to proceed as in [**8**, p. 22]. Let $I_n =]a_n, b_n[$. Using Lemma 2.1, there exists functions $u_n \in C^1(I_n)$ so that

$$|u_n - q| \leq \tfrac{1}{2}|q| \text{ on } I_n,$$

and $u_n(a_n) = u_n(b_n) = 0$. In particular,

$$\tfrac{1}{2}|q| \leq |u_n| \leq \tfrac{3}{2}|q| \text{ on } I_n. \tag{2.1}$$

Let $u : \mathbb{T} \to \mathbb{R}$ be defined by $u = u_n$ on I_n, $n = 1, 2, \ldots$, and $u = 0$ elsewhere on $\mathbb{T}$. Then $u \in C(\mathbb{T})$, $u \geq 0$, and by the left inequality in (2.1), $\log u \in L^1(\mathbb{T})$. Since $u \in C^1(\mathbb{T} \setminus Z(q))$ and $u|_{Z(q)} \equiv 0$, the outer function

$$g(z) = \sqrt{2}\, \exp \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|u(e^{it})|\, dt\right)$$

belongs by [**12**, p.52] to $A(\mathbb{D})$. It is clear that on $\mathbb{T}$ we have $|g|^2 = 2u \geq |q| \geq |f|$ and that g vanishes only on E. Moreover, $|f|/|g| \leq |g|$ shows that $f/g \in A(\mathbb{D})$ and that $Z(f/g) = Z(f)$. □

The following $\big(H^\infty, A(\mathbb{D})\big)$-multiplier type result will yield our final factorization result (Theorem 2.4), that will be central to our study of ideals of denominators.

THEOREM 2.3. *Let E be a closed subset of Lebesgue measure zero in $\mathbb{T}$ and let $f \in H^\infty$ be a function that has a continuous extension to $\mathbb{T} \setminus E$; i.e $f \in H^\infty_E$. Suppose that 0 does not belong to the cluster set of f at each point in E. Then there exists a function $h \in H^\infty_E$, invertible in H^∞, so that $fh \in A(\mathbb{D})$.*

Proof. Consider a peak function $p_E \in A(\mathbb{D})$ associated with E. By assumption, the ideal I generated by f and $1 - p_E$ in H^∞_E is proper. (Here we have used the corona theorem for H^∞_E [**2**].)

Since H^∞_E has the stable rank one ([**13**, Theorem 5.2]), there exist h invertible in H^∞_E and $g \in H^\infty_E$ such that $hf + g(1 - p_E) = 1$. Since the only points of discontinuity of g are located on E, we see that $g(1 - p_E) \in A(\mathbb{D})$. Thus $hf \in A(\mathbb{D})$. □

THEOREM 2.4. *Let $f \in A(\mathbb{D})$. Suppose that $Z(f) = E_1 \cup E_2$, where the E_j are two disjoint closed sets in $\overline{\mathbb{D}}$. Then there exist factors f_j of f in $A(\mathbb{D})$ such that $f = f_1 f_2$ and $Z(f_j) = E_j$.*

Proof. By assumption, $2\varepsilon := \operatorname{dist}(E_1, E_2) > 0$. Choose around each point $\alpha \in E_1 \cap \mathbb{T}$ a symmetric open arc $\mathcal{A} \subseteq \mathbb{T}$ with center α and length ε. Due to compactness, there are finitely many of these arcs whose union covers $E_1 \cap \mathbb{T}$. Let V be the union of these arcs. By combining two adjacent arcs, we may assume that V writes as $V = \cup_{n=1}^N]\alpha_j, \beta_j[$, the closures of the arcs $I_j :=]\alpha_j, \beta_j[$ being pairwise disjoint. We also have that $\overline{V} \cap E_2 = \emptyset$ as well as $E_1 \cap \partial V = \emptyset$.

We first consider the outer factor F of f. Note that

$$F(z) = \exp \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log |F(e^{it})| \frac{dt}{2\pi}.$$

Consider the factorization $F = F_1 F_2$, where

$$F_1(z) = \exp \int_{t: e^{it} \in V} \frac{e^{it}+z}{e^{it}-z} \log |F(e^{it})| \frac{dt}{2\pi},$$

and

$$F_2(z) = \exp \int_{t: e^{it} \in \mathbb{T}\setminus V} \frac{e^{it}+z}{e^{it}-z} \log |F(e^{it})| \frac{dt}{2\pi}.$$

Then the F_j have continuous extensions to every point in $\mathbb{T} \setminus \partial V$. Also $Z(F_j) = E_j \cap \mathbb{T}$. Applying Theorem 2.3, there exists an invertible function $h \in H^\infty_{\partial V}$ so that $G_1 := F_1 h \in A(\mathbb{D})$. Since $f = (F_1 h)(\frac{1}{h} F_2)$, we obtain that outside the zeros of F_1, that is outside E_1, the function $\frac{1}{h} F_2$ is continuous. Note that h is continuous on E_1 as well as F_2 itself. Hence, $G_2 := \frac{1}{h} F_2 \in A(\mathbb{D})$. Thus $F = G_1 G_2$ satisfies $Z(G_j) = E_j \cap \mathbb{T}$.

Now suppose that f has an inner factor $\Theta = BS_\mu$. Let

$$\sigma(\Theta) = \{\sigma \in \overline{\mathbb{D}} : \liminf_{z \to \sigma} |\Theta(z)| = 0\}$$

be the support of Θ. Note that $\sigma(\Theta) \subseteq Z(f) = E_1 \cup E_2$. Now we split the support of Θ into the corresponding parts $\Xi_1 := \sigma(\Theta) \cap E_1$ and $\Xi_2 := \sigma(\Theta) \cap E_2$ and write Θ as $\Theta_1 \Theta_2$. Then $f_j = \Theta_j G_j$ gives the desired factorization. □

3. *Ideals of denominators*

Notation: Let A be a commutative unital algebra. For $f_j \in A$, let

$$\mathfrak{I}(f_1, f_2, \dots, f_N) = \left\{ \sum_{j=1}^N g_j f_j : g_j \in A \right\}$$

denote the ideal generated by the functions f_j ($j = 1, \dots, N$). We also denote the principal ideal $\mathfrak{I}(f)$ by fA.

If $\gamma = n/d$ is a quotient of two elements n, d in $A \setminus \{0\}$, then

$$\mathfrak{D}(\gamma) = \{f \in A : f\gamma \in A\}$$

is the ideal of denominators generated by γ. If $\gamma \in A$, then it is easy to see that $\mathfrak{D}(\gamma) = A$.

Finally, if I is an ideal in $A(\mathbb{D})$, then $Z(I) = \bigcap_{f \in I} Z(f)$ denotes the zero set (or hull) of I.

The following two Lemmas are well known (see [**9**]) and work for quite general function algebras. For the reader's convenience we present simple proofs.

Lemma 3.1. *Let I be an ideal in $A(\mathbb{D})$ and let M be a maximal ideal containing I. Suppose that $I = IM$. Then I is not finitely generated.*

Proof. Suppose that $I = (f_1 m_1, \dots, f_N m_N)$ for some $f_j \in I$ and $m_j \in M$. Then

$$|f_k| \le C_k \sum_{j=1}^N |f_j m_j| \le C_k \left(\sum_{j=1}^N |f_j|^2 \right)^{1/2} \left(\sum_{j=1}^N |m_j|^2 \right)^{1/2}.$$

Thus, for $C = \sum_{k=1}^{N} C_k^2$,

$$\sum_{k=1}^{N} |f_k|^2 \leq C \left(\sum_{j=1}^{N} |f_j|^2 \right) \left(\sum_{j=1}^{N} |m_j|^2 \right).$$

Hence $\sum_{j=1}^{N} |m_j|^2 \geq 1/C$ on $\overline{\mathbb{D}} \setminus Z(I)$. Since $Z(I)$ is nowhere dense, we get this estimate to hold true on $\overline{\mathbb{D}}$. But this is a contradiction, since all the m_j vanish at some point. □

LEMMA 3.2. *Let I be an ideal in $A(\mathbb{D})$. Suppose that $Z(I) \subseteq \mathbb{D}$. Then I is generated by a finite Blaschke product.*

Proof. Due to compactness of $Z(I)$, we know that $Z(I)$ is finite (or empty). Let $Z(I) = \{a_1, \cdots, a_N\}$ and let m_n be the highest multiplicity of the zero a_n at which all functions in I vanish. We claim that I is generated by the Blaschke product B associated with these (a_n, m_n). In fact, the inclusion $I \subseteq \mathfrak{I}(B)$ is trivial, since B divides every function in I. By construction, $\bigcap_{f \in I} Z(f/B) = \emptyset$. Due to compactness, there are finitely many functions $f_j \in I$ so that $\bigcap_{j=1}^{n} Z(f_j/B) = \emptyset$. By the corona theorem for $A(\mathbb{D})$, we have that $1 \in \mathfrak{I}(f_1/B, \ldots, f_n/B)$. Thus $B \in I$. □

The following works for every commutative unital ring.

LEMMA 3.3. *Let n, d be two functions in A such that $\mathfrak{I}(n, d) = A$. Then $\mathfrak{D}(n/d) = dA$.*

Proof. Let $x, y \in A$ be such that $1 = xn + yd$. Then $f = x(fn) + (fy)d$. Now let $f \in \mathfrak{D}(n/d)$. Hence $fn = ad$ implies that $f = x(ad) + (fy)d \in dA$. The reverse inclusion is trivial, since $d \in \mathfrak{D}(n/d)$. □

Lemma 3.3 applies in particular to $A = A(\mathbb{D})$ if we assume that $Z(n) \cap Z(d) = \emptyset$.

COROLLARY 3.4. *Suppose that the greatest common divisor of two elements n and d in $A(\mathbb{D})$ is a unit. Then $\mathfrak{D}(n/d)$ is a principal ideal.*

Proof. Since $A(\mathbb{D})$ is a Pre-Bézout ring (see [**12**]) we have that $\mathfrak{I}(n, d) = A(\mathbb{D})$. The rest follows from Lemma 3.3 above. □

PROPOSITION 3.5. *Let B be a finite Blaschke product and let $f \in A(\mathbb{D})$, $f \not\equiv 0$. Then $\mathfrak{D}(B/f)$ is a principal ideal generated by a specific factor of f.*

Proof. Let b be the Blaschke product formed with the common zeros of B and f (multiplicities included). Consider the function $F = f/b$ and $B^* = B/b$. We claim that $\mathfrak{D}(B/f) = \mathfrak{I}(F)$. In fact,we obviously have that $\mathfrak{D}(B/f) = \mathfrak{D}(B^*/F)$. But F does not vanish at the zeros of B^*; so, by the corona theorem for $A(\mathbb{D})$, $\mathfrak{I}(B^*, F) = A(\mathbb{D})$. By Lemma 3.3, we get that $\mathfrak{D}(B/f) = \mathfrak{D}(B^*/F) = \mathfrak{I}(F)$. □

OBSERVATION 3.6. *Let I be an ideal in $A(\mathbb{D})$. Suppose that $f \in I$ and that $f = gh$, where $g, h \in A(\mathbb{D})$ and $Z(g) \cap Z(I) = \emptyset$. Then $h \in I$.*

This follows from the fact that the maximal ideal space is $\overline{\mathbb{D}}$: indeed, the assumption implies that the ideal generated by g and I is the whole algebra; hence $1 = ag + r$ where $a \in A(\mathbb{D})$ and $r \in I$. Thus $h = a(gh) + hr \in I$.

PROPOSITION 3.7. *Let $I = \mathfrak{D}(n/d)$ and $J = \mathfrak{D}(d/n)$ be ideals of denominators in $A(\mathbb{D})$. Suppose that $Z(J) \subseteq \mathbb{D}$. Then J and I are principal ideals.*

Proof. If $Z(J) \subseteq \mathbb{D}$, then, by Lemma 3.2, J is a principal ideal generated by a finite Blaschke product B. Hence, as we will show, I is a principal ideal, too. In fact, let $\gamma = n/d$. Suppose that $J = \mathfrak{D}(d/n) = BA(\mathbb{D})$. Since $n \in J$, we have that $n = BN$ for some $N \in A(\mathbb{D})$. Since $B \in J$, $Bd = kn = kBN$; so $d = kN$. Thus $\gamma = (BN)/(kN) = B/k$. Note that k and B have no common zeros inside $\mathbb{D}$, otherwise $J = \mathfrak{D}(k/B)$ would contain a factor of B. Thus $\mathfrak{I}(B, k) = A(\mathbb{D})$. Hence, by Lemma 3.3, $I = kA(\mathbb{D})$. □

Applying Theorem 2.4, we obtain the following:

PROPOSITION 3.8. *Let $I = \mathfrak{D}(n/d)$ be an ideal of denominators in $A(\mathbb{D})$. Suppose that $Z(I) \cap Z(n) = \emptyset$. Then I is a principal ideal.*

Proof. Let $I = \mathfrak{D}(n/d)$. Without loss of generality we may assume that n and d have no common zeros (otherwise we split the joint Blaschke product and use the fact that $A(\mathbb{D})$ has the F-property; that is that $uf \in A(\mathbb{D})$ implies that $f \in A(\mathbb{D})$ for any inner function u).

Note that by our assumption, $Z(I) \subseteq Z(d) \subseteq Z(n) \cup Z(I)$, and that this union is disjoint. By Theorem 2.4 we may factor d as $d = d_1 d_2$, where $Z(d_1) = Z(I)$ and $Z(d_2) \cap Z(I) = \emptyset$. We claim that $I = I_1 := \mathfrak{D}(n/d_1)$. In fact, let $f \in I_1$. Then $fn = gd_1$ for some $g \in A(\mathbb{D})$. Then $(d_2 f)n = g(d_1 d_2) = gd$, and hence $d_2 f \in \mathfrak{D}(n/d) = I$. But $Z(d_2) \cap Z(I) = \emptyset$. Thus by the Observation 3.6 above, we have that $f \in I$. So $\mathfrak{D}(n/d_1) \subseteq \mathfrak{D}(n/d)$.

To prove the reverse inclusion, let $f \in \mathfrak{D}(n/d)$. Then $fn = hd$ for some $h \in A(\mathbb{D})$. Hence $fn = (hd_2)d_1$. So $f \in \mathfrak{D}(n/d_1)$. We conclude that $\mathfrak{D}(n/d_1) = \mathfrak{D}(n/d)$. Since $Z(d_1) \cap Z(n) = \emptyset$, we obtain from Lemma 3.3 that $I_1 (= I)$ is a principal ideal. □

Recall that for $\alpha \in \overline{\mathbb{D}}$, $M(\alpha) = \{f \in A(\mathbb{D}) : f(\alpha) = 0\}$ is the maximal ideal associated with α.

Using Theorem 2.4 and its companion Proposition 3.8, we are now ready to prove our main result on the structure of finitely generated ideals of denominators in $A(\mathbb{D})$. We note that the result would hold for the Wiener algebra W^+ of all absolutely convergent power series in $\mathbb{D}$ as well, if Theorem 2.4 and Proposition 3.8 could be proven for W^+.

THEOREM 3.9. *Let $\gamma = n/d$ be a quotient in $A(\mathbb{D})$. Then the ideal of denominators, $\mathfrak{D}(\gamma) = \{f \in A(\mathbb{D}) : f\gamma \in A(\mathbb{D})\}$, is either a principal ideal or not finitely generated.*

Proof. Associate with $I := \mathfrak{D}(\gamma)$ the set $J = \{\gamma f : f \in \mathfrak{D}(\gamma)\}$. Then it is straightforward to check that J is an ideal in $A(\mathbb{D})$, too. In fact, $J = \mathfrak{D}(1/\gamma)$.

Suppose that J is not proper; then $Z(J) := \bigcap_{f \in \mathfrak{I}(\gamma)} Z(\gamma f) = \emptyset$. By compactness, there exist finitely many $f_j \in \mathfrak{D}(\gamma)$ so that $\bigcap_{j=1}^{n} Z(\gamma f_j) = \emptyset$. Hence $1 = \sum_{j=1}^{n} g_j(\gamma f_j)$ for some $g_j \in A(\mathbb{D})$. Then $1/\gamma \in A(\mathbb{D})$; hence $\gamma = 1/a$ for some $a \in A(\mathbb{D})$. Then $\mathfrak{D}(\gamma) = aA(\mathbb{D})$, the principal ideal generated by a.

Now suppose that $Z(J) \neq \emptyset$.

Case 1. $Z(J) \cap Z(I) \neq \emptyset$. Let $\alpha \in Z(I) \cap Z(J)$. Consider any $f \in I$. Then $fn = gd$ for some $g \in J$.

If $\alpha \in \mathbb{D}$, then $f = (z - \alpha)F$ and $g = (z - \alpha)G$. Hence $Fn = Gd$ and so $F \in I$. Thus $I = I \cdot M(\alpha)$.

If $\alpha \in \mathbb{T}$, then we use the fact that the maximal ideal $M(\alpha)$ contains an approximate unit and hence by the Cohen-Varopoulos factorization theorem [**15**], for any $f, g \in M(\alpha)$, there is a joint factor $h \in M(\alpha)$ of f and g, say $f = hF$ and $g = hG$ for $F, G \in A(\mathbb{D})$. Hence $Fn = Gd$ and again $F \in I$. Thus, also in this case, $I = I \cdot M(\alpha)$.

By Lemma 3.1 above, I cannot be finitely generated.

Case 2. $Z(I) \cap Z(J) = \emptyset$. Then there exist $f, g \in I$ such that $1 = f + \frac{n}{d}g$. Hence $d = df + ng$ and so $d(1 - f) = ng$. Thus $\gamma = \frac{n}{d} = \frac{1-f}{g}$. Without loss of generality, we may assume that I is proper. Let $\alpha \in Z(I)$. Since $g \in I$, we have that $Z(I) \subseteq Z(g)$. Hence $0 = g(\alpha)$ and (since $f \in I$), $f(\alpha) = 0$, too. So $Z(I) \cap Z(1 - f) = \emptyset$. By Proposition 3.8, I is a principal ideal. □

References

1. A. Browder, *Introduction to function algebras* (Benjamin, New York, 1969).
2. J. Détraz, 'Étude du spectre d'algèbres de fonctions analytiques sur le disque unité', (French), *C. R. Acad. Sci. Paris Sér. A-B* 269 (1969), A833–A835.
3. F. Forelli, 'A note on ideals in the disk algebra', *Proc. Amer. Math. Soc.* 84 (1982), 389–392.
4. F. Forelli, 'Divisibility problems in $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$', *Lect. Notes Math.* 1043 (1984), 396–398.
5. D. Gaier, 'Vorlesungen über Approximation im Komplexen', (German) *Lectures on approximation in the complex domain*, Birkhäuser Verlag, Basel-Boston, Mass., 1980.
6. J.B. Garnett, *Bounded analytic functions* (Academic Press, New York-London, 1981).
7. K. Hoffman, *Banach spaces of analytic functions* (*Reprint of the 1962 original*) (Dover, New York, 1988).
8. R. Mortini, 'The Chang-Marshall algebras', *Mitt. Math. Sem. Giessen* 185 (1988), 1–76.
9. R. Mortini and M. von Renteln, 'Ideals in the Wiener algebra W^+', *J. Austral. Math. Soc.* 46 (1989), 220–228.
10. A. Quadrat, *An Introduction to Control Theory* (*Course Notes given at Mathematics, Algorithms, Proofs, held at Castro Urdiales, Spain, 09-13 January, 2006*). Available at `www-sop.inria.fr/cafe/Alban.Quadrat/Stage/MAP1.pdf` .
11. M. von Renteln, 'Hauptideale und äussere Funktionen im Ring H^∞', *Archiv Math.* 28 (1977), 519–524.
12. M. von Renteln, 'Divisibility structure and finitely generated ideals in the disc algebra', *Monatsh. Math.* 82 (1976) 51–56.
13. A. Sasane, 'Properties of a subalgebra of $H^\infty(\mathbb{D})$ and stabilization', *IMA Journal on Mathematical Control and Information* 25 (2008) 1–21.
14. S. Treil, 'The stable rank of the algebra H^∞ equals 1', *J. Functional Analysis* 109 (1992) 130–154.
15. N. Th. Varopoulos, 'Sur les formes positives d'une algèbre de Banach', (French) *Comptes Rendus Acad. Sci. Paris* 258 (1964) 2465–2467, Groupe 1.

Raymond Mortini
Département de Mathématiques et LMAM,
Université Paul Verlaine,
Ile du Saulcy, F-57045 Metz
France

mortini@math.univ-metz.fr

Amol Sasane
Department of Mathematics,
London School of Economics,
Houghton Street, London WC2A 2AE
United Kingdom

A.J.Sasane@lse.ac.uk